These semi-private notes are constructed from the following books:

- R.Wald, "General Relativity" University of Chicago Press, 1984
- S.M.Carrol, "Spacetime and Geometry, An Introduction to General Relativity", Addison-Wesley, 2003.
- B.F.Schutz, "A First Course in General Relativity", Cambridge University Press, 1985.
- P.Townsed Black Holes

If you decide to use them to study or teach, please
(0) be careful and refer to the original books
(1) cite/refer to my website
(2) let me know and send feedbacks.

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SCHWARZSCHILD SOLUTION

Exact solution of GR for a spacetime:

- vacuum
- spherically symmetric
- static

Found by K. Schwarzschild in 1915.

Describes the spacetime outside a spherically symmetric mass distribution.

Provides us with the key GR predictions:

- Mercury perihelion precession
- Light bending
- Gravitational redshift
- Shapiro time delay

that can be tested in the Solar system (weak field limit).

Moreover, it includes some of the key and unexpected predictions and phenomena:

- Black holes
- Mass limit for compact star *
- Gravitational collapse *

* when combined with the proper solution for the interior of spherically symmetric mass distribution.

DERIVATION OF THE SOLUTION

A spherically symmetric and static metric

Def: A metric is <u>stationary</u> iff $\exists$ a timelike killing vector $T^a = (\partial_t)^a$

In the coordinates $x^\mu = (t, x^i)$ the metric is "time independent":

$$\partial_t g_{\alpha\beta} = 0$$

i.e.

$$g = -g_{00}(x^i)\,dt^2 + 2 g_{0i}(x^i)\,dx^i dt + g_{ij}(x^i)\,dx^i dx^j .$$

Def: A metric is said <u>static</u> iff it is stationary and invariant by time reversal:

$$t \mapsto -t$$

i.e.

$$g = -g_{00}(x^i)\,dt^2 + g_{ij}(x^i)\,dx^i dx^j =$$

$$= -N^2(x^i)\,dt^2 + \bar{g} \quad , \quad N \text{ smooth function on } \Sigma, \text{ "lapse function"}$$

In other terms the manifold can be written: $M_g = \mathbb{R} \times \Sigma_{\bar{g}}$

and the Killing vector T^a is orthogonal to the hypersurfaces $\Sigma_{\bar{g}}$.

$t = \text{const}$ defines the hypersurface Σ_t with normal:

$$n_\mu = -N\,(dt)_\mu = (-N, 0, 0, 0)$$

or

$$n^\mu = g^{\mu\nu} n_\nu = \left(\tfrac{1}{N}, 0, 0, 0\right)$$

$$\Rightarrow \forall\, s^a \in T_p\Sigma \quad , \quad n^a s_a = 0$$

and Σ_t is spacelike.

Remark :
- Stationary $\to$ invariance by time Translations
- Static $\to$ invariance also by time reflections

Impose now spherical symmetry on $\mathcal{E}$.

Def : A spacetime is spherically symmetric iff there exist coordinates
$$x^\mu = (t, r, \theta, \varphi)$$
such that :

(i) the surfaces $t = \text{const} = \bar{t}$ $\quad r = \text{const} = \bar{r}$ are 2-spheres

(ii) the metric tensor can be written
$$g = -e^{2\alpha(t,r)}dt^2 + e^{2\beta(t,r)}dr^2 + e^{2\gamma(t,r)}r^2 d^2\Omega$$
with
$$d^2\Omega = d^2\theta + \sin^2\theta \, d\varphi \, .$$

Observations

- The metric above has an obvious killing vector :
$$R_{(3)} = \partial_\varphi \quad : \quad \text{rotations about the axis } z = r\cos\theta$$
and two less obvious ones :
$$R_{(1)} = -\sin\varphi\,\partial_\theta - \cot\theta\cos\varphi\,\partial_\varphi$$
$$R_{(2)} = -\cos\varphi\,\partial_\theta + \cot\theta\sin\varphi\,\partial_\varphi$$
corresponding to rotations about
$$x = r\sin\theta\cos\varphi$$
$$y = r\sin\theta\cos\varphi$$

the existence of these 3 Killing vectors can be taken as alternative definition of spherically symmetric metric.

