GR — Exercise sheet 4

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Special Relativity

Exercise 1.1: Change of perspective

Using the tensor transformation law applied to $F_{\mu\nu}$, show how the electric and magnetic fields $\vec{E}$ and $\vec{B}$ transform under

(a) A rotation about the $z-$axis.

(b) A boost along the $y-$axis.

Manifolds and tangent vectors

Exercise 2.1: Jacobi identity

Consider three smooth vector fields X, Y, Z defined on a manifold $\mathcal{M}$.

(a) Show that $[X+Y, Z] = [X, Z] + [Y, Z]$.

(b) Show that the Jacobi identity holds, i.e.

$$[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0. \tag{1}$$

Exercise 2.2: Curves in spherical coordinates

Consider $\mathbb{R}^3$ as a manifold with a flat Euclidean metric, and coordinates $\{x, y, z\}$. Introduce spherical polar coordinates $\{\rho, \theta, \phi\}$, related to $\{x, y, z\}$ as follows:

$$\begin{aligned} x &= \rho \sin\theta \cos\phi \\ y &= \rho \sin\theta \sin\phi \\ z &= \rho \cos\theta. \end{aligned}$$

(a) Show that the metric components $g_{\mu\nu}$ in spherical coordinates are given by

$$ds^2 = d\rho^2 + \rho^2 d\theta^2 + \rho^2 \sin^2 \theta d\phi^2 \tag{2}$$

(b) A particle moves along a parametrized curve given by

$$\gamma^i(\lambda) = (\cos \lambda, \sin \lambda, \lambda). \tag{3}$$

Express the path of the curve in the $\{\rho, \theta, \phi\}$ system.

(c) Calculate the components of the tangent vectors to the curve in both coordinate systems. Show for at least one of the components that the vector transformation law holds.

Exercise 2.3: Transform the metric of spacetime

The spacetime metric of special relativity is

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2. \tag{4}$$

Find the components of $g_{\mu\nu}$ and $g^{\mu\nu}$ (respectively the metric and the inverse metric) in "rotating coordinates" defined by

$$\begin{aligned}
t' &= t \\
x' &= (x^2 + y^2)^{1/2} \cos(\phi - \omega t) \\
y' &= (x^2 + y^2)^{1/2} \sin(\phi - \omega t) \\
z' &= z.
\end{aligned}$$

Where $\tan \phi = y/x$.