These semi-private notes are constructed from the following books:

- R.Wald, "General Relativity" University of Chicago Press, 1984
- S.M.Carrol, "Spacetime and Geometry, An Introduction to General Relativity", Addison-Wesley, 2003.
- L.D.Landau, "The Classical Theory of Fields", Course of Theoretical Physics, Vol. 2. Classical Theory, Butterworth-Heinemann, 1980.
- B.F.Schutz, "A First Course in General Relativity", Cambridge University Press, 1985.
- B.F.Schutz, "Geometrical Methods of Mathematical Physics", Cambridge University Press, 1980.

If you decide to use them to study or teach, please
(0) be careful and refer to the original books
(1) cite/refer to my website
(2) let me know and send feedbacks.

SB 2018
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SPECIAL RELATIVITY

INERTIAL REF. SYSTEM = Ref. system in which a freely moving body* moves at constant velocity.

(* No forces are acting on the body)

POSTULATES OF SR :

- Principle of relativity : All laws of nature are identical in all inertial systems of reference.

  → Physics laws must be invariant with respect to tranformation of coordinates from one inertial system to another.

- The speed of light in vacuum is the same in all inertial system its value is finite :

$$\boxed{c \cong 2.99\ldots \cdot 10^{10} \text{ cm } s^{-1}}$$

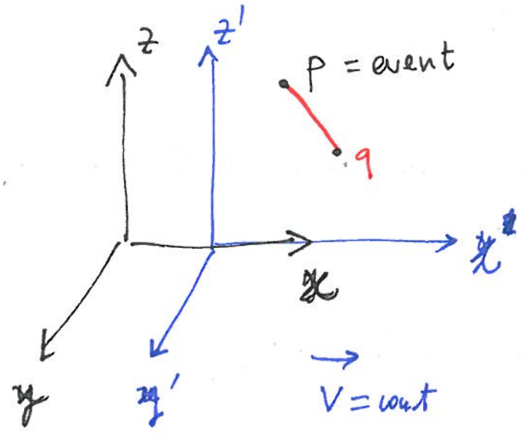
(The speed of propagation of interactions is universal constant, c is the max speed at which interaction can propagate)

Observations

- time is not absolute ; time intervals can have different values from ref. system to another.

- without notion of "simultaneity" one cannot define distances (spatial intervals) in an ref. system - invariant way !

<u>Example</u> : Distances in pre-rel. physics vs. SR



$$\begin{cases} t' = t & \text{(absolute time)} \\ x' = x - vt \\ y' = y \\ z' = z \end{cases}$$

<u>Galileian transformation</u>

$\overline{pq}$ = invariant

$$s^2 = \delta x^2 + \delta y^2 + \delta t^2 \qquad x_i = (x_1, x_2, x_3) = (x, y, z)$$

$$= \sum_{i=1}^{3} \delta x_i^2$$

$$= \sum_i \sum_j \delta_{ij} (\delta x_i)(\delta x_j) \qquad \delta_{ij} = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} = \mathrm{diag}(1,1,1)$$

$$= \delta_{ij} (\delta x^i)(\delta x^j) \qquad \text{— sum understood —}$$

$$= (\delta x_i)(\delta x^i) \qquad \text{— " —}$$

- Works because there's a single time and we need to care only about translation and rotations between spatial reference systems.

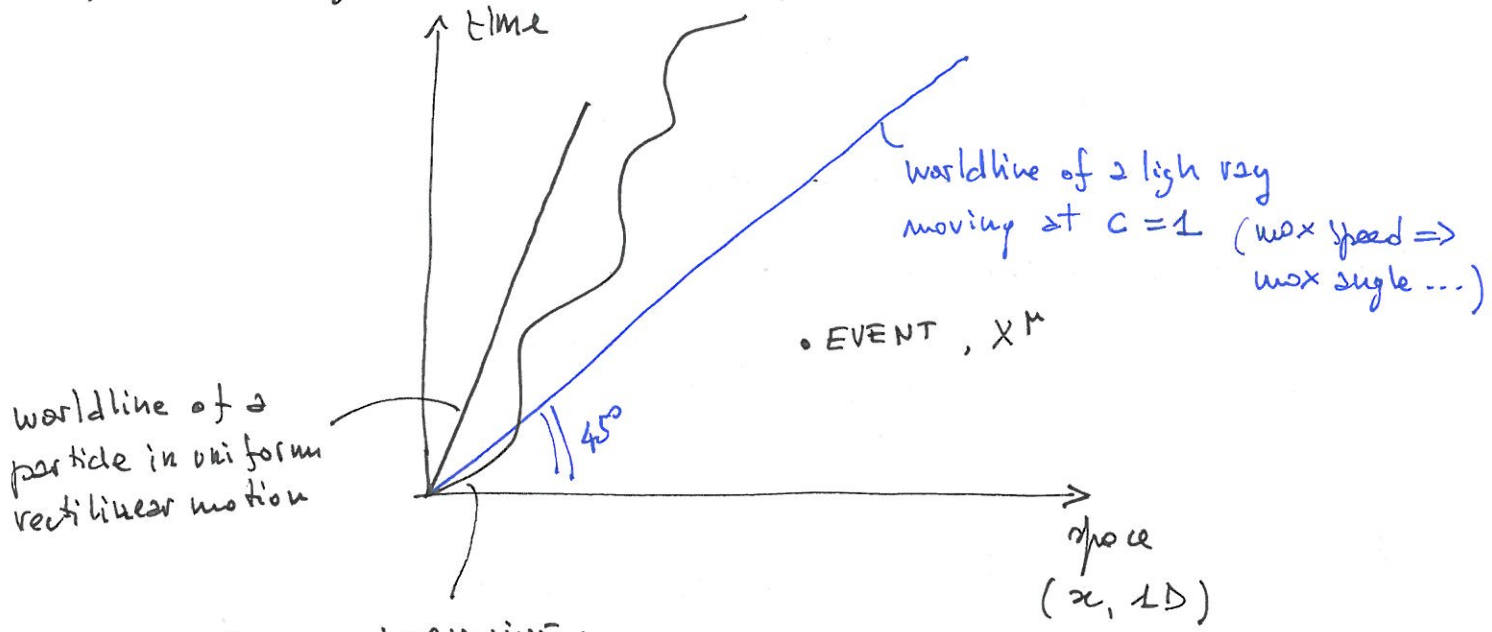
- In SR : Is there some invariant interval ?

