GR — Exercise sheet 5

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Manifolds

Exercise 1.1: Infinite cylinder as manifold

Consider the infinite cylinder with the main axis coincident with the $x-$axis. It can be thought as the set $C \equiv \mathbb{S}^1 \times \mathbb{R} \equiv \{(x,\ y,\ z) \in \mathbb{R}^3 : y^2 + z^2 = 1\}$. A subset of C can be mapped to a subset of $\mathbb{R}^2$ using the following map:

$$\psi(x, y, z) = e^x(y, z). \tag{1}$$

Show that C is a manifold.

Forms

Exercise 2.1: Differential of forms

Given a $p-$form ω and a $q-$form v, the following relation holds:

$$d(\omega \wedge v) = d\omega \wedge v + (-1)^p\ \omega \wedge dv \tag{2}$$

Where $\wedge$ is the wedge product between forms defined as

$$(\omega \wedge v)_{\mu_1 \cdots \mu_p \nu_1 \cdots \nu_q} \equiv \frac{(p+q)!}{p!q!}\omega_{[\mu_1 \cdots \mu_p} v_{\nu_1 \cdots \nu_q]} \tag{3}$$

d is the exterior derivative.

$$(d\omega)_{\mu_1 \cdots \mu_p} \equiv (p+1)\partial_{[\alpha}\omega_{\mu_1 \cdots \mu_p]} \tag{4}$$

and p is the rank of ω. Given $v = v_\mu\ dx^\mu$

(a) Show that Eq. 2 is satisfied for $\omega = \omega_\nu\ dx^\nu$.

(b) Show that Eq. 2 is satisfied for $\omega = \omega_{\rho\sigma}\ dx^\rho \wedge dx^\sigma$.

Covariant derivative

Exercise 3.1: Covariant derivative

Given the expression for the components of the covariant derivative of a vector

$$\nabla_\mu v^\nu = \partial_\mu v^\nu + \Gamma^\nu_{\mu\rho} v^\rho \ , \tag{5}$$

show that if one formally assumes

$$\nabla_\mu \omega_\nu = \partial_\mu \omega_\nu + \hat{\Gamma}^\rho_{\mu\nu} \omega_\rho \ , \tag{6}$$

then $\hat{\Gamma} = -\Gamma$.

Exercise 3.2: Christoffel symbols

Derive how $\Gamma^\rho_{\mu\nu}$ transforms under a coordinate transformation. Show that the difference between two Christoffel symbols $S^\rho_{\mu\nu} := \Gamma^\rho_{\mu\nu} - \tilde{\Gamma}^\rho_{\mu\nu}$ transforms as the components of a $(1, 2)$ tensor. Discuss the above result in terms of the equation $C^c_{ab}\omega_c := \tilde{\nabla}_a \omega_b - \nabla_a \omega_b$