GR — Exercise sheet 9

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Light bending

Exercise 1.1: Light bending (2 points)

Consider the motion of photons (light rays) in a static, central gravitational field with potential $\phi(x, y, z) = \phi(r)$.

- Start with a Newtonian calculation: derive the equation of motion for the Lagrangian written in spherical coordinates and with a central potential:

$$L = \frac{1}{2}[(\frac{dr}{dt})^2 + r^2(\frac{d\varphi}{dt})^2] - \phi(r) \ .$$

  Write the Euler-Lagrange EOM in terms of $u = 1/r$ and introduce the angular momentum $\ell := r^2 d\varphi/dt$ (integral of motion, say why). Specify the equations for $\phi = -GM/r$.

- In GR the equations of motion for a light ray are

$$\frac{1}{1+2\phi}e^2 - \frac{1}{1+2\phi}\dot{r}^2 - \frac{\ell^2}{r^2} = 0, \quad (1)$$

  and the constants of motion e and ℓ correspond to energy and angular momentum, respectively. A dot denotes differentiation with respect to proper time. We are interested in orbital trajectories $r(\varphi)$. Use the definition $\ell := r^2\dot{\varphi}$ and $\phi = -m/r$ with $m = GM/c^2$ to show that Eq. (1) can be written as the orbit equation

$$u'' + u = 3mu^2, \quad (2)$$

  where $u = u(\varphi) := 1/r(\varphi)$, and primes denote differentiation with respect to φ.

- Verify that the right-hand side of Eq. (2) is small when evaluated using solar parameters.

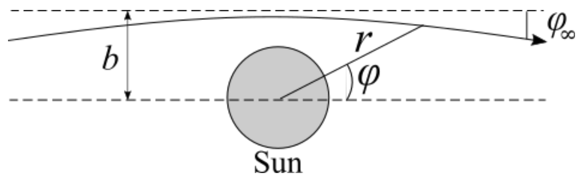

Figure 1: Geometry of light bending problem.

- Note Eq. (2) is similar to the Newtonian equation derived above. Neglecting the term $3mu^2$, verify that Eq. (2) describes a straight light path $u(\varphi) = b^{-1}\sin\varphi$.
- Use the Newtonian (0th-order) solution in the right-hand side of Eq. (2) to obtain a 1st-order perturbation equation for the orbit. Solve it by finding a particular solution and the solution of the homogeneous equation.
- For large r, $\varphi \to \varphi_\infty$. Show that $\varphi_\infty = -2m/b$.
- Define the total deflection angle $\delta := 2|\varphi_\infty|$, and obtain Einstein's famous prediction $\delta = 1.75''$ for a light ray grazing the surface of the Sun.
- **(BONUS: 1 extra point!)** Can you find an argument to obtain Eq. 1 (or a similar one...) from the weak metric? [Hint: do not perform full the calculation of geodesics, think about symmetries and the GR Lagrangian. What do you obtain?]

Gravitational waves

Exercise 2.1: Orders of magnitude

(a) Use the quadrupole formula and dimensional analysis to obtain the following estimate for gravitational-wave amplitude

$$h \sim \left(\frac{R}{D}\right)\left(\frac{GM}{c^2 R}\right)\left(\frac{v}{c}\right)^2 \ .$$

Above, all quantities on the RHS refer to the source: R is the typical size, D is the distance to the observer, M is the mass of the source, and v its typical speed.

(b) Evaluate the above formula for the following events:

- A car crashing few meters from a GW detector;
- A supernova exploding in the galaxy and detected on Earth;
- A black hole collision at cosmological distance and detected on Earth.

[You will need to do a quick search for the characteristic numbers of these sources].

Exercise 2.2: STF projector

Show that if $\bar{h}_{\mu\nu}$ is a plane wave propagating along $\hat{n}$ (unit vector), then the wave in the TT gauge can be computed as

$$h_{ij}^{\rm TT} = \Lambda_{ij}{}^{kl} h_{kl}$$

with

$$\Lambda_{ij}{}^{kl} = P_i{}^k P_j{}^l - \frac{1}{2} P_{ij} P^{kl} \tag{3}$$

$$P_{ij} = \delta_{ij} - n_i n_j \tag{4}$$

Note that the above transformation is general and can be used to transform any symmetric tensor into its symmetric-transverse-traceless part (STF).

Steps:

(a) Show that P_{ij} (which is symmetric) is (i) a projector, i.e. $P_{ij} = P_i{}^k P_{kj}$; (ii) is transverse, i.e. $n^i P_{ij} = 0$; (iii) its trace is 2.

(b) Show that $\Lambda_{ij}{}^{kl}$ is a projector, i.e. $\Lambda_{ij}{}^{kl}\Lambda_{klmn} = \Lambda_{ijmn}$; is transverse in all indexes, i.e. $n^i\Lambda_{ijkl} = 0$, $n^j\Lambda_{ijkl} = 0$, etc.; and is traceless with respect to the pairs of indices ij and kl, respectively, ie. $\Lambda^i{}_{ikl} = 0 = \Lambda_{ijk}{}^k$.

(c) Show that Λ_{ijkl} is symmetric under interchange of pairs ij, kl, by writting down its explicit form.

Exercise 2.3: GW150914

The first GW detected on Earth in 2015 had an amplitude of $h \sim 10^{-21}$ and was detected with an apparatus with $L \sim 4$ km. Estimate the relative distance variation that has been measured.