- Remark : s^2 is a quadratic form of the coordinates. Euclidean distance and scalar product in $\mathbb{R}^3$.

SPACETIME DIAGRAMS & INVARIANT INTERVAL

Spacetime = 4D continuum of events labelled by coordinates $x^{\mu} = (t, x, y, z)$
$\mu = 0, 1, 2, 3$

Spacetime diagram:

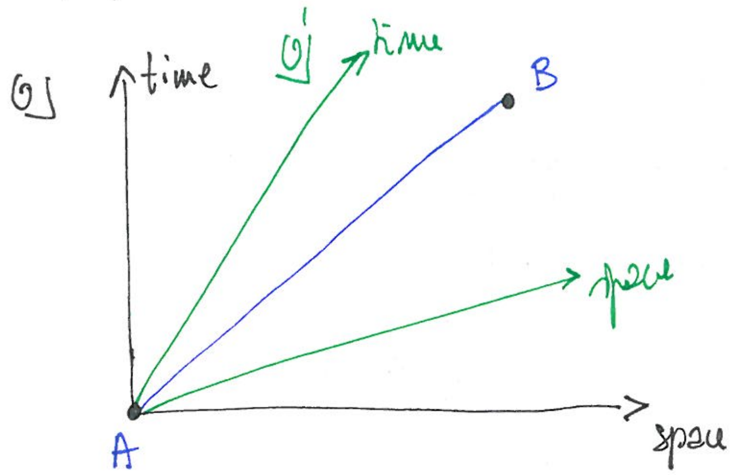


WORLD LINE:

$x^{\mu} = x^{\mu}(s) \quad s \in \mathbb{R}$

"What is "invariant" in this diagram?"

→ light propagates at the same speed in any ref. sys:



Take 2 events:

A: Emission of light at (t_1, x_1, y_1, z_1) in O
B: Arrival of light at (t_2, x_2, y_2, z_2) in O

Take another ref. sys. observing the same events:

A: (t_1', x_1', y_1', z_1') in O'

B: (t_2', x_2', y_2', z_2') in O'

Light propagate at c so the distance is:

$$\overline{AB} = c\,(t_2-t_1) = \left[(X_2-X_1)^2 + (Y_2-Y_1)^2 + (Z_2-Z_1)^2\right]^{1/2}$$
$$= \left[\sum_{i=1}^{3}(X_2^i - X_1^i)^2\right]^{1/2}$$

$$\rightarrow \quad -c\,(t_2-t_1) + \left[\sum_i (X_2^i - X_1^i)^2\right]^{1/2} = 0$$

But the same must hold for O':

$$\rightarrow \quad -c\,(t_2'-t_1') + \left[\sum_i (X_2^{i\prime} - X_1^{i\prime})^2\right]^{\frac{1}{2}} = 0$$

$$\Rightarrow \quad \boxed{S_{12}^2 \equiv -c^2(t_2-t_1)^2 + \sum_i (X_2^i - X_1^i)^2}$$

is invariant interval for light-like events.

Infinitesimal version:

$$ds^2 = -c^2dt^2 + dx^2 + dy^2 + dz^2$$

Observations

- Quadratic form similar to Euclidean distance but with a "$-$"
- We could write it as:

$$ds^2 = -c^2dt^2 + \sum_{i=1}^{3}(dx^i)^2$$

$$= \sum_{\mu=0}^{4}(dx^\mu)^2 \qquad x^\mu \equiv (ct, x, y, z) = (x^0, x^1, x^2, x^3)$$

$$= \sum_{\alpha,\beta} \eta_{\alpha\beta}\, dx^\alpha dx^\beta \qquad \eta_{\alpha\beta} = \mathrm{diag}(-1,1,1,1)$$

$$= dx_\beta dx^\beta$$

$\rightarrow$ distance from a scalar product (not Euclidean) in $\mathbb{R}^4$

- $ds = 0 = ds'$
  ds infinitesimal } <u>in general</u> : $ds^2 = a\, ds'^2$

they must be proportional (infinitesimal of same order)

with : $a = a(|\vec{V}|)$, $\vec{V}$ = relative velocity $\mathcal{O}-\mathcal{O}'$.

In fact:

- homogeneity of spacetime $\Rightarrow$ a cannot depend on (t, x^i)
  otherwise different points would not be equivalent;

- isotropy of space $\Rightarrow$ a cannot depend on $\vec{V}$ direction
  otherwise there would be a favorite direction.

In general:

$$\boxed{ds^2 = ds'^2}$$

the spacetime interval $ds^2 \equiv \eta_{\alpha\beta}\, dx^\alpha dx^\beta$ between 2 events is the same in all inertial systems.

PROOF :

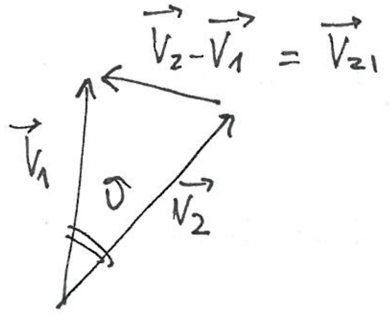
Take 3 inertial systems : $\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}$
with rel. velocities :

$$\begin{cases} V_1 : \mathcal{O}_1 - \mathcal{O} \\ V_2 : \mathcal{O}_2 - \mathcal{O} \\ V_{12} : \mathcal{O}_1 - \mathcal{O}_2 \end{cases}$$

we have :

$$ds^2 = a(V_1)\, ds_1^2 = a(V_2)\, ds_2^2$$
$$ds_1^2 = a(V_{12})\, ds_2^2$$

$$\rightarrow \quad a(V_{12}) = \frac{a(V_2)}{a(V_1)}$$



— $|\vec{V}_{12}|$ depends on $|\vec{V}_1|$, $|\vec{V}_2|$ and the ϑ angle

$\Rightarrow$ $a(V_{12})$ depends on ϑ

— $\dfrac{a(V_1)}{a(V_2)}$ do not depend on ϑ !

— $\dfrac{a(V_1)}{a(V_2)} = a(V_{12}) \Rightarrow a(V) = \text{const} \equiv K$

— which constant? $K \equiv 1$ because:

$1 = \dfrac{\cancel{K}}{\cancel{K}} = K$, the only value compatible ... $\square$

<u>Observations</u>

- d^2s $\begin{cases} = 0 & : \text{light-like or null events} \\ < 0 & : \text{time-like} \quad \text{—"—} \\ > 0 & : \text{space-like} \quad \text{—"—} \end{cases}$

symbol! $ds^2 = I$

this conceptual subdivision of events (characterization) is independent on ref. sys. (absolute!)

