GR — Exercise sheet 8

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Killing vectors

Exercise 1.1: Killing vectors and the Riemann tensor

Show that

(a) If K^a is a Killing vector, then

$$\nabla_a \nabla_b K^c = R^c{}_{bad} K^d \ .$$

[Hint: you will need Killing's equation, the Bianchi identity, and some creativity rewriting zero.]

(b) If U^a is the tangent vector to an affinely parametrized geodesic and K^a a Killing vector as before, then $U^a K_a$ is constant along that geodesic.

Linearized metric theory in general relativity

Exercise 1.2: Linearized gravity

Consider the metric of flat spacetime with a small perturbation added to it: $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, where $\eta_{\mu\nu} = \mathrm{diag}[-1, 1, 1, 1]$ is the Minkowski metric and $|h_{\mu\nu}| \ll 1$. The inverse metric is given by $g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu} + \mathcal{O}(h^2)$, where $h^{\mu\nu} = \eta^{\mu\alpha}\eta^{\nu\beta}h_{\alpha\beta}$. Throughout this exercise we will neglect terms of order h^2 and higher orders, hence the use of the word "linear". Our goal here is to obtain the Einstein equation for this spacetime using linear perturbation theory.

(a) Start by showing that

$$\Gamma^\alpha_{\beta\gamma} = \frac{1}{2}\eta^{\alpha\rho}\left(-\partial_\rho h_{\beta\gamma} + \partial_\gamma h_{\beta\rho} + \partial_\beta h_{\gamma\rho}\right) + \mathcal{O}(h^2)\,.$$

(b) Next, show that

$$R_{\mu\nu\rho\sigma} = \frac{1}{2}\left(\partial_\rho\partial_\nu h_{\mu\sigma} + \partial_\sigma\partial_\mu h_{\nu\rho} - \partial_\rho\partial_\mu h_{\nu\sigma} - \partial_\sigma\partial_\nu h_{\mu\rho}\right) + \mathcal{O}(h^2)\,.$$

(c) Then, show that

$$R_{\mu\nu} = \frac{1}{2}\left(\partial_\rho\partial_\nu h^\rho{}_\mu + \partial_\rho\partial_\mu h^\rho{}_\nu - \partial_\mu\partial_\nu h - \Box h_{\mu\nu}\right) + \mathcal{O}(h^2)\,,$$

where $\Box = \partial_\mu\partial^\mu$ is the d'Alembertian operator in flat spacetime and $h = h^\mu{}_\mu = \eta^{\mu\nu}h_{\mu\nu}$ is the trace of the perturbation term.

(d) From the Ricci tensor, obtain

$$R = \partial_\mu\partial_\nu h^{\mu\nu} - \Box h + \mathcal{O}(h^2)\,,$$

(e) Finally, put all of this together and arrive at

$$G_{\mu\nu} = \frac{1}{2}\left(\partial_\rho\partial_\nu h^\rho{}_\mu + \partial_\rho\partial_\mu h^\rho{}_\nu - \partial_\mu\partial_\nu h - \Box h_{\mu\nu} - \eta_{\mu\nu}\partial_\rho\partial_\sigma h^{\rho\sigma} + \eta_{\mu\nu}\Box h\right) + \mathcal{O}(h^2)\,.$$

Exercise 1.3: Gauge invariance in linearized gravity

Show that the above linearized $R_{\mu\nu\rho\sigma}$ is invariant under the coordinate (gauge) transformation $x^\mu \to x^\mu + \xi^\mu(x^\nu)$. Recall that, under this transformation, we have

$$h_{\mu\nu} \to h_{\mu\nu} - \partial_\mu\xi_\nu - \partial_\nu\xi_\nu\,,$$

where $\xi_\mu = \eta_{\mu\nu}\xi^\nu$ and $|\partial_\mu\xi_\nu| \ll 1$.

Schwarzschild spacetime

Exercise 1.4: Schwarzschild spacetime

Welcome to Schwarzschild spacetime! Let us begin by considering a general, spherically symmetric spacetime for which the metric can be written as follows

$$g = -e^{2\alpha(r)}dt^2 + e^{2\beta(r)}dr^2 + r^2 d\theta^2 + r^2\sin^2\theta\, d\phi^2\,.$$

Using the machinery you have learned, compute the nonzero components of

(a) The Christoffel symbols, e.g., $\Gamma^t_{tr} = \partial_r\alpha$.

(b) The Riemann tensor, e.g., $R^t{}_{\theta t\theta} = -re^{-2\beta}\partial_r\alpha$ [6 of them should suffice considering the symmetries of Riemann].

(c) The Ricci tensor, e.g., $R_{\theta\theta} = e^{-2\beta}\left[r(\partial_r\beta - \partial_r\alpha) - 1\right] + 1$. [Hint: four nonzero components in total].