- Restrict to : $t = \bar{t}$ and $r = \bar{r}$

the 2-sphere line element is :
$$ds^2 = e^{2\gamma(\bar{t},\bar{r})}\,\bar{r}^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right)$$

The <u>area</u> of the 2-spheres is :
$$A = e^{2\gamma(\bar{t},\bar{r})}\,\bar{r}^2$$

One typically defines a radial coordinate such that the area is:

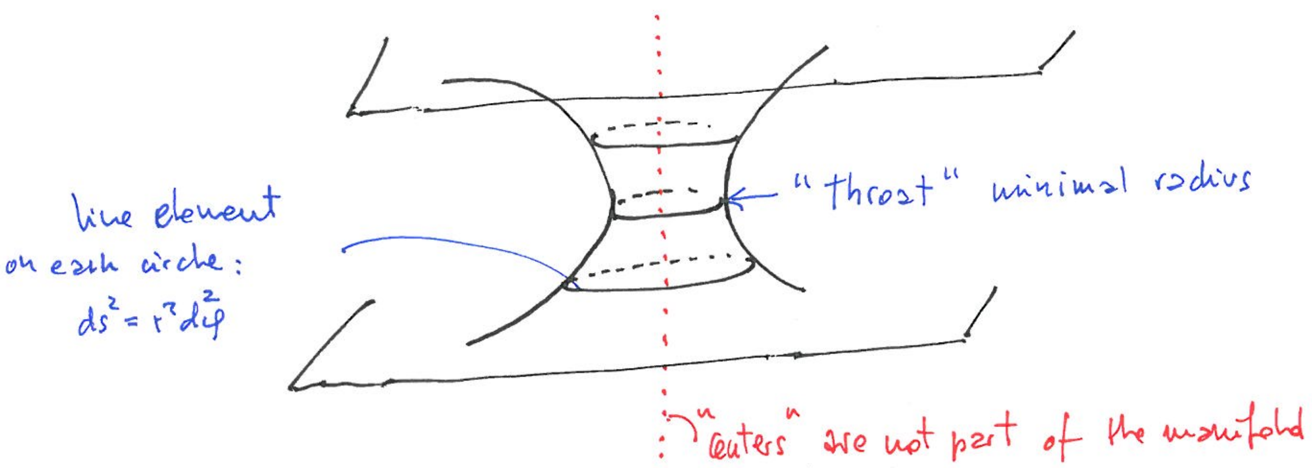
$$\boxed{A = 4\pi r^2} \quad \text{i.e.} \quad \underbrace{r^2 = e^{2\gamma(t,\bar{r})}\,\bar{r}^2}_{\text{(coordinate transformation)}}$$

the coordinate r is called "areal radius".

It is important to realise that r does <u>not</u> represent, in general, a "distance from center to surface" of 2-spheres.

"r" it is defined only by the properties of the 2-sphere (area); the center is not a point of the 2-sphere and might not belong to the manifold

An example can be given in 2D (θ = const, 2-spheres $\to$ circles):



Putting things together (and writing $N = e^{\alpha}$) a static and spherically symmetric metric has the form:

$$g = -e^{2\alpha(r)} dt^2 + e^{2\beta(r)} dr^2 + r^2 d\Omega^2$$

[Note we have used "2" and "exp" so far to make sure that the metric coefficients are positive and the signature is $(-,+,+,+)$]

Physical interpretation of metric coefficients.

- Consider a photon of 4-momentum p^{μ} moving on geodesics of g.

An observer at rest has 4-velocity:

$$u^{\alpha} = (u^0, u^i) = (u^0, \vec{0})$$

$$-1 = u_{\alpha} u^{\alpha} = g_{\alpha\beta} u^{\alpha} u^{\beta} = g_{00} u^0 u^0 \Rightarrow u^0 = e^{\alpha} > 0 \,,$$

then the observed energy of the photon is:

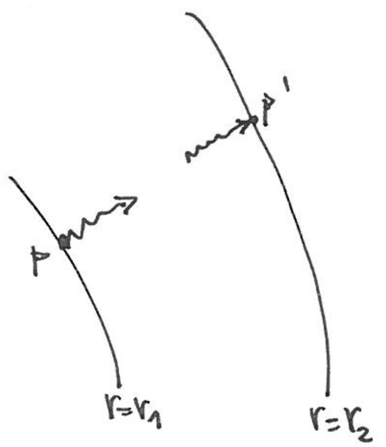
$$E = -u^{\alpha} p_{\alpha} = -u^0 p_0 = -e^{\alpha} p_0$$

Because the metric has a Killing vector there exist a constant of motion:

$$T^{\alpha} p_{\alpha} = \text{const} \equiv \mathcal{E}$$

but $T^{\alpha} = (1, \vec{0}) \Rightarrow p_0 = \mathcal{E}$ in every point!