- What kind of events occur at one point of space ?

$$ds^2 = -c^2dt^2 + dl^2$$

if they occur at the same point we must have $dl'^2 = 0$ in some reference frame

$$ds^2 = -c^2dt^2 + dl^2 = -c^2dt'^2 + dl'^2 = -c^2dt'^2 < 0$$

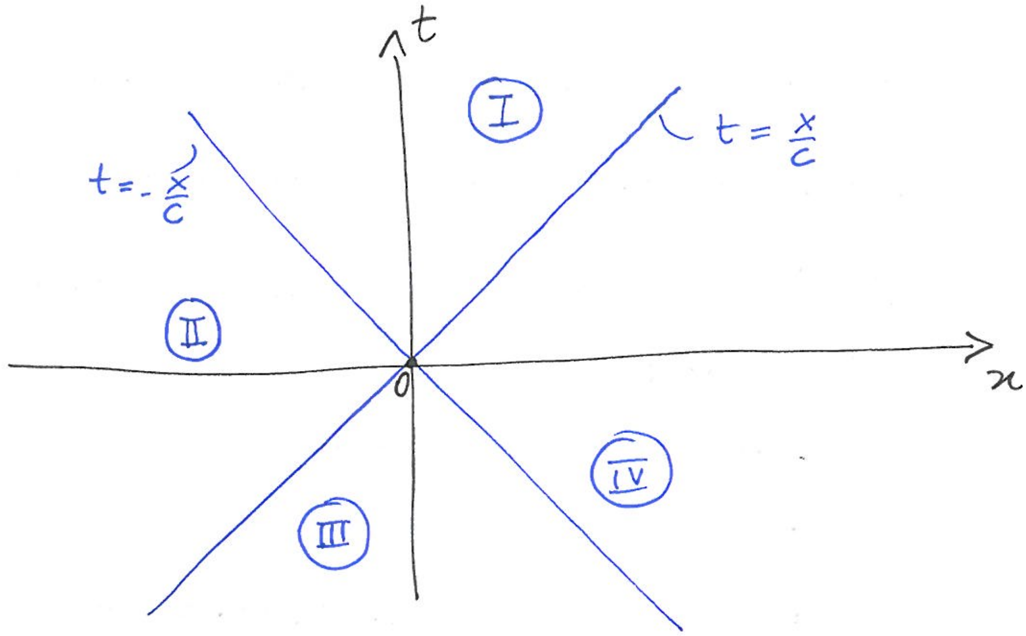
$\rightarrow$ <u>time like</u> events can happen at same point in space.

- What kind of events occur at the same time?

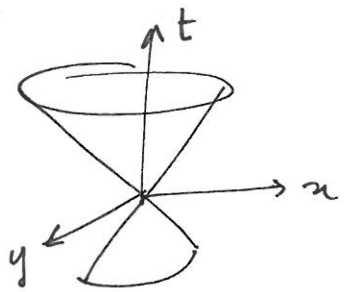
$$ds^2 = -c^2dt^2 + dl^2 = dl^2 > 0$$

$\rightarrow$ spacelike events.

More on spacetime diagram:



- $t = \pm \frac{x}{c}$ define light rays passing by 0
- in 4D (3 spatial dimensions) $-c^2t^2 + x^2 + y^2 + z^2 = 0$ is a <u>cone</u> with axis "t". A representation in 2D is:



- Region (I): $t > x \Rightarrow -t^2 + x^2 < 0 \Rightarrow$ timelike events
  $t > 0 \Rightarrow$ all events occur <u>after</u> event 0 $\Bigg\} \Rightarrow$

  it is <u>impossibile</u> to find in (I) events that are simultaneous to 0.

$\rightarrow$ (ABSOLUTE) FUTURE OF EVENT 0.

- Region (III) : $t^2 > x^2 \Rightarrow$ events are timelike
  $t < 0 \Rightarrow$ —"— occur <u>before</u> 0

→ no events are simultaneous to 0

→ PAST OF 0.

- Region (II), (IV) : events are spacelike.

→ events occur at different points in space in every ref. sys.

→ for any event in those regions there exist a ref. sys. Such that:

  - the event occur before 0
  - —"— after 0
  - —"— simultaneously to 0

  i.e. "before", "after", "simultaneously" is a concept that depends on the ref. sys.

→ spacelike events are causally disconnected from 0 since nothing can propagate beyond the ligh cone.

LORENTZ TRANSFORMATIONS

Principle of relativity $\Rightarrow$ inertial ref. sys. must be connected by transformations that leave ds^2 invariant.

Consider the formula:

$$ds^2 = \sum_{\alpha,\beta} \eta_{\alpha\beta} \, dx^\alpha dx^\beta \qquad \alpha,\beta = 0,1,2,3$$

$$= \eta_{\alpha\beta} \, dx^\alpha dx^\beta$$

- $x^\alpha \longrightarrow x^\alpha + a^\alpha = x^{\alpha'}$ $\qquad a^\alpha = (a^0, a^1, a^2, a^3) \in \mathbb{R}^4$

  4D translations $\rightarrow$ $dx^\alpha = dx^{\alpha'}$ ✓

- Linear transformations : $x^{\mu'} = \Lambda^{\mu'}{}_\nu \, x^\nu$, where $\Lambda^{\mu'}{}_\nu$ is a 4x4 matrix.

  Let us use first matrix notation :

$$dx^\alpha = \text{column vector} \equiv \overline{dx}$$

$$\left.\begin{aligned} ds^2 &= (\overline{dx})^T \eta \, (\overline{dx}) \\ \overline{dx}' &= \Lambda (\overline{dx}) \end{aligned}\right\} \quad ds^2 = (\overline{dx})^T \eta \, (\overline{dx}) = (\overline{dx}')^T \eta \, (\overline{dx}') = (\overline{dx})^T \Lambda^T \eta \Lambda \, (\overline{dx})$$

$$\Rightarrow \boxed{\eta = \Lambda^T \eta \Lambda} \qquad \underline{\text{Lorentz transformations}}$$

in components :

$$\boxed{\eta_{\alpha\beta} = \Lambda^{\alpha'}{}_\alpha \Lambda^{\beta'}{}_\beta \, \eta_{\alpha'\beta'}}$$

observations

- In the 3D spatial sector (don't change the time coords):

$$\mathbb{1}_{3\times 3} = R^T \, \mathbb{1}_{3\times 3} \, R \quad \rightarrow \quad O(3) \ \underline{\text{Rotations}}$$

Example:

$$\Lambda^{\mu'}{}_{\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad , \quad \text{rot. about } \hat{z}$$

- $\mathbb{1}_{3\times 3} = \text{diag}(1,1,1)$: matrix defining Euclidean scalar product and distances.

