GR — Exercise sheet 11

Sarp Akcay
[sarp.akcay@uni-jena.de, Abbeanum, office 202]
(Return date: 20.01.2020)

12.01.2020

1 Mercury perihelion

Exercise 1.1: Perturbative solution of precession equation

Consider the equation

$$u'' + u = A + Bu^2 \ , \tag{1}$$

with A, B constants. For $B = 0$ the solutions are Newtonian ellipses $u_N(\phi) = A(1 + e\cos\phi)$ characterized by their eccentricity e.

(a) Let $u = u_N + v$, and derive an equation for v.

(b) Linearize the equation for v, $v'' + v = s$, where s is a term that does not depend on v.

(c) The linearized equation corresponds to a forced oscillator; its solution is given by the general solution of the homogeneous equation plus a particular solution of the full equation.

(d) Verify that each of the equations on the left has the particular solution on the right:

$$v'' + v = C \qquad\qquad v = C \tag{2}$$

$$v'' + v = C\cos\phi \qquad\qquad v = \frac{C}{2}\phi\sin\phi \tag{3}$$

$$v'' + v = C\cos^2\phi \qquad\qquad v = \frac{C}{2} - \frac{C}{6}\cos(2\phi) \tag{4}$$

(e) Write down the solution of the linearized equation for v, and discuss the effect of each of the three terms.

2 Recap: Perfect fluid

Exercise 2.1: Perfect fluid

Consider the perfect fluid model for the stress energy tensor

$$T = (\rho + p) u \otimes u + p g \ ,$$

where u is the 1-form associated to the fluid's 4-velocity, ρ, p are scalars and g is the metric. Use the general definition of the stress-energy tensor to

(a) Justify that ρ is the energy density measured by an observer comving with the fluid.

(b) Argument that p is the pressure in the fluid's comoving frame.

(c) Calculate the expression for the momentum density of the matter and of the stress tensor of the matter.

(d) Introduce the properly the speed of light c and discuss that in the slow velocity limit $|T_{00}| \gg |T_{0i}| \gg |T_{ij}|$.

[Hint: consider an observer with 4-velocity v and a set of 3 unit vectors e_i perpendicular to v and use the definition of $T(.,.)$]

3 Recap: Parallel Transport

Exercise 3.1: Parallel transport on the 2-torus

Consider the 2-torus

$$T^2 = \left\{ (x, y, z) \in \mathbb{R}^3 | \left(\sqrt{x^2 + y^2} - R \right)^2 + z^2 = r^2 \right\} ,$$

where r is the radius of the circular section of the torus, and R is the distance between the center of the torus and the center of the circular section. r and R are constants such that $r < R$. The parametric representation of the torus reads

$$\begin{cases} x(\theta, \phi) & = (R + r\cos\theta)\cos\phi, \\ y(\theta, \phi) & = (R + r\cos\theta)\sin\phi, \\ z(\theta, \phi) & = r\sin\theta, \end{cases}$$

where $\theta, \phi \in [0, 2\pi)$.

(a) Find the line element ds^2 using the expressions above, and thus the components of the metric induced on the torus in coordinates (θ, ϕ).

(b) Write down the parallel transport equation around the curve $\gamma^i(\phi) = (\theta_0, \phi)$, $\theta_0 = cnst$, in terms of the Christoffel symbols, showing explicitly which terms vanish. **Hint:** First you need to find the tangent vector to the curve with the correct normalization factor.

(c) Derive the two coupled non-vanishing differential equations for the parallel transport and solve them.
Hint: Once you get the first-order system of equations, take the derivative of each one and substitute in the other one.