Hence, the energy of a photon emitted at $r = r_1$ and absorbed at $r = r_2$ is:



$$\frac{E(r_1)}{E(r_2)} = \frac{e^{\alpha(r_1)} p_0(r_1)}{e^{\alpha(r_2)} p_0(r_2)} = \frac{\cancel{\mathcal{E}}}{\cancel{\mathcal{E}}} e^{\alpha_1 - \alpha_2}$$

$\Rightarrow$ Redshift: $z = e^{\alpha_1 - \alpha_2} - 1$.

if $r_2 \to +\infty$, then $z = e^{\alpha} - 1$

- For any <u>isolated system</u> the gravitational field far from the source must be ~ 0. Thus:

$$g \to \eta$$

and

$$\lim_{r \to +\infty} \alpha = \lim_{r \to +\infty} \beta = 0 .$$

Moreover, from the weak-field solution we must have:

$$e^{2\alpha} \sim -\left(1 - \frac{2GM}{c^2 r}\right) \qquad e^{2\beta} \sim \left(1 - \frac{2GM}{c^2 r}\right)$$

Determination of α, β from EFE

In vacuum : $R_{\alpha\beta} = 0$

$$\left.\begin{array}{l} R_{tt} = 0 \\ R_{rr} = 0 \end{array}\right\} \text{ combine these equations in : } \quad 0 = e^{2(\beta-\alpha)} R_{tt} - R_{rr} =$$

$$= \frac{2}{r}\left(\partial_r \alpha + \partial_r \beta\right)$$

$$\rightarrow \quad \boxed{\alpha = -\beta + \text{const}} \quad (\times)$$

The constant can be re-absorbed into a redefinition of the time coordinate:

$$e^{\alpha} = e^{-\beta} e^{c} \quad \rightarrow \quad e^{2\alpha} dt^2 = e^{-2\beta} e^{2c} dt^2 \longmapsto e^{-2\beta} dt^2 \qquad (t \rightarrow e^{c} t)$$

$$R_{\theta\theta} = 0 \quad : \quad e^{2\alpha}\left(2r\partial_r \alpha + 1\right) = 1$$

$$\partial_r \left(r e^{2\alpha}\right) = 1 \quad \text{with solution : } \quad \boxed{e^{2\alpha} = 1 + \frac{R}{r}} \; (\times\times)$$

for some constant R.

Verify that with the choice $(\times)$ and $(\times\times)$, $R_{tt} \equiv 0$ and $R_{rr} \equiv 0$.

The metric is thus :

$$g = -\left(1 + \frac{R}{r}\right) dt^2 + \left(1 + \frac{R}{r}\right)^{-1} dr^2 + r^2 d\Omega^2$$

The constant R can be fixed if we assume that the metric desribes an <u>isolated body</u> : in this case we can impose the asymptotic condition that the metric matches the weak-field metric

$$g_{00}(r\to+\infty) = -\left(1+\frac{R}{r}\right)$$

$$g_{rr}(r\to+\infty) = \left(1+\frac{R}{r}\right)^{-1} \approx \left(1-\frac{R}{r}\right)$$

Compare to static, weak-field:

$$g_{00}^{Newt} = -(1+2\phi) = -\left(1-2\frac{GM}{c^2 r}\right)$$

$$g_{rr}^{Newt} = (1-2\phi) = \left(1+2\frac{GM}{c^2 r}\right)$$

Hence:

$$\boxed{R_s = 2M} \qquad \underline{\text{Schwarzschild radius}}$$

Final result:

$$\boxed{g^{Sch} = -\left(1-\frac{2M}{r}\right)dt^2 + \left(1-\frac{2M}{r}\right)^{-1} dr^2 + r^2 d\Omega^2}$$

<u>Observations</u>

- $g^{Sch} \to \eta$ for $M \to 0$
- $g^{Sch} \to g^{Newt}$ for $r \to +\infty$: <u>Asymptotically flat</u>
- Metric coefficients are singular for $r=0$ and $r=R_s$.

  Q: physical or coordinate singularities?

  A sufficient condition to verify that a singularity is physical is to find a <u>scalar</u> of the curvature that diverges.

While the Ricci scalar is of no use here, one can compute:

$$R^{\alpha\beta\gamma\delta} R_{\alpha\beta\gamma\delta} = 12 M^2 r^{-6}$$

that indicate $r=0$ is a singular point.

