GR — Exercise sheet 9

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Gravitational waves (GWs)

Exercise 1.1: GW effects on geodesics

Use the geodesics equation to show that in the TT gauge and linear order in $h_{\mu\nu}$, a particle initially at rest ($dx^i/d\tau = 0$, with τ proper time) before the wave arrival remains at rest after the arrival. This does not mean GWs have no effect. For a GW propagating along the $\hat{z}$ direction, show that the proper distance between two events $(t, x_1, 0, 0)$ and $(t, x_2, 0, 0)$ with $L = x_2 - x_1$, is given by

$$s \simeq L\left[1 + \frac{1}{2}h_+(t)\right] = L\left[1 + \frac{1}{2}A_+ \cos(\omega t)\right] \; .$$

The above formula leads to the well-known expression

$$\frac{\delta L}{L} \sim \frac{h}{2}$$

used for estimating the relative variation of distances due to the GW, where h is the so called (dimensionless) strain amplitude of the wave.

The first GW detected on Earth in 2015 (GW150914) had an amplitude of $h \sim 10^{-21}$ and was detected with an apparatus with $L \sim 4$ km. Estimate the relative distance variation that has been measured.

Exercise 1.2: Orders of magnitude

Use the quadrupole formula and dimensional analysis to obtain the following estimate for gravitational-wave amplitude

$$h \sim \left(\frac{R}{D}\right)\left(\frac{GM}{c^2 R}\right)\left(\frac{v}{c}\right)^2 \; .$$

Above, all quantities on the RHS refer to the source: R is the typical size, D is the distance to the observer, M is the mass, and v is speed. Evaluate the above formula for the following events:

- A car crashing few meters from a GW detector;
- A supernova exploding in the galaxy and detected on Earth;
- A black hole collision at cosmological distance and detected on Earth.

[You will need to do a quick search for the characteristic numbers of these sources].

Exercise 1.3: STF projector

Show that if $\bar{h}_{\mu\nu}$ is a plane wave propagating along $\hat{n}$ (unit vector), then the wave in the TT gauge can be computed as

$$h_{ij}^{\mathrm{TT}} = \Lambda_{ij}{}^{kl} h_{kl}$$

with

$$\Lambda_{ij}{}^{kl} = P_i{}^k P_j{}^l - \frac{1}{2} P_{ij} P^{kl} \tag{1}$$

$$P_{ij} = \delta_{ij} - n_i n_j \tag{2}$$

Note that the above transformation is general and can be used to transform any symmetric tensor into its symmetric-transverse-traceless part (STF).

Steps:

- Show that P_{ij} (which is symmetric) is (i) a projector, i.e. $P_{ij} = P_i{}^k P_{kj}$; (ii) is transverse, i.e. $n^i P_{ij} = 0$; (iii) its trace is 2.
- Show that $\Lambda_{ij}{}^{kl}$ is a projector, i.e. $\Lambda_{ij}{}^{kl} \Lambda_{klmn} = \Lambda_{ijmn}$; is transverse in all indexes, i.e. $n^i \Lambda_{ijkl} = 0$, $n^j \Lambda_{ijkl} = 0$, etc.; and is traceless with respect to the pairs of indices ij and kl, respectively, ie. $\Lambda^i{}_{ikl} = 0 = \Lambda_{ijk}{}^k$.
- Show that $\Lambda_{ij}{}^{kl}$ is symmetric under interchange of pairs ij, kl, by writting down its explicit form.