GR — Exercise sheet 8

Sebastiano Bernuzzi
[sebastiano.bernuzzi@uni-jena.de, Abbeanum, office 202]
(Return date: 16.12.19)

08.12.2019

Fields on manifolds

Exercise 1.1: Stress-energy tensor for a scalar field

(a) Given the action of matter for a scalar field ϕ on an arbitrary 4-dimensional Lorenzian manifold with metric g_{ab} (whose determinant is denoted by g),

$$S_{\mathrm{M}}[g_{ab}, \phi] = -\frac{1}{2} \int d^4x \sqrt{-g} \left\{ \nabla_c \phi \nabla^c \phi + V(\phi) \right\},$$

where V is an arbitrary potential, derive the explicit form of the stress-energy tensor, defined as

$$T_{ab} := -\frac{2}{\sqrt{-g}} \frac{\delta S_M}{\delta g^{ab}} \ .$$

(b) Derive conservation equations from $\nabla_a T^{ab} = 0$, and discuss them.

Linearized GR

Exercise 2.1: Linearized gravity

Consider the metric of flat spacetime with a small perturbation added to it: $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, where $\eta_{\mu\nu} = \mathrm{diag}[-1, 1, 1, 1]$ is the Minkowski metric and $|h_{\mu\nu}| \ll 1$. The inverse metric is given by $g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu} + \mathcal{O}(h^2)$, where $h^{\mu\nu} = \eta^{\mu\alpha}\eta^{\nu\beta}h_{\alpha\beta}$. Throughout this exercise we will neglect terms of order h^2 and higher orders, hence the use of the word “linear”. Our goal here is to obtain the Einstein equation for this spacetime using linear perturbation theory.

(a) Start by showing that

$$\Gamma^{\alpha}_{\beta\gamma} = \frac{1}{2} \eta^{\alpha\rho} \left(-\partial_\rho h_{\beta\gamma} + \partial_\gamma h_{\beta\rho} + \partial_\beta h_{\gamma\rho} \right) + \mathcal{O}(h^2) \, .$$

(b) Next, show that

$$R_{\mu\nu\rho\sigma} = \frac{1}{2}\left(\partial_\rho\partial_\nu h_{\mu\sigma} + \partial_\sigma\partial_\mu h_{\nu\rho} - \partial_\rho\partial_\mu h_{\nu\sigma} - \partial_\sigma\partial_\nu h_{\mu\rho}\right) + \mathcal{O}(h^2)\,.$$

(c) Then, show that

$$R_{\mu\nu} = \frac{1}{2}\left(\partial_\rho\partial_\nu h^\rho{}_\mu + \partial_\rho\partial_\mu h^\rho{}_\nu - \partial_\mu\partial_\nu h - \Box h_{\mu\nu}\right) + \mathcal{O}(h^2)\,,$$

where $\Box = \partial_\mu\partial^\mu$ is the d'Alembertian operator in flat spacetime and $h = h^\mu{}_\mu = \eta^{\mu\nu}h_{\mu\nu}$ is the trace of the perturbation term.

(d) From the Ricci tensor, obtain

$$R = \partial_\mu\partial_\nu h^{\mu\nu} - \Box h + \mathcal{O}(h^2)\,,$$

(e) Finally, put all of this together and arrive at

$$G_{\mu\nu} = \frac{1}{2}\left(\partial_\rho\partial_\nu h^\rho{}_\mu + \partial_\rho\partial_\mu h^\rho{}_\nu - \partial_\mu\partial_\nu h - \Box h_{\mu\nu} - \eta_{\mu\nu}\partial_\rho\partial_\sigma h^{\rho\sigma} + \eta_{\mu\nu}\Box h\right) + \mathcal{O}(h^2)\,.$$

Exercise 2.2: Gauge invariance in linearized gravity

Show that the above linearized $R_{\mu\nu\rho\sigma}$ is invariant under the coordinate (gauge) transformation $x^\mu \to x^\mu + \xi^\mu(x^\nu)$. Recall that, under this transformation, we have

$$h_{\mu\nu} \to h_{\mu\nu} - \partial_\mu\xi_\nu - \partial_\nu\xi_\nu\,,$$

where $\xi_\mu = \eta_{\mu\nu}\xi^\nu$ and $|\partial_\mu\xi_\nu| \ll 1$.