None of the curvature scalars diverge at $r = R_S$. We will study the behaviour of the metric under coordinate change below...

Meantime one can note that for the Sun:

$$R_\odot \sim 10^6 M_\odot \gg R_S$$

$\Rightarrow$ R_S is located in the <u>interior</u> of the Sun where the solution (vacuum) is not valid. Schwartzschild metric and coordinates can be safely used for Solar system applications!

BIRKHOFF THEOREM

"The Schwarzschild metric is the unique vacuum solution in spherical symmetry"

the statement above does not mention "static" ... in fact

0. Define a spherically symmetric spacetime, in general, as one that admits 3 Killing vectors such that:

$$[R_{(1)}, R_{(2)}] = R_{(3)}$$
$$[R_{(2)}, R_{(3)}] = R_{(1)}$$
$$[R_{(3)}, R_{(1)}] = R_{(2)}$$

1. Any spherically symmetric spacetime can be foliated in 2-spheres, the most general form of the metric is:

$$g = -e^{2\alpha(t,r)} dt^2 + e^{2\beta(t,r)} dr^2 + r^2 d\Omega^2$$

2. Use EFE in vacuum for the metric above and show that the "time dependence" in the coefficients α, β can be removed.

Take: $\quad R_{tr} = 0 \quad \Rightarrow \quad \partial_t \beta = 0 \quad \Rightarrow \quad \beta = \beta(r)$

$$\begin{cases} \partial_t R_{\theta\theta} = 0 \\ R_{tr} = 0 \end{cases} \Rightarrow \partial_t \partial_r \alpha = 0 \Rightarrow \alpha(t,r) = \alpha(r) + f(t)$$

but the $f(t)$ term can be re-absorbed in the definition of time:

$$t \mapsto e^{f(t)} t$$

the last step proves that

"Any spherically symmetric vacuum spacetime is static",

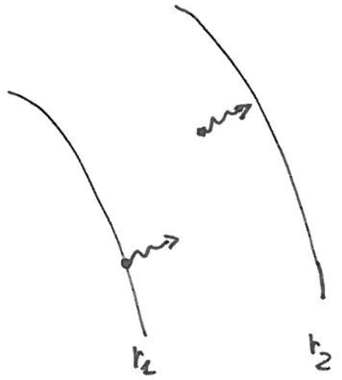
Birkhoff theorem.

Observation

- The result apply for any exterior spherically symmetric solution.
  For example, the exterior of a spherically symmetric body that is contracting (gravitational collapse) is static.

- Exercise: Do black holes suck? Discuss.

Exercise: Gravitational redshif of photons



Consider a photon emitted at $r=r_1$ and absorbed at $r=r_2$.
An observer with 4-velocity u^μ : $u^\mu u_\mu = -1$
stationary in Schwarzschild coordinates : $u^\mu = (u^0, \vec{0})$
measures a frequency :

$$\omega = - u_\mu p^\mu = - u_\mu \frac{dx^\mu}{d\lambda}$$

where $x^\mu(\lambda)$ is the orbit of the photon (null geodesic).

$$\left.\begin{array}{l} -1 = u^\mu u_\mu \\ u^\mu = (u^0, \vec{0}) \end{array}\right\} \longrightarrow \quad u^0 = \left(1 - \frac{2M}{r}\right)^{-1/2}$$

Hence:

$$\hbar \omega = - u_0 p^0 = - u_0 \frac{dt}{d\lambda} = + \left(1 - \frac{2M}{r}\right)^{+1/2} \frac{dt}{d\lambda} = \left(1 - \frac{2M}{r}\right)^{-1/2} E$$

as we shall show later that:

$$\frac{dt}{d\lambda} = \left(1 - \frac{2M}{r}\right)^{-1} E \, .$$

Take the ratio:

$$\frac{\omega_2}{\omega_1} = \left(\frac{1 - \frac{2M}{r_1}}{1 - \frac{2M}{r_2}} \right)^{1/2}$$

in the limit $r \gg 2M$:

$$\frac{\omega_2}{\omega_1} \approx 1 - \frac{M}{r_1} + \frac{M}{r_2} = 1 + \phi_1 + \phi_2 = 1 + \Delta\phi$$

that coincides with the weak field / Newtonian limit.