- $\Lambda = \text{diag}(-1,1,1,1)$: defines Lorentzian scalar product

- For Lorentz transformations we have:
  - Rotations xy, yz, xz
  - "Rotations" tx, ty, tz ... let us find them!

Consider the most general linear transformation that preserve $ds^2 = -t^2 + x^2$:

$$\boxed{c = 1}$$

$$\begin{cases} x' = x\cosh\varphi - t\sinh\varphi \\ t' = -x\sinh\varphi + t\cosh\varphi \end{cases}$$

(verify at home!)

To find the meaning of the angle φ, consider the motion of point $x'=0$ as observed by O' moving at speed V w.r.t. O:

$$x'=0 \Rightarrow \quad 0 = x\cosh\varphi - t\sinh\varphi \quad \Rightarrow \frac{x}{t} = \frac{\sinh\varphi}{\cosh\varphi} = \tanh\varphi$$

$$= V \quad \left(\frac{V}{c}\right)$$

$$\Rightarrow \begin{cases} \cosh\varphi = \left[1-\left(\frac{V}{c}\right)^2\right]^{-1/2} \equiv \gamma & \text{LORENTZ FACTOR} \\ \sinh\varphi = \frac{V}{c^2}\left[1-\left(\frac{V}{c}\right)^2\right]^{-1/2} = \frac{V}{c^2}\gamma \end{cases}$$

[verify at home]

$$\Rightarrow \begin{cases} t' = \gamma\left(t - \frac{V}{c^2}x\right) \\ x' = \gamma\,(x - Vt) \end{cases}$$

Boost transformation in x-direction.

Observations

- $c \to +\infty$ $(c \gg V)$ $\Rightarrow$ $\gamma \to 1$, $\frac{V}{c^2}\gamma \to 0$ $\Rightarrow \begin{cases} t' = t \\ x' = x - Vt \end{cases}$

  $\rightarrow$ Lorentz transformations contain Galileian transf. as a limit : $V \ll c$ (slow velocity)

- $\left(\frac{V}{c}\right)^2 \leqslant 1 \Rightarrow \boxed{\gamma \geqslant 1}$

- Assuming $\left(\frac{V}{c}\right)^2 > 1 \Rightarrow \gamma \in \mathbb{C}$ (Transf. undefined in $\mathbb{R}^4$)

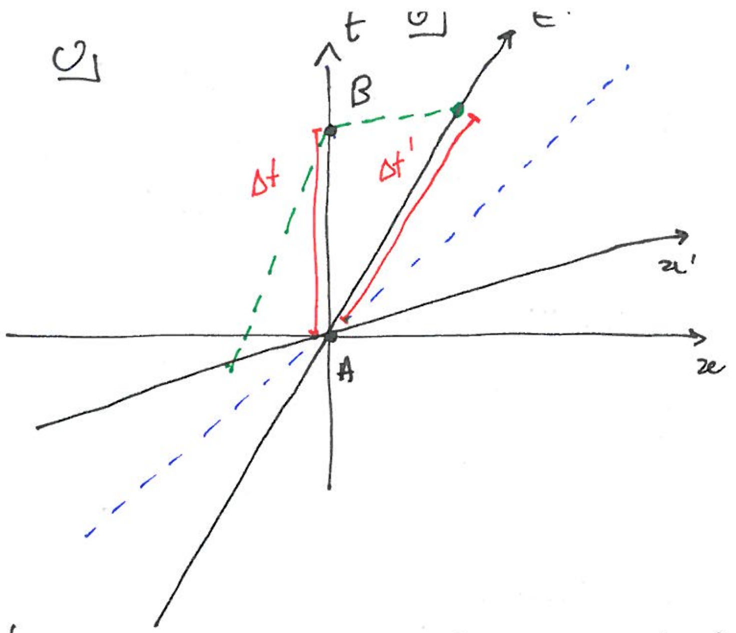
- $V = c \Rightarrow \gamma = \infty$ (den. $= 0$) : No observer (ref. sys.) can move at c!

- Lorentz transformations $\rightarrow$ time dilation (and lenght contraction)

  - time interval measured by obs. at rest with clock : $d\tau$, PROPER TIME ($\Delta\tau = \Delta t$).

  - time interval measured by obs. that look at the motion of the object carrying the clock :

$$\begin{cases} t_A' = \gamma\left(t_A - \frac{V}{c^2} x_A\right) \\ t_B' = \gamma\left(t_B - \frac{V}{c^2} x_B\right) \end{cases}$$



$x_A = x_B = 0 \Rightarrow$

$$\boxed{\Delta t' = \gamma \Delta t > \Delta t}$$

time elapsed if you move with the clock (proper time)

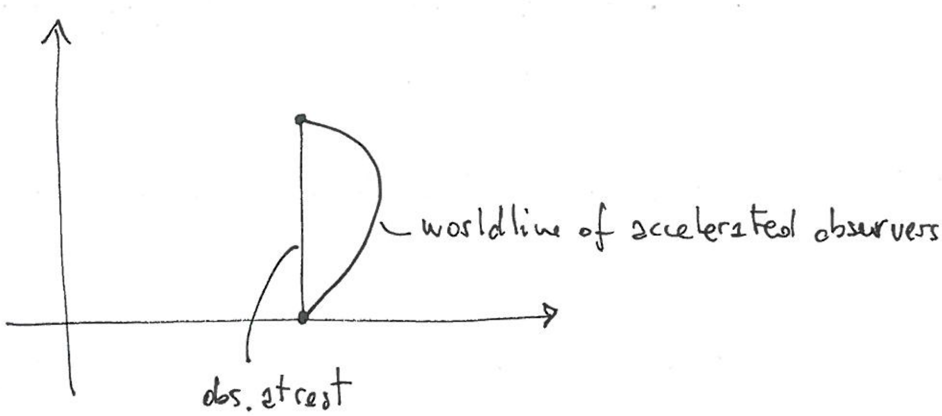
time elapsed according an observer that see you moving with the clock

- "time is running <u>slower</u> for moving clocks"
- Proper time is the <u>minimal</u> time interval measured by <u>inertial frames</u>.

