# GR — Exercise sheet 7

Sarp Akcay
[sarp.akcay@uni-jena.de, Abbeanum, office 202]
(Return date: 10.12.2018)

03.12.2018

## Christoffel symbols

**Exercise 1.1:** *Geodesic equations of motion using the Lagrangian approach*
Let's do Robertson-Friedmann-Walker cosmology as a precursor to later chapters. Consider the following 1+1 dimensional metric

$$g = -dt^2 + a(t)^2(1 - kr^2)^{-1}dr^2$$

(a) Using the Lagrangian approach, derive the geodesic equations for the $t$ and $r$ coordinates from Lagrange's equations.

(b) Check the geodesic equations by computing the Christoffel coefficients ${\Gamma^\alpha}_{\beta\gamma}$ for this spacetime and inserting them into the standard geodesic equation.

## Riemann tensor

**Exercise 1.2:** *Riemann tensor symmetries*
Show that in $n$ dimensions the Riemann tensor has $n^2(n^2 - 1)/12$ independent components. [There are many ways to solve this problem. One is to consider particular cases, e.g., $n = 2, 3$, and generalize from these.]

**Exercise 1.3:** *Riemann tensor*
Compute ${R_{abc}}^d$, $R_{ab}$ and $R$ for $(M, g)$ with

- $g = dx^2 + dy^2$
- $g = a^2(d\theta^2 + \sin\theta d\phi^2)$ with $a \in \mathbb{R}^+$ (2-sphere of radius $a$)
- $g = r^2 d\phi^2 + dz^2$ (Cylinder)

Note: though you may find some of the results in this exercise trivially disappointing. Think of these as practice rounds for similar calculations in more interesting spacetimes.

## Parallel transport

**Exercise 1.4:** *Parallel transport on the 2-sphere*
Consider the parallel transport of a vector along the $\theta = \theta_0$ curve, $\mathcal{C}$, on the 2-sphere of radius $R$ given by the metric

$$ds^2 = R^2 d\theta^2 + R^2 \sin^2\theta\, d\phi^2.$$

By answering the questions below, you will teach yourself how to parallel transport a given vector along this particular curve on the 2-sphere.

(a) Determine the unit tangent vector to $\mathcal{C}$.

(b) Show that along $\mathcal{C}$, the parallel transport equation for a vector $v^i$ is given by

$$\partial_\phi v^i + \Gamma^i_{j\phi} v^j = 0\,.$$

(c) Using your results from previous assignments on computing Christoffel symbols for the 2-sphere, obtain the two coupled differential equations for the components of $v^i$

$$\frac{\partial v^\theta}{\partial \phi} = \ldots,$$
$$\frac{\partial v^\phi}{\partial \phi} = \ldots.$$

(d) Solve these differential equations using the initial conditions $v^i(\phi = 0) = (v_0^\theta, v_0^\phi)^T$. [Hint: $v^\theta(\phi) = v_0^\theta \cos(\phi\cos\theta_0) + \ldots$]

(e) Finally, compute the inner product $\mathbf{u}\cdot\mathbf{v}$ to show that

$$\mathbf{u}\cdot\mathbf{v} = R v_0^\phi, \qquad \text{at the equator, i.e., } \theta_0 = \frac{\pi}{2},$$
$$\mathbf{u}\cdot\mathbf{v} = R\left(\theta_0 v_0^\phi \cos\phi - v_0^\theta \sin\phi\right), \qquad \text{near the North pole, i.e., } \theta_0 \ll 1\,.$$