GEODESICS

The equations of motion in Schwarzschild spacetime can be found by the general procedure of minimizing the Lagrangian

$$L = g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} .$$

These calculations lead to a system of 2nd order <u>coupled</u> ODEs for the $\ddot{x}^\mu$:

$$\frac{d^2t}{d\lambda^2}, \quad \frac{d^2r}{d\lambda^2}, \quad \frac{d^2\theta}{d\lambda^2}, \quad \frac{d^2\varphi}{d\lambda^2}$$

Solutions to the geodesics equations can be found using the conserved quantities associated to the killing vectors; for each of the killing vector one has:

$$K_\mu \frac{dx^\mu}{d\lambda} = \text{constant of motion} .$$

Additionally, the Lagrangian is constant along geodesics. Let:

$$\epsilon \equiv - g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} = \begin{cases} -g_{\mu\nu} u^\mu u^\nu = +1 & \text{for massive particles } (\lambda = \tau) \\ 0 & \text{for light (null geodesics)} \\ -1 & \text{for spacelike geodesics} \end{cases}$$

The third property that allows one to simplify solutions is the fact that the <u>motion is on a plane</u>, exactly as in Newtonian gravity. The geodesic $\ddot{\theta}$ is:

$$\frac{d^2\theta}{d\lambda^2} + \frac{2}{r} \frac{d\theta}{d\lambda} \frac{dr}{d\lambda} - \sin\theta \cos\theta \left(\frac{d\varphi}{d\lambda} \right)^2 = 0$$

One immediately sees that if one chooses:

$$\begin{cases} \theta(\lambda=0) = \frac{\pi}{2} \\ \theta'(\lambda=0) = 0 \end{cases} \Rightarrow \begin{cases} \frac{d^2\theta}{d\lambda} = 0 \\ \frac{d\theta}{d\lambda} = 0 \end{cases} \Rightarrow \boxed{\theta = \frac{\pi}{2} \quad \forall \lambda}$$

Calculate the constants of motion from Killing vector:

$$T^a = (\partial_t)^a \quad : \quad T^\mu = (1, \vec{0}) \qquad \text{"time symmetry"}$$

$$T_\mu = \left[-\left(1-\frac{2M}{r}\right), \vec{0}\,\right] = (-A, \vec{0}), \quad A \equiv 1-\frac{2M}{r}$$

$$R_z^{\ a} = (\partial_\varphi)^a \quad : \quad R^\mu = (0,0,0,1) \qquad \text{"}\hat{z}\text{-rotations"}$$

$$R_\mu = (0,0,0, r^2 \sin^2\theta)$$

Hence:

$$E \equiv -T_\mu \frac{dx^\mu}{d\lambda} = A\frac{dt}{d\lambda} = \left(1-\frac{2M}{r}\right)\frac{dt}{d\lambda} \qquad \text{integral of geodesics } \ddot{t}$$

$$\ell \equiv R_\mu \frac{dx^\mu}{d\lambda} = r^2 \frac{d\varphi}{d\lambda}\bigg|_{\theta=\frac{\pi}{2}} \qquad \text{integral of geodesics } \ddot{\varphi} \text{ for } \theta=\frac{\pi}{2}$$

<u>Observations</u>

- For $r \gg M$, $E \simeq u_0 = \frac{p_0}{m}$ is the energy per unit mass of a particle measured by a static obs. (SR).

In general, E is interpreted for timelike geodesics as the <u>total</u> energy per unit mass of the particle relative to a static observer at infinity = energy required by such observer to put the particle in the given orbit starting from infinity.

Similarly, for null geodesics, E is the total energy of a photon.

Note that $E \equiv -T_\mu \frac{dx^\mu}{d\lambda}$ is different from $-u_\mu \frac{dx^\mu}{d\lambda}$ where u_μ is the velocity of an observer ($u_\mu u^\mu = -1$). u^μ is not a Killing vector (has normalization). The quantity $-u_\mu \frac{dx^\mu}{d\lambda}$ does not include the contribution due to the gravitational potential energy; the latter, however, is defined only in presence of a timelike Killing vector via E (total energy).

- ℓ is interpreted as angular momentum per unit mass in case of timelike geodesics. It generalizes the Kepler law to GR. For photons, $\hbar\ell$ is again interpreted as angular momentum.

