GR — Exercise sheet 4

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(Return date: 18.11.19)

10.11.2019

Tangent vectors, Metric, p-forms

Exercise 1.1: Play with commutators

Consider two smooth vector fields $v,\ w$ defined on a manifold $\mathcal{M}^n$. In coordinate basis they read:

$$v = \sum_{\alpha=1}^{n} v^\alpha \frac{\partial}{\partial x^\alpha}, \qquad w = \sum_{\beta=1}^{n} w^\beta \frac{\partial}{\partial x^\beta}. \tag{1}$$

(a) Show that the composition $v \circ w \equiv v(w)$ is not a vector field. (Hint: apply it to smooth "test" function $f,\ g$ and show that this combination does not fulfil the definition of a tangent vector).

(b) Using the previous result, show that the commutator $[v,\ w] \equiv v(w) - w(v)$ is a vector field.

(c) Derive the components of the commutator of the two vector fields in coordinate basis.

Exercise 1.2: Jacobi identity

Consider three smooth vector fields $X,\ Y,\ Z$ defined on a manifold $\mathcal{M}$.

(a) Show that $[X+Y, Z] = [X, Z] + [Y, Z]$.

(b) Show that the Jacobi identity holds, i.e.

$$[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0. \tag{2}$$

Exercise 1.3: Curves in spherical coordinates

Consider $\mathbb{R}^3$ as a manifold with a flat Euclidean metric, and coordinates $\{x, y, z\}$. Introduce spherical polar coordinates $\{\rho, \theta, \phi\}$, related to $\{x, y, z\}$ as follows:

$$x = \rho \sin\theta \cos\phi$$
$$y = \rho \sin\theta \sin\phi$$
$$z = \rho \cos\theta.$$

(a) Show that the metric components $g_{\mu\nu}$ in spherical coordinates are given by

$$ds^2 = d\rho^2 + \rho^2 d\theta^2 + \rho^2 \sin^2\theta d\phi^2 \tag{3}$$

(b) A particle moves along a parametrized curve given by

$$\gamma^i(\lambda) = (\cos\lambda, \sin\lambda, \lambda). \tag{4}$$

Express the path of the curve in the $\{\rho, \theta, \phi\}$ system.

(c) Calculate the components of the tangent vectors to the curve in both coordinate systems. Show for at least one of the components that the vector transformation law holds.

Exercise 1.4: Forms determinant and coordinate transformation

Consider a n-dimensional manifold M and the space of p-forms $\Lambda_p M$. Prove the following statements about p-forms.

(a) The dimension of $\Lambda_p M$ is $n!/p!(n-p)!$.

(b) Calculate explicitely how the component of a 2-form in a 2-manifold transforms under coordinate transformation.

(c) Derive a formula for the determinant of a $N \times N$ matrix in terms of the antisymmetric Levi-Civita $\epsilon_{i_1 \dots i_N}$.

(d) Calculate how the determinant of the metric change under coordinate transformation.

BONUS (1)

Exercise 1.5: Differential of forms

Given a $p-$form ω and a $q-$form v, the following relation holds:

$$d(\omega \wedge v) = d\omega \wedge v + (-1)^p \; \omega \wedge dv \tag{5}$$

Where $\wedge$ is the wedge product between forms defined as

$$(\omega \wedge v)_{\mu_1 \cdots \mu_p \nu_1 \cdots \nu_q} \equiv \frac{(p+q)!}{p!q!} \omega_{[\mu_1 \cdots \mu_p} v_{\nu_1 \cdots \nu_q]} \tag{6}$$

d is the exterior derivative.

$$(d\omega)_{\mu_1 \cdots \mu_p} \equiv (p+1) \partial_{[\alpha} \omega_{\mu_1 \cdots \mu_p]} \tag{7}$$

and p is the rank of ω. Given $v = v_\mu \; dx^\mu$

(a) Show that Eq. 5 is satisfied for $\omega = \omega_\nu \; dx^\nu$.

(b) Show that Eq. 5 is satisfied for $\omega = \omega_{\rho\sigma} \; dx^\rho \wedge dx^\sigma$.