GR — Exercise sheet 12

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1 Compact stars

Exercise 1.1: Schwarzschild spacetime: Interior solution

To compute the interior solution of Einstein Equations (EE) for spherical symmetric spacetime, we can start from the expression of the metric

$$ds^2 = -e^{2\alpha(r)}dt^2 + e^{2\beta(r)}dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta\, d\phi^2 \,.$$

and use it to compute the left hand side of Einstein Equations in presence of matter

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}R g_{\mu\nu} = 8\pi G T_{\mu\nu}.$$

(a) We model the star as a perfect fluid, so the energy-momentum tensor has the form

$$T_{\mu\nu} = \rho(r) u_\mu u_\nu + P(r)(u_\mu u_\nu + g_{\mu\nu})$$

Where u^μ is the velocity of the fluid which, for static fluids, has to be time-like; ρ is the energy density and P is the pressure. Derive the expression for u^μ that allows to interpret ρ as a total energy-mass density ($T_{00} \propto \rho$ only) and write down the components of $T_{\mu\nu}$.

(b) Given the only non-vanishing components of the Einstein Tensor

$$\begin{aligned}
G_{tt} =& \frac{1}{r^2} e^{2(\alpha-\beta)} \left(2r\partial_r\beta - 1 + e^{2\beta}\right) \\
G_{rr} =& \frac{1}{r^2} \left(2r\partial_r\alpha + 1 - e^{2\beta}\right) \\
G_{\theta\theta} =& r^2 e^{-2\beta} \left[\partial_r^2\alpha + (\partial_r\alpha)^2 - \partial_r\alpha\partial_r\beta + \frac{1}{r}(\partial_r\alpha - \partial_r\beta)\right] \\
G_{\phi\phi} =& \sin^2\theta\; G_{\theta\theta}
\end{aligned}$$

write down the $tt,\ rr,\ \theta\theta$ components of the Einstein Equations.

(c) To start with the calculation is convenient to define the function

$$m(r) = \frac{1}{2G}(r - re^{-2\beta}).$$

Use the tt component of the EEs to show that

$$m(r) = 4\pi \int_0^r \rho(r')r'^2 dr'.$$

(d) The Schwarzschild mass M is defined as $M = m(R)$, where R is the radius of the star. Explain why this is not the total integrated energy density $\bar{M}$ and find the expression of $\bar{M}$ in terms of $\rho(r),\ m(r),\ R$.
Hint: think about the volume element in a curved space.

(e) Use now the rr equation to derive $\frac{d\alpha}{dr}$ in terms of $m(r)$.

(f) Show that the energy-momentum conservation law $\nabla_\mu T^{\mu\nu} = 0$, leads to the relation

$$(\rho + P)\frac{d\alpha}{dr} = -\frac{dP}{dr}.$$

Hint: the only non-trivial term corresponds to the $\nu = r$ component. Use the Christoffel symbols for a spherical spacetime you have been given in the notes.

Finally, combine the last two equations to get

$$\frac{dP}{dr} = -\frac{(\rho + P)[Gm(r) + 4\pi G r^3 P]}{r[r - 2Gm(r)]}.$$

This is the Tolman-Oppenheimer-Volkoff (TOV) equation, i.e. the equation of hydrostatic equilibrium. Coupling it with the so-called Equation of State $P = P(\rho)$ we get a closed system of equations which models the star.

Exercise 1.2: The Newtonian limit of TOV equation

(a) Explain why, in this context, the Newtonian limit is reached when $P \ll \rho, m(r) \ll r$. How does the term $[Gm(r) + 4\pi G r^3 P]$ become in this limit?
Hint: restore the physical units.

(b) Starting from the result of the previous exercise, derive the TOV equation in the Newtonian limit.

Exercise 1.3: Buchdal's theorem for an incompressible star

The simplest way to model a star is to take an incompressible fluid, i.e. to consider a fluid with a constant density up to the star's surface of radius R as follows

$$\rho(r) = \begin{cases} \rho_*, & r < R \\ 0, & r \geq R \end{cases}$$

(a) Calculate the expression of $m(r)$ for this density, where $r \in [0, \infty)$.

(b) Use the TOV equations to verify that

$$P(r) = \rho_* \left[\frac{R\sqrt{R - 2GM} - \sqrt{R^3 - 2GMr^2}}{\sqrt{R^3 - 2GMr^2} - 3R\sqrt{R - 2GM}} \right]$$

(c) A consequence of the formula above is the existence of a maximum value for the mass that can be squeezed inside a sphere of radius R, beyond which the *central pressure* of the star would become infinite. Find the value of M_{max}.