Consider now the equation $\quad \epsilon = -g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}$ for $\theta = \frac{\pi}{2}$:

$$-\left(1-\frac{2M}{r}\right)\left(\frac{dt}{d\lambda}\right)^2 + \left(1-\frac{2M}{r}\right)^{-1}\left(\frac{dr}{d\lambda}\right)^2 + r^2\left(\frac{d\varphi}{d\lambda}\right)^2 = -\epsilon$$

multiply by A :

$$\underbrace{-A^2\left(\frac{dt}{d\lambda}\right)^2}_{-E^2} + \left(\frac{dr}{d\lambda}\right)^2 + A\,\underbrace{r^2\left(\frac{d\varphi}{d\lambda}\right)^2}_{\ell^2 r^{-2}} = -A\epsilon$$

$$\left(\frac{dr}{d\lambda}\right)^2 + A\left(\frac{\ell^2}{r^2}+\epsilon\right) = E^2$$

which can be written

$$\frac{1}{2}\left(\frac{dr}{d\lambda}\right)^2 + V(r) = \frac{1}{2}E^2$$

with:

$$\boxed{\begin{aligned} V(r) &\equiv \frac{1}{2}\left(1-\frac{2M}{r}\right)\left(\frac{\ell^2}{r^2}+\epsilon\right) = \\ &= \frac{\epsilon}{2} - \epsilon\frac{M}{r} + \frac{\ell^2}{2r^2} - \frac{M\ell^2}{r^3} \end{aligned}}$$

The equation above is the same equation for a Newtonian particle of unit mass $(m=1)$ and energy $\frac{1}{2}E^2$ moving in the central potential $V(r)$. Note that for $\epsilon = 1$:

$$V(r) \sim \text{constant} + \frac{1}{r}\text{ Newtonian potential} + \text{centrifugal potential} + \text{GR term}$$

and that the GR term is $\mathcal{O}(r^{-3})$ and vanish before the others for $r \gg M$, hence $V(r) \sim V_{Newton}(r)$ for $r \gg M$ (as it should!).

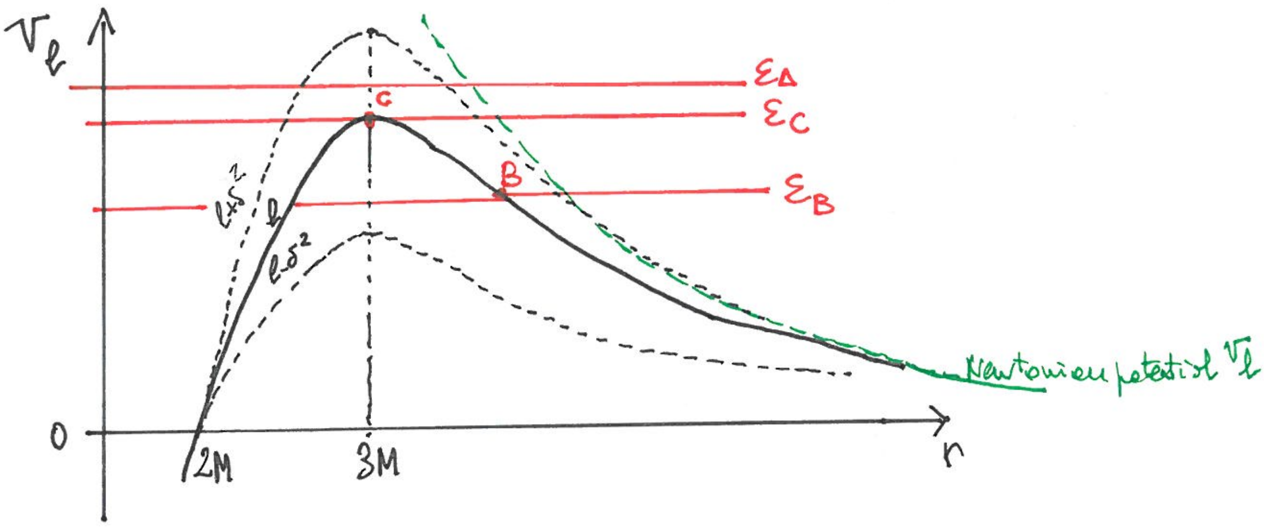
Recall that $V(r)$ is <u>exact</u> in GR.

