GR — Exercise sheet 10

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(Return date: 13.01.2020)

03.01.2020

1 Killing vectors

Exercise 1.1: Killing vectors of the 2-sphere

Compute the three Killing vectors $R_{(3)}, R_{(2)}, R_{(1)}$ of the 2-sphere with respect to the metric

$$g = d\vartheta^2 + \sin^2\vartheta d\varphi^2,$$

and verify that they satisfy the algebra

$$\begin{aligned}
\left[R_{(3)}, R_{(2)}\right] &= R_{(1)}, \\
\left[R_{(2)}, R_{(1)}\right] &= R_{(3)}, \\
\left[R_{(1)}, R_{(3)}\right] &= R_{(2)}.
\end{aligned}$$

A manifold is said to be spherically symmetric if and only if it admits three Killing vectors sastifying the above commutators. **Hint**: A possible way to solve this is to consider the rotational symmetry of $\mathbb{R}^3$ in Cartesian and spherical coordinates (as you computed in previous exercises), and find first the Killing vector corresponding to rotations about the z-axis ($R_{(3)}$) and then the others.

2 Schwarzschild spacetime

Exercise 2.1: Schwarzschild spacetime

Consider the general, spherically symmetric, stationary spacetime metric

$$g = -e^{2\alpha(r)}dt^2 + e^{2\beta(r)}dr^2 + r^2 d\theta^2 + r^2\sin^2\theta\, d\phi^2\,,$$

and recall the corresponding calculation of the Riemann tensor.

(a) In vacuum, the Eisntein field equations reduce to $R_{\mu\nu} = 0$. Show that these equations imply $\exp\left[2\alpha(r)\right] = \exp\left[-2\beta(r)\right]$. You can fix integration constants by rescaling the time coordinate.

(b) Using your previous result show that

$$e^{2\alpha} = 1 - \frac{R_S}{r},$$

where R_S is an integration constant called *Schwarzschild radius.* In physical units, $R_S = 2GM/c^2$ which becomes $R_S = 2M$ in geometric units. You must justify the minus sign.

(c) Now, put all of this together and rewrite the original metric g above in terms of Schwarzschild coordinates $\{t, r, \theta, \phi\}$ and the mass M. Later you will learn that R_S marks the causal boundary (called *event horizon*) of a Schwarzschild black hole.

3 Tests of weak-field general relativity

Exercise 3.1: Shapiro time delay

Suppose that a radar signal is transmitted from point 1 with coordinates $(r_1, \vartheta = \pi/2, \varphi_1)$ to point 2 with $(r_2, \vartheta = \pi/2, \varphi_2)$ [see Fig. 1], and then reflected from point 2 back to point 1. Calculate the time delay of the signal along the circuit due to the presence of the Sun.

This calculation can be performed in the weak-field metric and also for a generic spherically symmmetric spacetime. You will do the generic calculation using the equation of motion for light rays in the Schwarzschild metric

$$\left(1 - \frac{2m}{r}\right)^{-3} \left(\frac{dr}{dt}\right)^2 = \left(1 - \frac{2m}{r}\right)^{-1} - \left(\frac{L}{E}\right)^2 \frac{1}{r^2}, \tag{1}$$

where $m = GM/c^2$, and the constants of motion E and L correspond to energy and angular momentum of the photon. Follow the steps (a) to (c).

(a) The derivative dr/dt vanishes at the radius $r = r_0$ of closest approach to the Sun. Use this fact in Eq. (1) to show that the coordinate time which the light requires to go from r_0 to r (or reverse) is

$$t(r, r_0) = \int_{r_0}^{r} \frac{dr}{1 - 2m/r} \left[1 - \frac{1 - 2m/r}{1 - 2m/r_0} \left(\frac{r_0}{r}\right)^2\right]^{-1/2}. \tag{2}$$

(b) The weak gravitational field allows you to treat the term $2m/r$ in the integrand of Eq. (2) as small. Under this assumption, obtain

$$t(r, r_0) \simeq \sqrt{r^2 - r_0^2} + 2m \ln\left(\frac{r + \sqrt{r^2 - r_0^2}}{r_0}\right) + m \left(\frac{r - r_0}{r + r_0}\right)^{1/2}.$$

(c) Along the circuit point 1 – point 2 – point 1, compute the *Shapiro delay in coordinate time* $\Delta t := 2\left(t(r_1, r_0) + t(r_2, r_0) - \sqrt{r_1^2 - r_0^2} - \sqrt{r_2^2 - r_0^2}\right)$.

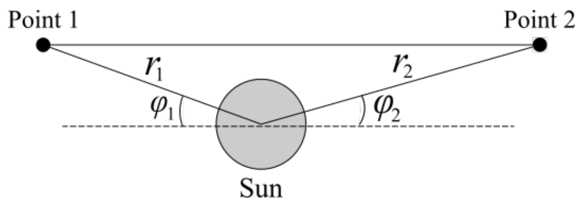

Figure 1: Geometry of the Shapiro delay. The Shapiro time delay is one of the fundamental test of GR in the Solar system. Moreover it plays an important role in the calculation of the time of arrival of radio pulses from pulsars sources. The Shapiro time delay has to be taken into account to analyze the famous Hulse&Taylor pulsar data.