This concern same world line, different inertial ref. frames.

Consider <u>one</u> ref. sys. and different worldlines:

— The interval AB can correspond to proper time of an <u>accelerated</u> obs.



particle at rest : $\Delta\tau = \Delta t$

moving / accelerated particle : $-d\tau^2 = -dt^2 + d^2\ell$

$$= -dt^2\left[1 - \left(\frac{d\ell}{dt}\right)^2\right] = -dt^2\left[1 - \left(\frac{V}{c}\right)^2\right]$$

$$\Rightarrow \int_{t_A}^{t_B} d\tau = \int_{t_A}^{t_B} dt \underbrace{\left[1-\left(\frac{v}{c}\right)^2\right]^{1/2}}_{\gamma^{-1} \leq 1}$$

$\rightarrow$ Proper time is maximum for non-accelerated observers.

Remark

- t_A, t_B are the same integration limits
- timelike worldlines
- Consequence of the "$-$" in η (SR spacetime is not Euclidean)
- Signature choices do not affect this conclusion
- This fact will carry over when we will study geodesics in spacetime
- "Twin paradox"

"SR DEFINITION" OF 4-VECTORS AND TENSORS

$x^\alpha = (ct, x, y, z)$ $\qquad \alpha = 0,1,2,3 \qquad$ 4D "vector" (→ vector components...)

The scalar product : $\eta_{\alpha\beta} x^\alpha x^\beta = x_\alpha x^\alpha$ is invariant under Lorentz transformations.

Def: 4-VECTOR (components) = A set of 4 quantities A^α that transform like

$$A^{\alpha'} = \Lambda^{\alpha'}{}_{\alpha} A^\alpha \quad , \quad \text{if} \quad x^{\alpha'} = \Lambda^{\alpha'}{}_{\alpha} x^\alpha$$

Example: x-boost $(c=1)$

$$\begin{cases} A^{0'} = \gamma(A^0 - vA^1) \\ A^{1'} = \gamma(A^1 - vA^0) \\ A^{2'} = A^2 \\ A^{3'} = A^3 \end{cases}$$

Remark :

- $\eta_{\alpha\beta} A^\alpha A^\beta = A_\alpha A^\alpha$ is invariant : SCALAR
- A^α : contravariant vector components
- $A_\alpha \equiv \eta_{\alpha\beta} A^\beta = \sum_\beta \eta_{\alpha\beta} A^\beta = (-A^0, A^1, A^2, A^3)$:

  covariant vector components , "co-vectors"

- $A_\alpha A^\alpha$
  - 0 : null vector
  - <0 : timelike vector
  - >0 : spacelike vector

- How to calculate the components of a vector from those of a co-vector?

In matrix notation: $\underline{A} \equiv \eta \bar{A} \Rightarrow \bar{A} = \eta^{-1} \underline{A}$

In component notation: $A^\alpha = (\eta^{-1})^{\alpha\beta} A_\beta \equiv \eta^{\alpha\beta} A_\beta$, i.e.

$$\eta_{\alpha\beta}\eta^{\beta\nu} = \eta^{\beta\nu}\eta_{\alpha\beta} = \delta^\nu_\alpha = \mathbb{1}_{4\times4}$$

- Note that:

$$\frac{\partial x^{\alpha'}}{\partial x^\alpha} = \frac{\partial}{\partial x^\alpha}\left(\Lambda^{\alpha'}{}_\beta x^\beta\right) = \Lambda^{\alpha'}{}_\beta \frac{\partial x^\beta}{\partial x^\alpha} = \Lambda^{\alpha'}{}_\beta \delta^\beta_\alpha = \Lambda^{\alpha'}{}_\alpha$$

the notation:

$$A^{\alpha'} = \frac{\partial x^{\alpha'}}{\partial x^\alpha} A^\alpha \quad \text{and} \quad A_\alpha = \frac{\partial x^{\alpha'}}{\partial x^\alpha} A_{\alpha'}$$

is often used to indicate how the components change w.r.t. coordinate change more explicitly.

~ · ~

<u>Def</u>. TENSOR (k,l) = object with components $T^{\alpha_1\ldots\alpha_k}{}_{\beta_1\ldots\beta_l}$ that under coordinate transformation change as:

$$T^{\alpha_1'\ldots\alpha_k'}{}_{\beta_1'\ldots\beta_l'} = \frac{\partial x^{\alpha_1'}}{\partial x^{\alpha_1}}\frac{\partial x^{\alpha_2'}}{\partial x^{\alpha_2}}\cdots\frac{\partial x^{\beta_1}}{\partial x^{\beta_1'}}\cdots\frac{\partial x^{\beta_l}}{\partial x^{\beta_l'}} T^{\alpha_1\ldots\alpha_k}{}_{\beta_1\ldots\beta_l}$$

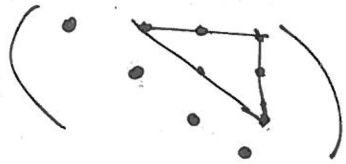
<u>Examples</u>

- $T_{\alpha\beta}$ is a tensor of type $(0,2)$; $T_{\alpha\beta} = \frac{\partial x^{\alpha'}}{\partial x^\alpha}\frac{\partial x^{\beta'}}{\partial x^\beta} T_{\alpha'\beta'}$
- $T_{\alpha\beta}$ is <u>symmetric</u> tensor $(0,2)$ iff $T_{\alpha\beta} = T_{\beta\alpha}$
- $T_{\alpha\beta}$ is <u>antisymm</u> tensor $(0,2)$ iff $T_{\alpha\beta} = -T_{\alpha\beta}$

- How many components (independent)? $\dim = 4$

$$T_{\alpha\beta} \Rightarrow 4 \times 4 = 16$$

$$T_{(\alpha\beta)} \equiv \frac{1}{2}\left(T_{\alpha\beta} + T_{\beta\alpha}\right) \quad \text{symm.} \rightarrow$$

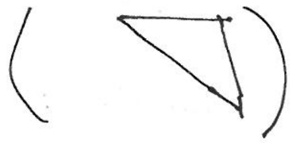


diagonal + lower (upper) part

$\quad 4 \qquad\qquad 6$

$$\Rightarrow 10$$

$$T_{[\alpha\beta]} \equiv \frac{1}{2}\left(T_{\alpha\beta} - T_{\beta\alpha}\right) \quad \text{antysimm} \rightarrow$$



$$\Rightarrow 6$$

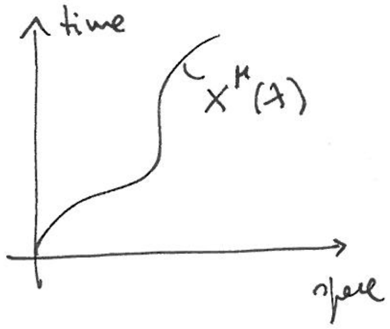
<u>Remarks</u>

- Vectors and tensors will be defined as <u>coordinate independent</u> objects, what changes are the components ...