Discussion on orbits

$$\dot{r}^2 = \frac{E^2}{2} - V \geq 0 \quad \Rightarrow \quad \text{Motion is restricted to radii } r : \; V < \frac{E^2}{2}$$

Note the acceleration is $\ddot{r} = -\frac{dV}{dr}$; let $\mathcal{E} \equiv \frac{1}{2}E^2$ (DO NOT CONFUSE $\mathcal{E}$ WITH ϵ)

$\boxed{\epsilon = 0}$ Photons



Facts:

- Potential $V_\ell(2M) = 0$
- Potential has a max at $r = 3M$ $(\ell > 0)$
- $V(r \to 0) \to -\infty$ (Newtonian : $+\infty$)

For a given ℓ :

- photon with energy $\mathcal{E}_A$ moves from $r \sim +\infty$ down to $r = 2M$ and $r = 0$
- photon with energy $\mathcal{E}_B$ moves from $r \sim +\infty$ down to radius $r = r_B$,
  at that point : $V = \mathcal{E} \to \dot{r} = 0$ TURNING POINT
  the photon moves back to larger r : hyperbolic orbit.
- photon with energy $\mathcal{E}_C$ is at a maximum of V_ℓ : $\frac{dV}{dr} = 0$
  circular orbit at $\boxed{0 = \frac{dV}{dr} = \epsilon M r^2 - \ell^2 r + 3M\ell^2}$ $\quad \epsilon = 0$

$\boxed{r_c = 3M}$

- A circular orbit exist $\forall \ell$
- The orbit is unstable (maximum)

The minimum energy for an incoming photon to "pass" the potential barrier is:

$$\mathcal{E} = \frac{1}{2}E^2 = V(r=3M) = \frac{\ell^2}{2(3M)^2} - \frac{M\ell^2}{(3M)^3} = \frac{\ell^2 M}{2(3M)^3}$$

or

$$b_c^2 \equiv \frac{\ell^2}{E^2} = 27M^2$$

b is the <u>impact parameter</u> of the light ray in flat spacetime.

In GR, any photon with

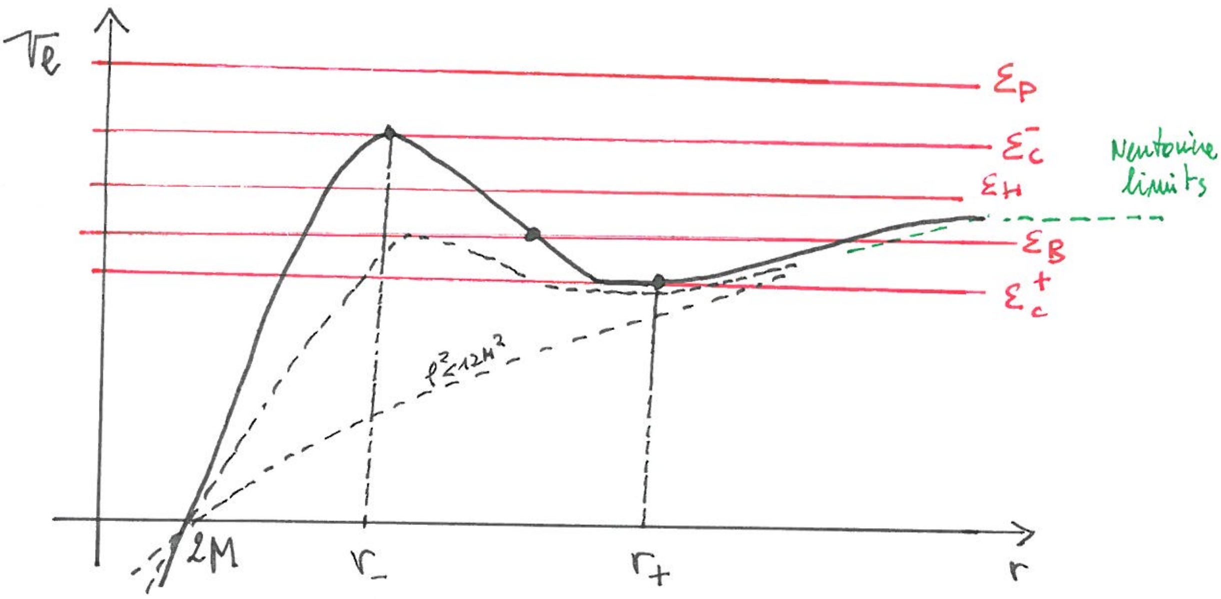
$$b \leq b_c = 3^{3/2} M$$

will be "captured" and will move towards $r=2M$ (and then $r=0$.)

