# GR — Exercise sheet 7

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## Parallel transport

### Exercise 1.1: Parallel transport on the 2-sphere

Consider the parallel transport of a vector along the $\theta = \theta_0$ curve, $\mathcal{C}$, on the 2-sphere of radius $R$ given by the metric

$$ds^2 = R^2 d\theta^2 + R^2 \sin^2\theta \, d\phi^2.$$

By answering the questions below, you will teach yourself how to parallel transport a given vector along this particular curve on the 2-sphere.

(a) Determine the unit tangent vector, $u^i$, to $\mathcal{C}$.

(b) Show that along $\mathcal{C}$, the parallel transport equation for a vector $v^i$ is given by

$$\partial_\phi v^i + \Gamma^i_{j\phi} v^j = 0\,.$$

(c) Using your results from previous assignments on computing Christoffel symbols for the 2-sphere, obtain the two coupled differential equations for the components of $v^i$

$$\frac{\partial v^\theta}{\partial \phi} = \ldots,$$
$$\frac{\partial v^\phi}{\partial \phi} = \ldots.$$

(d) Solve these differential equations using the initial conditions $v^i(\phi = 0) = (v_0^\theta, v_0^\phi)^T$. [Hint: $v^\theta(\phi) = v_0^\theta \cos(\phi\cos\theta_0) + \ldots$]

(e) Finally, compute the inner product $\mathbf{u} \cdot \mathbf{v}$ to show that

$$\mathbf{u} \cdot \mathbf{v} = R v_0^\phi, \qquad \text{at the equator, i.e., } \theta_0 = \frac{\pi}{2},$$
$$\mathbf{u} \cdot \mathbf{v} = R \left( \theta_0 v_0^\phi \cos\phi - v_0^\theta \sin\phi \right), \qquad \text{near the North pole, i.e., } \theta_0 \ll 1\,.$$

## Geodesics

### Exercise 2.1: Geodesic equation and affine parametrization

The geodesic equation could be formulated as

$$t^a \nabla_a t^b = \alpha t^b \ , \ \alpha \in \mathbb{R}$$

instead of

$$t^a \nabla_a t^b = 0 \ .$$

(a) Show that any curve with tangent vector $t^\mu = \dot{x}^\mu(\sigma)$ that satisfies the first equation can be reparametrized to $t^\mu = x^\mu(\lambda)$ so to satisfy the second equation (i.e. it is possible to set $\alpha = 0$ and take the second equation as definition).

(b) The parametrization that satisfies the second equation is called *affine parametrization*. Show that all other affine parameters are linear combination $a\lambda + b$ with $a, b \in \mathbb{R}$.

## Fields on manifolds

### Exercise 3.1: Calculations with symmetric/antisymmetric tensors

Given the $(0,2)$ tensors $T_{ab}$ (generic), $A_{ab} = A_{[ab]}$ (antisymm.), $S_{ab} = S_{(ab)}$ (symm.), and the Ricci $R_{ab}$ tensor (symm.), prove the following Key properties:

(a) $S^{ab} A_{ab} = 0$

(b) $[\nabla_a, \nabla_b] T^{ab} = R_{ab}(T^{ab} - T^{ba}) = R_{ab} 2 T^{[ab]} = 0$

(c) $0 = [\nabla_a, \nabla_b] A^{ab} = 2 \nabla_a \nabla_b A^{ab}$.

Note: from these properties many physics result follow!

## Killing vectors

### Exercise 4.1: Killing vectors and the Riemann tensor

Show that

(a) If $K^a$ is a Killing vector, then

$$\nabla_a \nabla_b K^c = R^c{}_{bad} K^d \ .$$

[Hint: you will need Killing's equation, the Bianchi identity, and some creativity rewriting zero.]

(b) If $U^a$ is the tangent vector to an affinely parametrized geodesic and $K^a$ a Killing vector as before, then $U^a K_a$ is constant along that geodesic.