- The quantities $\eta_{\alpha\beta}$, $\eta^{\alpha\beta}$, $\delta^{\alpha}{}_{\beta}$ <u>are</u> tensor components, but for the moment should be considered "special" $\rightarrow$ they do not change under coordinate transformation.

RELATIVISTIC MECHANICS

KINEMATIC



Worldline parametrized by $\lambda \in \mathbb{R}$: $x^\mu(\lambda)$

Def: 4-VELOCITY $\boxed{u^\alpha \equiv \frac{dx^\alpha}{d\lambda} = \dot{x}^\alpha}$

λ generic parameter, but for timelike worldlines one can use proper time : $\boxed{ds^2 = -d\tau^2}$ $(c=1)$.

How to calculate proper time :

$$\tau = \int \sqrt{-\eta_{\alpha\beta}\, dx^\alpha dx^\beta} = \int \sqrt{-\eta_{\alpha\beta}\dot{x}^\alpha \dot{x}^\beta}\, d\lambda \rightarrow \underbrace{\tau(\lambda) \rightarrow \lambda(\tau)}_{\text{REPARAMETRIZATION}} \text{ (if parametrization is good ---)}$$

$$u^\alpha = \frac{dx^\alpha}{d\tau} \Rightarrow u^\alpha u_\alpha = \eta_{\alpha\beta} u^\alpha u^\beta = \eta_{\alpha\beta}\dot{x}^\alpha \dot{x}^\beta = \eta_{\alpha\beta}\frac{dx^\alpha dx^\beta}{d\tau\, d\tau} = \frac{\eta_{\alpha\beta} dx^\alpha dx^\beta}{d\tau^2} = -1.$$

$\rightarrow$ timelike worldline parametrized by τ have normalized 4-velocity.

Remarks

- u^α is dimensionless
- u^α parametrized by τ is normalized
- 4-velocity components in rest frame : $u^\alpha = (1,0,0,0)$
- 4-velocity components for generic observers : $u^\alpha = (\gamma, \gamma v^i) = (\gamma, \gamma \vec{v})$

Def : 4-ACCELERATION $\boxed{a^\alpha \equiv \frac{d^2x^\alpha}{d\lambda} = \frac{du^\alpha}{d\lambda} = \ddot{x}^\alpha}$

Property : $0 = \frac{d}{d\tau}(u^\alpha u_\alpha) = \frac{d}{d\tau}(\eta_{\alpha\beta}\dot{x}^\alpha \dot{x}^\beta) = 2\eta_{\alpha\beta}\ddot{x}^\alpha \dot{x}^\beta = 2 u_\alpha a^\alpha$

$\rightarrow$ Acceleration is "orthogonal" to velocity.

DYNAMICS OF PARTICLES

Motion of bodies must be described by a Lagrangian and an action.

L, Lagrangian $\qquad [L] = E$

S, action $\qquad S = \int L\,dt \qquad [S] = ET = ML^2T^{-1}$

Let us find S.

$$S = \int \underbrace{dS}_{\substack{\text{INFINITESIMAL} \\ \text{LORENTZ INVARIANT}}} = \underset{\substack{\uparrow \\ \text{A constant}}}{k} \int \underset{\substack{\nwarrow \\ \text{Spacetime invariant}}}{ds}$$

For a timelike worldline (= motion of particles):

$$ds^2 = -c^2dt^2 + d\ell^2 = -c^2d\tau^2 < 0$$

$\rightarrow$ one can take proper time (or $\sqrt{-ds^2}$):

$$S \equiv k\int\sqrt{-\eta_{\alpha\beta}\,dx^\alpha dx^\beta} = \int\sqrt{-\eta_{\alpha\beta}\,\dot{x}^\alpha\dot{x}^\beta}\,d\lambda = kc\int d\tau$$

$$= kc\int dt\,\underbrace{\sqrt{1-\frac{v^2}{c^2}}}_{\gamma^{-1}}$$

$$d\tau^2 = dt^2 - \frac{d\ell^2}{c^2}$$

Determination of k from the Newtonian limit:

$L = \frac{1}{2}mv^2$: Lagrangian of a free particle in Newtonian dynamics.

$$\sqrt{1-x^2} \approx 1 - \frac{x^2}{2} + \mathcal{O}(x^4) \qquad x \ll 1 \qquad \Rightarrow$$

$$L = Kc\sqrt{1-\frac{v^2}{c^2}} \approx Kc - Kc\,\frac{1}{2}\frac{v^2}{c^2} = Kc - \frac{1}{2}\frac{K}{c}v^2 \Rightarrow \boxed{K = mc}$$

there is still a sign ambiguity: if one requires S to have a minimum (not only an extremum) one fixes the sign...

$$S \equiv -mc^2 \int \sqrt{-\eta_{\alpha\beta}\dot{x}^\alpha \dot{x}^\beta}\, d\lambda$$

$$L \equiv -mc^2\sqrt{-\eta_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta} = -mc^2\sqrt{1-\frac{v^2}{c^2}}$$

Remarks:

- Lorentz invariant
- Parametrization invariant (action)

$$\lambda \to \lambda' \quad ; \quad \frac{d}{d\lambda} \to \frac{d}{d\lambda'}\frac{d\lambda'}{d\lambda} \quad ; \quad \int \to \int$$

Given the Lagrangian one can calculate the conjugate mom. and the Hamiltonian:

- $$\vec{p} = \frac{\partial L}{\partial \vec{v}} = -mc^2\,\frac{1}{2}\,\frac{1\cdot 2}{\sqrt{1-\frac{v^2}{c^2}}}\left(-\frac{\vec{v}}{c^2}\right) = \gamma m \vec{v}$$ ($\rightarrow$ = 3-vector Euclidean)
- $$H = \vec{p}\cdot\vec{v} - L = \gamma m v^2 - \left(-mc^2\sqrt{1-\frac{v^2}{c^2}}\right) =$$

$$= \gamma m v^2 + mc^2\gamma^{-1} = m\underbrace{\left(\gamma v^2 + \gamma^{-1}c^2\right)}_{(\circ)} = \gamma mc^2$$

$$(\circ)\quad \gamma v^2 + \gamma^{-1}c^2 = \frac{v^2 + c^2\left(1-\frac{v^2}{c^2}\right)}{\sqrt{1-\frac{v^2}{c^2}}} = \gamma c^2 \qquad ⊛$$

Remarks:

- Newtonian limit ($v \ll c$): $\begin{cases} \vec{p} \simeq m\vec{v} \\ H \simeq mc^2 + \dfrac{mv^2}{2} \end{cases}$
- $v = c \Rightarrow \vec{p} \to +\infty$

– Particle at rest ($\gamma = 1$) : $H = E = mc^2$

→ the energy of the particle has a contribution from the mass!

