GR — Exercise sheet 13

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1 Cosmology

Exercise 1.1: Cosmological constant

The Hilbert-Einstein action for vacuum spacetime can be modified by adding the cosmological constant

$$S = \frac{1}{16\pi G}\int \sqrt{-g}(R - 2\Lambda)d^4x \ ,$$

- Derive the modified Einstein field equations (EFEs) by varying S.
- The vacuum EFE with cosmological constant can be written in the form

$$G_{ab} = 8\pi G T^{\text{vac}}_{ab} \ .$$

  Where the additional term $\propto \Lambda$ is often interpreted as *vacuum energy* ρ_{vac}. Comparing T^{vac}_{ab} with the perfect fluid expression, what is the expression for $\rho_{\text{vac}}(\Lambda)$?

- Write down the components of T^{vac}_{ab} and discuss the properties of the fluid described by this tensor.

Exercise 1.2: Robertson-Walker metric

Consider the RW metric

$$ds^2 = -dt^2 + a^2\left(\frac{dr^2}{1-\kappa r^2} + r^2 d\Omega^2\right)$$

- For $\kappa = +1$ is possible to better understand the geometrical features of the metric defining $r = \sin\chi$. Write down the expression of ds^2 in this case.

- Defining

$$
\begin{aligned}
X &= a \sin\chi \sin\theta \cos\phi \\
Y &= a \sin\chi \sin\theta \sin\phi \\
Z &= a \sin\chi \cos\theta \\
W &= a \cos\chi
\end{aligned}
$$

  The spatial part of the metric takes the form of the 3-sphere embedded in R^4

$$ds^2 = -dt^2 + dX^2 + dY^2 + dZ^2 + dW^2.$$

  Show that a is the radius of the sphere. What portion of the 3-sphere defined above is covered by the initial coordinates $(r,\ \theta,\ \phi)$?

- Derive the FRW equations for the case $\kappa = \pm 1$. [Hint: write the trace reverse EFE, and use the expressions of the Ricci tensor that you have from the notes.]

Exercise 1.3: Static Universe

Consider the FRW equations with the cosmological constant:

$$
\begin{cases}
\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3}\rho + \frac{\Lambda}{3} - \frac{\kappa}{a^2} \\
\frac{\ddot{a}}{a} = -\frac{4\pi}{3}(\rho + 3P) + \frac{\Lambda}{3}
\end{cases}
$$

Show that static solutions ($\dot{a} = 0 = \ddot{a}$) are possible only for $\kappa > 0$ and $\Lambda > 0$. [Hint: do first the dust case $P = 0$]