GR — Exercise sheet 3

Francesco Zappa
[francesco.zappa@uni-jena.de, Abbeanum, office 219]
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Special Relativity

Exercise 1.1: Hyperbolic orthogonality

Two four-vectors a^μ, b^ν are orthogonal if $a_\mu b^\mu = 0$. Show that (a) the sum of two vectors can be spacelike, null or timelike independently of the nature of the vectors, and (b) non-spacelike vectors which are orthogonal to a given nonzero null vector, must be multiples of the null vector. (c) Find four linearly independent null vectors in the Minkowski space.

Exercise 1.2: Having fun with indices

Imagine we have a tensor $X^{\mu\nu}$ and a vector V^μ, with components

$$X^{\mu\nu} = \begin{pmatrix} 2 & 0 & 1 & 0 \\ -2 & 2 & 2 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & -2 \end{pmatrix}_{\mu\nu} , \qquad V^\mu = (-2, \; -1, \; 0, \; 2)_\mu. \tag{1}$$

Defined in the Minkowki spacetime, where the metric tensor is $\eta_{\mu\nu} = diag(-1, \; 1, \; 1, \; 1)$. Using the metric tensor to raise/lower the indices ($X_{\mu\nu} = \eta_{\mu\rho}\eta_{\nu\sigma}X^{\rho\sigma}$), calculate the components of the following quantities:

(a) $X^\mu{}_\nu$

(b) $X_\mu{}^\nu$

(c) $X^{(\mu\nu)}$

(d) $X_{[\mu\nu]}$

(e) $X^\lambda{}_\lambda$

(f) $V^\mu V_\mu$

(g) $V_\mu X^{\mu\nu}$

Manifolds

Exercise 2.1: Curves in a manifold

In Euclidean three-space, let p be the point with coordinates $(x,\ y,\ z) = (1,\ 0,\ -1)$. Consider the following curves:

$$x^i(\lambda) = (\lambda,\ (\lambda-1)^2,\ -\lambda)$$
$$x^i(\mu) = (\cos\mu,\ \sin\mu,\ \mu-1)$$
$$x^i(\sigma) = (\sigma^2,\ \sigma^3+\sigma^2,\ \sigma).$$

(a) Verify that all the curves above pass through p.

(b) Calculate the components of the tangent vectors to the curves at p in the coordinate basis $\{\partial_x,\ \partial_y,\ \partial_z\}$.

(c) Let $f = x^2 + y^2 - yz$. Calculate $df/d\lambda,\ df/d\mu,\ df/d\sigma$.

Exercise 2.2: Infinite cylinder as manifold

Consider the infinite cylinder with the main axis coincident with the $x-$axis. It can be thought as the set $C \equiv \mathbb{S}^1 \times \mathbb{R} \equiv \{(x,\ y,\ z) \in \mathbb{R}^3 : y^2 + z^2 = 1\}$. A subset of C can be mapped to a subset of $\mathbb{R}^2$ using the following map:

$$\psi(x, y, z) = e^x(y, z). \tag{2}$$

Show that C is a manifold.