"mass – energy equivalence"

– $$\left.\begin{aligned} p^2 &= \gamma^2 m^2 v^2 \\ H^2 &= \gamma^2 m^2 c^4 \\ &(*) \end{aligned}\right\} \rightarrow \quad (i) \quad p^2 = \frac{v^2}{c^4} E^2 \Rightarrow \boxed{p = \frac{v}{c^2} E}$$

$$v = c \quad (m = 0) \Rightarrow p = \frac{E}{c}$$

$$(ii) \quad \underbrace{\gamma^2 v^2}_{= \frac{p^2}{m^2}} = \underbrace{\gamma^2 c^2}_{= \frac{H^2}{m^2 c^2}} - c^2$$

$$\Rightarrow \quad H^2 = m^2 c^2 + c^2 p^2$$

$$\boxed{H = c\sqrt{p^2 + m^2 c^2}}$$ Relativistic Hamiltonian

Newtonian limit: $H \approx mc^2 + \frac{p^2}{2m}$

EQUATIONS OF MOTION

$$0 = \delta S \Leftrightarrow \frac{\partial L}{\partial x^\alpha} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}^\alpha} = 0 \qquad L = L(\dot{x}^\alpha; x^\alpha)$$

Euler–Lagrange equations:

$$0 = \underbrace{\frac{\partial L}{\partial x^\mu}}_{=0} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}^\mu} = -\frac{d}{dt}\left[\frac{1}{2}\left(-\eta_{\alpha\beta} u^\alpha u^\beta\right)^{-1/2} 2\eta_{\mu\nu} u^\nu\right] = 0$$

$$L = \left(-\eta_{\alpha\beta}\dot{x}^\alpha \dot{x}^\beta\right)^{1/2} = \left(-\eta_{\alpha\beta} u^\alpha u^\beta\right)^{1/2} \quad (*)$$

$$\Rightarrow \quad \frac{d}{d\tau} u^\mu = \ddot{x}^\mu = a^\mu = 0 \longrightarrow \text{"Free particle EOM"}.$$

Def: 4-MOMENTUM $\qquad p^\mu \equiv mc\, u^\mu$

$$= mc\,(\gamma, \gamma \vec{v}) = \left(\frac{E}{c}, \vec{p}\right)$$

Remarks :

- $p_\mu p^\mu = m^2 c^2 \underbrace{u_\mu u^\mu}_{=-1} = -m^2 c^2$

  $\Rightarrow \quad \dfrac{E^2}{c^2} = p^2 + m^2 c^2 \quad \checkmark$

- Definition is consistent with : $\quad p_\mu \equiv -\dfrac{\partial S}{\partial x^\mu}$.

- In presence of forces : $\quad a^\mu = F^\mu$

(*) Example

$$\frac{\partial}{\partial u^\mu}\left(\eta_{00} u^0 u^0 + \eta_{11} u^1 u^1\right) = \eta_{00} \frac{\partial u^0}{\partial u^\mu} u^0 + \eta_{00} u^0 \frac{\partial u^0}{\partial u^\mu} + \eta_{11} \frac{\partial u^1}{\partial u^\mu} u^1 + \eta_{11} u^1 \frac{\partial u^1}{\partial u^\mu} =$$

$$= \eta_{00} u^0 + \eta_{00} u^0 + \eta_{11} u^1 + \eta_{11} u^1 = 2\left(\eta_{00} u^0 + \eta_{11} u^1\right)$$

$$\frac{\partial u^0}{\partial x^\mu} = \delta^0_\mu$$

DYNAMICS OF FIELDS

Classical fields : $\phi = \phi(x^\mu)$, $\underbrace{\Phi_I(x^\mu)}_{\text{set of fields}}$ $I = 1, \dots, n$

Action for classical fields is the functional :

$$S[\Phi_I] \equiv \int L\, dt = \int dt \int d^3x\, \mathcal{L} \quad , \quad \text{where}$$

$\mathcal{L}[\Phi_I, \partial_\mu \Phi_I]$ is a Lorentz scalar called LAGRANGIAN DENSITY.

the "equations of motion" for the fields follows from $\delta S = 0$:

- vary the fields :

$$\Phi_I \longrightarrow \Phi_I + \delta\Phi_I$$
$$\partial_\mu \Phi_I \longrightarrow \partial_\mu \Phi_I + \delta(\partial_\mu \Phi_\mu) = \partial_\mu \Phi_I + \partial_\mu(\delta\Phi_I)$$

- vary the Lagrangian and expand : $\mathcal{L}[\Phi_I, \partial_\mu \Phi_I] \longrightarrow \mathcal{L}[\Phi_I + \delta\Phi_I, \partial_\mu\Phi_I + \partial_\mu(\delta\Phi_I)]$

$$\mathcal{L}[\Phi + \delta\Phi, \partial\Phi + \partial(\delta\Phi)] \simeq \mathcal{L}[\Phi, \partial\Phi] + \frac{\partial\mathcal{L}}{\partial\Phi}\delta\Phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi)}\partial_\mu(\delta\Phi)$$

(drop "I"...)

