GR — Exercise sheet 5

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Christoffel symbols

Exercise 1.1: Christoffel symbols for weak field metric

Compute the Christoffel symbols for the static, weak-field metric

$$g = -(1+2\phi)dt^2 + (1-2\phi)\left(dx^2 + dy^2 + dz^2\right)$$

with $\phi = \phi(x, y, z)$ and $\phi/c^2 \ll 1$ [discard contributions $\mathcal{O}(\phi/c^2)^2$].

Covariant derivative

Exercise 2.1: Covariant derivative

Given the expression for the components of the covariant derivative of a vector

$$\nabla_\mu v^\nu = \partial_\mu v^\nu + \Gamma^\nu_{\mu\rho} v^\rho \ , \tag{1}$$

show that if one formally assumes

$$\nabla_\mu \omega_\nu = \partial_\mu \omega_\nu + \hat{\Gamma}^\rho_{\mu\nu} \omega_\rho \ , \tag{2}$$

then $\hat{\Gamma} = -\Gamma$.

Exercise 2.2: Christoffel symbols and tensors

Derive how $\Gamma^\rho_{\mu\nu}$ transforms under a coordinate transformation. Show that the difference between two Christoffel symbols $S^\rho_{\mu\nu} := \Gamma^\rho_{\mu\nu} - \tilde{\Gamma}^\rho_{\mu\nu}$ transforms as the components of a $(1, 2)$ tensor. Discuss the above result in terms of the equation $C^c_{ab}\omega_c := \tilde{\nabla}_a \omega_b - \nabla_a \omega_b$

Riemann tensor

Exercise 3.1: Riemann tensor symmetries

Show that in n dimensions the Riemann tensor has $n^2(n^2-1)/12$ independent components. [There are many ways to solve this problem. One is to consider particular cases, e.g., $n = 2, 3$, and generalize from these.]

Exercise 3.2: Riemann tensor in coordinate basis

Show that in a coordinate basis, the components of the Riemann tensor can be written as

$$R^\rho{}_{\sigma\mu\nu} = \partial_\mu \Gamma^\rho_{\nu\sigma} - \partial_\nu \Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\alpha}\Gamma^\alpha_{\nu\sigma} - \Gamma^\rho_{\nu\alpha}\Gamma^\alpha_{\mu\sigma}\,.$$