GR — Exercise sheet 3

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Special Relativity

Exercise 1.1: Having fun with indices

Imagine we have a tensor $X^{\mu\nu}$ and a vector V^{μ}, with components

$$X^{\mu\nu} = \begin{pmatrix} 2 & 0 & 1 & 0 \\ -2 & 2 & 2 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & -2 \end{pmatrix}_{\mu\nu}, \qquad V^{\mu} = (-2, \ -1, \ 0, \ 2)_{\mu}. \tag{1}$$

Defined in the Minkowki spacetime, where the metric tensor is $\eta_{\mu\nu} = diag(-1, \ 1, \ 1, \ 1)$. Using the metric tensor to raise/lower the indices ($X_{\mu\nu} = \eta_{\mu\rho}\eta_{\nu\sigma}X^{\rho\sigma}$), calculate the components of the following quantities:

(a) $X^{\mu}{}_{\nu}$

(b) $X_{\mu}{}^{\nu}$

(c) $X^{(\mu\nu)}$

(d) $X_{[\mu\nu]}$

(e) $X^{\lambda}{}_{\lambda}$

(f) $V^{\mu}V_{\mu}$

(g) $V_{\mu}X^{\mu\nu}$

Manifolds and tangent vectors

Exercise 2.1: The circle as manifold

Consider a circle of radius a and centered at the origin

$$S^1 = \left\{ p = (x_1,\ x_2) \in \mathbb{R}^2 : \sum_{i=1}^{2} x_i^2 = a^2 \right\}. \tag{2}$$

(a) Find explicitly the expressions of the stereographic projections $\varphi_\pm : O_\pm \mapsto U_\pm \subseteq \mathbb{R}$, where $O_\pm$ are respectively $S^1 \setminus N \equiv (0,\ a)$ and $S^1 \setminus S \equiv (0,\ -a)$.

(b) Show that all the conditions which define a manifold are satisfied by $\mathcal{M} = \{S^1, O_\pm, \varphi_\pm\}$ so proving that $\mathcal{M}$ is a manifold.

(c) Give an example of a set that is not a manifold and justify.

Exercise 2.2: Curves in a manifold

In Euclidean three-space, let p be the point with coordinates $(x,\ y,\ z) = (1,\ 0,\ -1)$. Consider the following curves:

$$\begin{aligned} x^i(\lambda) &= (\lambda,\ (\lambda-1)^2,\ -\lambda) \\ x^i(\mu) &= (\cos\mu,\ \sin\mu,\ \mu - 1) \\ x^i(\sigma) &= (\sigma^2,\ \sigma^3+\sigma^2,\ \sigma). \end{aligned}$$

(a) Verify that all the curves above pass through p.

(b) Calculate the components of the tangent vectors to the curves at p in the coordinate basis $\{\partial_x,\ \partial_y,\ \partial_z\}$.

(c) Let $f = x^2 + y^2 - yz$. Calculate $df/d\lambda,\ df/d\mu,\ df/d\sigma$.

Exercise 2.3: Play with commutators

Consider two smooth vector fields $v,\ w$ defined on a manifold $\mathcal{M}^n$. In coordinate basis they read:

$$v = \sum_{\alpha=1}^{n} v^\alpha \frac{\partial}{\partial x^\alpha}, \qquad w = \sum_{\beta=1}^{n} w^\beta \frac{\partial}{\partial x^\beta}. \tag{3}$$

(a) Show that the composition $vw \equiv v(w)$ is not a vector field. (Hint: apply it to smooth "test" function $f,\ g$ and show that this combination does not fulfil the definition of a tangent vector).

(b) Using the previous result, show that the commutator $[v,\ w] \equiv v(w) - w(v)$ is a vector field.

(c) Derive the components of the commutator of the two vector fields in coordinate basis.