- vary the action : $S \rightarrow S + \delta S$

$$\delta S = \int d^4x \left[\frac{\partial\mathcal{L}}{\partial\Phi}\delta\Phi + \underbrace{\frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi)}\partial_\mu(\delta\Phi)}_{\text{p.p.}}\right] =$$

$$= -\int d^4x\, \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi)}\right)\delta\Phi + \int d^4x\, \underbrace{\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi)}\delta\Phi\right)}_{\text{total derivative}}$$

$$\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi)}\delta\Phi\right]_{\text{boundary}} = 0$$

$$\uparrow$$

$$\delta\Phi\Big|_{\text{boundary}} = 0$$

→ Euler-Lagrange equations for fields:

$$\frac{\partial \mathcal{L}}{\partial \Phi_I} - \partial_\mu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi_I)} \right] = 0$$

Example: Scalar field

$$\mathcal{L} \equiv -\frac{1}{2} \eta^{\alpha\beta} (\partial_\alpha \phi)(\partial_\beta \phi) - V(\phi)$$

$$\frac{\partial \mathcal{L}}{\partial \phi} = -\frac{dV}{d\phi} \quad ; \quad \frac{\partial \mathcal{L}}{\partial(\partial_\alpha \phi)} = -\eta^{\alpha\beta}(\partial_\alpha \phi) \Rightarrow$$

$$-\frac{dV}{d\phi} + \underbrace{\partial_\beta (\eta^{\alpha\beta} \partial_\alpha \phi)} = 0$$

$$\eta^{\alpha\beta} \partial_\alpha \partial_\beta \equiv \Box \qquad \longrightarrow \qquad \Box \phi - \frac{dV}{d\phi} = 0 .$$

Exercise:
- Find dimensions of $\mathcal{L}$ in natural units $c = G = \hbar = 1$
- Specify to $V(\phi) = \frac{1}{2} m^2 \phi^2$: Klein-Gordon equation

ELECTRODYNAMICS

Maxwell equations :

$$\begin{cases} \mathrm{curl}(\vec{B}) - \partial_t \vec{E} = 4\pi \vec{J} & (i) \\ \mathrm{div}(\vec{E}) = 4\pi \rho & (ii) \\ \mathrm{curl}(\vec{E}) - \partial_t \vec{B} = 0 & (iii) \\ \mathrm{div}(\vec{B}) = 0 & (iv) \end{cases}$$

Introducing a scalar ϕ and vector $\vec{A}$ potential, (iii) (iv) $\Rightarrow$

$$\vec{E} = \mathrm{grad}(\phi) - \partial_t \vec{A}$$
$$\vec{B} = \mathrm{curl}(\vec{A})$$

with the "gauge ambiguity" :

$$\phi \longrightarrow \phi - \partial_t \chi$$
$$\vec{A} \longrightarrow \vec{A} + \mathrm{grad}(\chi) .$$

Def: 4-potential $\quad A^\alpha \equiv (\phi, A^i) \qquad \alpha = 0,1,2,3$
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad i = 1,2,3$

Note: $A_\alpha = \eta_{\alpha\beta} A^\beta = (-\phi, A^i)$

In Lorentz gauge $\partial_\alpha A^\alpha = 0 \longrightarrow$ Maxwell equations read:

$$\boxed{\Box A^\alpha = 4\pi J^\alpha}$$

with : $\quad J^\alpha = (\rho, J^i) \quad$ 4-current.

Def: Faraday/Maxwell tensor $\quad F_{\alpha\beta} \equiv \partial_\alpha A_\beta - \partial_\beta A_\alpha$

Properties :

- $F_{\alpha\beta} = -F_{\beta\alpha}$ Antisymm.
- $F_{0i} = \partial_0 A_i - \partial_i A_0 = \partial_t A_i - \partial_i \phi = -E_i \qquad i = 1,2,3$
- $F_{ij} = \partial_i A_j - \partial_j A_i = \underbrace{\epsilon_{ijk}}_{\text{cyclic permutations}} B^k$

- $$F_{\alpha\beta} = \begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ +E_1 & 0 & +B_3 & -B_2 \\ +E_2 & -B_3 & 0 & +B_1 \\ +E_3 & +B_2 & -B_1 & 0 \end{pmatrix}$$

- Lorentz transformation : $F_{\alpha'\beta'} = \Lambda^{\alpha}{}_{\alpha'} \Lambda^{\beta}{}_{\beta'} F_{\alpha\beta}$

  [verify ; it is a tensor ...]

  Matrix notation : $F' = \Lambda F \Lambda^T$

  — Rotations :

  $$\vec{E}' = R\vec{E} \quad ; \quad \vec{B}' = R\vec{B}$$

  — Boost in direction "x" at speed V :

$$\begin{cases} E'_x = E_x \\ E'_y = \gamma\left(E_y - V B_z\right) \\ E'_z = \gamma\left(E_z + V B_y\right) \end{cases} ; \quad \begin{cases} B'_x = B_x \\ B'_y = \gamma\left(B_y + V E_z\right) \\ B'_z = \gamma\left(B_z - V E_y\right) \end{cases}$$

Moxwell epuations in terms of the $F_{\alpha\beta}$:

$$F^{\alpha\beta} = \eta^{\mu\alpha}\eta^{\nu\beta}F_{\mu\nu} = \begin{pmatrix} 0 & +E_1 & +E_2 & +E_3 \\ -E_1 & & & \\ -E_2 & & & \\ -E_3 & & & \end{pmatrix} \rightarrow \begin{aligned} F^{00} &= 0 \\ F^{0i} &= E^i \\ F^{ij} &= \epsilon^{ijk}B_k \end{aligned}$$

Compute derivatives of $F^{\alpha\beta}$ and read off eqs (i) and (ii) :

$$\partial_i F^{0i} = \partial_i(\eta^{00}\eta^{ii}F_{0i}) = \partial_i E^i = \mathrm{div}(\vec{E})$$

$$\partial_\mu F^{i\mu} = \underbrace{\partial_0 F^{i0}}_{-E^i} + \underbrace{\partial_j F^{ij}}_{\epsilon^{ijk}B_k} = -\partial_t\vec{E} + \mathrm{curl}(\vec{B})$$

$$\boxed{\partial_\mu F^{\nu\mu} = 4\pi J^\mu}$$

Similarly, eqs (iii) and (iv) can be combined in:

$$\boxed{\partial_{[\mu}F_{\nu\lambda]} = \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} + \partial_\lambda F_{\mu\nu} = 0}$$

Remarks

- $F_{\alpha\beta}$ is a tensor $\rightarrow$ equations transform "correctly" under Lorentz transf.

- Charge conservation follows from antisymmetry :

$$0 = \partial_\mu\partial_\nu F^{\mu\nu} = 4\pi\,\partial_\nu J^\nu$$

- Lagrangian density $\mathcal{L} = \mathcal{L}[A_\mu, \partial_\nu A_\mu]$:

$$\boxed{\mathcal{L} = -\frac{1}{4}F_{\alpha\beta}F^{\alpha\beta} + \frac{1}{c}J_\mu A^\mu}$$