# GR — Exercise sheet 12

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# 1 Compact stars

#### Exercise 1.1: Schwarzschild spacetime: Interior solution

To compute the interior solution of Einstein Equations (EE) for spherical symmetric spacetime, we can start from the expression of the metric

$$ds^{2} = -e^{2\alpha(r)}dt^{2} + e^{2\beta(r)}dr^{2} + r^{2}d\theta^{2} + r^{2}\sin^{2}\theta \,d\phi^{2}$$

and use it to compute the left hand side of Einstein Equations in presence of matter

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi GT_{\mu\nu}$$

(a) We model the star as a perfect fluid, so the energy-momentum tensor has the form

$$T_{\mu\nu} = \rho(r)u_{\mu}u_{\nu} + P(r)(u_{\mu}u_{\nu} + g_{\mu\nu})$$

Where  $u^{\mu}$  is the velocity of the fluid which, for static fluids, has to be time-like;  $\rho$  is the energy density and P is the pressure. Derive the expression for  $u^{\mu}$  that allows to interpret  $\rho$  as a total energy-mass density ( $T_{00} \propto \rho$  only) and write down the components of  $T_{\mu\nu}$ .

(b) Given the only non-vanishing components of the Einstein Tensor

$$G_{tt} = \frac{1}{r^2} e^{2(\alpha - \beta)} \left( 2r\partial_r \beta - 1 + e^{2\beta} \right)$$

$$G_{rr} = \frac{1}{r^2} \left( 2r\partial_r \alpha + 1 - e^{2\beta} \right)$$

$$G_{\theta\theta} = r^2 e^{-2\beta} \left[ \partial_r^2 \alpha + (\partial_r \alpha)^2 - \partial_r \alpha \partial_r \beta + \frac{1}{r} (\partial_r \alpha - \partial_r \beta) \right]$$

$$G_{\phi\phi} = \sin^2 \theta \ G_{\theta\theta}$$

write down the tt, rr,  $\theta\theta$  components of the Einstein Equations.

(c) To start with the calculation is convenient to define the function

$$m(r) = \frac{1}{2G}(r - re^{-2\beta}).$$

Use the tt component of the EEs to show that

$$m(r) = 4\pi \int_0^r \rho(r') r'^2 dr'$$

- (d) The Schwarzschild mass M is defined as M = m(R), where R is the radius of the star. Explain why this is not the total integrated energy density  $\overline{M}$  and find the expression of  $\overline{M}$  in terms of  $\rho(r)$ , m(r), R. Hint: think about the volume element in a curved space.
- (e) Use now the rr equation to derive  $\frac{d\alpha}{dr}$  in terms of m(r).
- (f) Show that the energy-momentum conservation law  $\nabla_{\mu}T^{\mu\nu} = 0$ , leads to the relation

$$(\rho + P)\frac{d\alpha}{dr} = -\frac{dP}{dr}$$

*Hint*: the only non-trivial term corresponds to the  $\nu = r$  component. Use the Christoffel symbols for a spherical spacetime you have been given in the notes.

Finally, combine the last two equations to get

$$\frac{dP}{dr} = -\frac{(\rho+P)[Gm(r)+4\pi Gr^3 P]}{r[r-2Gm(r)]}.$$

This is the Tolman-Oppenheimer-Volkoff (TOV) equation, i.e. the equation of hydrostatic equilibrium. Coupling it with the so-called Equation of State  $P = P(\rho)$  we get a closed system of equations which models the star.

## Exercise 1.2: The Newtonian limit of TOV equation

- (a) Explain why, in this context, the Newtonian limit is reached when  $P \ll \rho, m(r) \ll r$ . How does the term  $[Gm(r) + 4\pi Gr^3 P]$  become in this limit? *Hint*: restore the physical units.
- (b) Starting from the result of the previous exercise, derive the TOV equation in the Newtonian limit.

#### Exercise 1.3: Buchdal's theorem for an incompressible star

The simplest way to model a star is to take an incompressible fluid, i.e. to consider a fluid with a constant density up to the star's surface of radius R as follows

$$\rho(r) = \begin{cases} \rho_*, & r < R\\ 0, & r \ge R \end{cases}$$

- (a) Calculate the expression of m(r) for this density, where  $r \in [0, \infty)$ .
- (b) Use the TOV equations to verify that

$$P(r) = \rho_* \left[ \frac{R\sqrt{R - 2GM} - \sqrt{R^3 - 2GMr^2}}{\sqrt{R^3 - 2GMr^2} - 3R\sqrt{R - 2GM}} \right]$$

(c) A consequence of the formula above is the existence of a maximum value for the mass that can be squeezed inside a sphere of radius R, beyond which the *central pressure* of the star would become infinite. Find the value of  $M_{\text{max}}$ .