GR — Exercise sheet 11

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1 Tests of weak-field general relativity

Exercise 1.1: *Light deflection*
Consider a light ray trajectory in a weak, spherically symmetric, static gravitational field, passing close to a mass M (for instance our Sun) with impact parameter b. See Fig. 1. Calculate the deflection angle for small light deviations. Follow steps (a) to (e).

(a) The equation of motion for a light ray in Schwarzschild spacetime is

$$\frac{1}{1-2m/r}E^2 - \frac{1}{1-2m/r}\dot{r}^2 - \frac{L^2}{r^2} = 0, \tag{1}$$

where $m = GM/c^2$, and the constants of motion E and L correspond to energy and angular momentum, respectively. A dot denotes differentiation with respect to proper time. We are interested in orbital trajectories $r(\varphi)$. Use the definition $L := r^2\dot{\varphi}$ to show that Eq. (1) can be written as the orbit equation

$$u'' + u = 3mu^2, \tag{2}$$

where $u = u(\varphi) := 1/r(\varphi)$, and primes denote differentiation with respect to φ.

(b) Verify that the right-hand side of Eq. (2) is small when evaluated using solar parameters.

(c) Neglecting the term $3mu^2$, verify that Eq. (2) describes a straight light path $u(\varphi) = b^{-1}\sin\varphi$. Use this 0th-order solution in the right-hand side of Eq. (2) to obtain a 1st-order perturbation equation for the orbit. Solve it by finding a particular solution and the solution of the homogeneous equation.

(d) For large r, $\varphi \to \varphi_\infty$. Show that $\varphi_\infty = -2m/b$.

(e) Define the total deflection angle $\delta := 2|\varphi_\infty|$, and obtain Einstein's famous prediction $\delta = 1.75''$ for a light ray grazing the surface of the Sun.

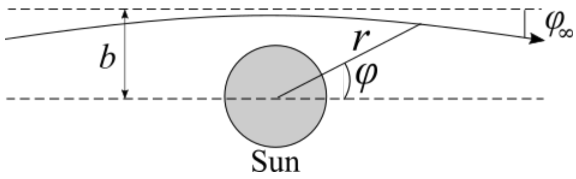

Figure 1:

Exercise 1.2: *Shapiro time delay*

Suppose that a radar signal is transmitted from point 1 with Schwarzschild coordinates $(r_1, \vartheta = \pi/2, \varphi_1)$ to point 2 with $(r_2, \vartheta = \pi/2, \varphi_2)$ [see Fig. 2], and then reflected from point 2 back to point 1. Calculate the time delay of the signal along the circuit due to the presence of the Sun. Follow the steps (a) to (d).

(a) Use the definition $E := \dot{t}\,(1 - 2m/r)$, where the dot denotes derivative with respect to proper time, to show that Eq. (1) can be written as

$$\left(1 - \frac{2m}{r}\right)^{-3} \left(\frac{dr}{dt}\right)^2 = \left(1 - \frac{2m}{r}\right)^{-1} - \left(\frac{L}{E}\right)^2 \frac{1}{r^2}. \tag{3}$$

(b) The derivative dr/dt vanishes at the radius $r = r_0$ of closest approach to the Sun. Use this fact in Eq. (3) to show that the coordinate time which the light requires to go from r_0 to r (or reverse) is

$$t(r, r_0) = \int_{r_0}^{r} \frac{dr}{1 - 2m/r} \left[1 - \frac{1 - 2m/r}{1 - 2m/r_0} \left(\frac{r_0}{r}\right)^2\right]^{-1/2}. \tag{4}$$

(c) The weak gravitational field allows you to treat the term $2m/r$ in the integrand of Eq. (4) as small. Under this assumption, obtain

$$t(r, r_0) \simeq \sqrt{r^2 - r_0^2} + 2m \ln\left(\frac{r + \sqrt{r^2 - r_0^2}}{r_0}\right) + m \left(\frac{r - r_0}{r + r_0}\right)^{1/2}.$$

(d) Along the circuit point 1 – point 2 – point 1, compute the *Shapiro delay in coordinate time* $\Delta t := 2\left(t(r_1, r_0) + t(r_2, r_0) - \sqrt{r_1^2 - r_0^2} - \sqrt{r_2^2 - r_0^2}\right)$.

Exercise 1.3: *Perturbative solution of precession equation*

Consider the equation

$$u'' + u = A + Bu^2\ , \tag{5}$$

with A, B constants. For $B = 0$ the solutions are Newtonian ellipses $u_N(\phi) = A(1 + e\cos\phi)$ characterized by their eccentricity e.

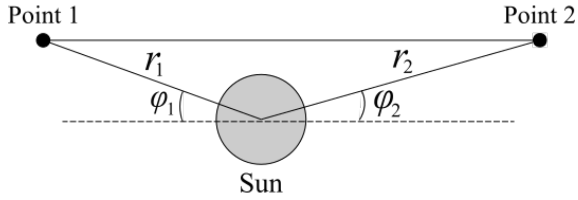

Figure 2:

(a) Let $u = u_N + v$, and derive an equation for v.

(b) Linearize the equation for v, $v'' + v = s$, where s is a term that does not depend on v.

(c) The linearized equation corresponds to a forced oscillator; its solution is given by the general solution of the homogeneous equation plus a particular solution of the full equation.

(d) Verify that each of the equations on the left has the particular solution on the right:

$$v'' + v = C \qquad v = C \tag{6}$$

$$v'' + v = C\cos\phi \qquad v = \frac{C}{2}\phi\sin\phi \tag{7}$$

$$v'' + v = C\cos^2\phi \qquad v = \frac{C}{2} - \frac{C}{6}\cos(2\phi) \tag{8}$$

(e) Write down the solution of the linearized equation for v, and discuss the effect of each of the three terms.