One can define a capture <u>cross-section</u>:

$$\sigma = \pi b_c^2 = 27M^2\pi$$

$\boxed{\epsilon=1}$ Particles



Feats:

- $V_\ell(2M) = 0 \quad \forall \ell$
- V_ℓ has max and a min (r_-, r_+) for sufficiently large values of ℓ
- $V_\ell(r \to 0) \to -\infty$

For a given ℓ :

- Incoming particles with $\mathcal{E}_P$ "move-in" to $r=2M$ (and $r\to 0$). <u>Plunge orbits</u>.
- Incoming particles with $\mathcal{E}_H$ are on <u>hyperbolic</u> orbits.
- Particles with energy $\mathcal{E}_C$ are on <u>circular</u> orbits :

$$\boxed{0 = \frac{dV_e}{dr} = \epsilon M r^2 - \ell^2 r + 3M\ell^2} \qquad \epsilon = 1$$

Roots :

$$r_\pm = \frac{\ell^2 \pm \sqrt{\ell^2(\ell^2 - 12M^2)}}{2M}$$

<u>if $\ell^2 < 12M^2$:</u> V has <u>no</u> extrema

An incoming particle ($\dot{r} \leq 0$) reaches $r=2M$ and $r\to 0$

<u>if $\ell^2 > 12M^2$:</u> r_- : maximum $\to$ <u>unstable</u> circular orbit

r_+ : minimum $\to$ <u>stable</u> circular orbit

For $\ell^2 \gg 12M^2$: $(r_-, r_+) \approx \left(3M, \frac{\ell^2}{M}\right)$

"photon limit" "Newton limit"

For ℓ progressively smaller the 2 roots get closer until...

<u>if $\ell^2 = 12M^2$:</u> $r_+ = r_- = 6M$ a single <u>stable</u> circular orbit :

LAST STABLE ORBIT (LSO)

INNERMOST STABLE CIRCULAR ORBIT (ISCO)

<u>Summary of circular orbits</u> :

- Stable : $r_+ \geq 6M$ with frequency : $\Omega^2 = \left(\frac{d\varphi}{d\lambda}\right)^2 = \frac{\ell^2}{r_+^4}$
- Unstable : $3M < r_- < 6M$

- Particles with energy $\mathcal{E}_B$ are in <u>bound</u> orbit $r_{min} \leq r \leq r_{max}$ (not circular).

if $r \simeq r_+ + \delta r$, then the orbit oscillates in radius about r_+

$$\ddot{r} = -\frac{dV}{dr} \Rightarrow \delta\ddot{r} = -\left.\frac{d^2V}{dr^2}\right|_{r_+} \delta r$$

The oscillation frequency is $\omega^2 = \left.\frac{d^2V}{dr^2}\right|_{r_+} = \frac{M(r_+ - 6M)}{r_+^3 (r_+ - 3M)} = \frac{r_+ - 6M}{r_+} \Omega^2$

(Remember one can eliminate "ℓ" using the circular orbit equation: $\ell^2 = \frac{Mr^2}{r-3M}$).

One can verify that:

<u>For $r \gg M$</u> : $\omega^2 \approx \Omega^2$, the orbit is <u>closed</u>

And the particle return to the same radius after one period. Newtonian bound orbits are closed ellipses.

<u>But in general</u>: Bound orbits are <u>not</u> closed $\rightarrow$ precession!

$$\omega_p \equiv \Omega - \omega = \left[-\left(1 - \frac{6M}{r_+}\right)^{1/2} + 1 \right] \Omega$$

$$\approx \frac{3M^{3/2}}{r_+^{5/2}} \quad \text{for } r \gg M \quad \text{(leading order term)}$$

The term above is responsible for Mercury precession.

Exercise: Mercury perihelion precession

Consider the radial geodesic:

$$\frac{d^2r}{d\lambda^2} + \frac{1}{2}\frac{d\beta}{dr}\left(\frac{dr}{d\lambda}\right)^2 = e^{-\beta}\left(\frac{d\varphi}{d\lambda}\right)^2 + \frac{1}{2}e^{\alpha-\beta}\frac{d\beta}{dr}\left(\frac{dt}{d\lambda}\right)^2$$

and:

- include constants of motion
- multiply by $\left(\frac{d\lambda}{d\varphi}\right)^2$
- change variable to $u = \frac{1}{r}$

$\Big\}\rightarrow$ Get an equation for the orbit $u = u(\varphi)$

$$\boxed{\frac{d^2u}{d\varphi^2} + u = \frac{M}{\ell^2} + \underbrace{3Mu^2}_{\text{GR term}}}$$

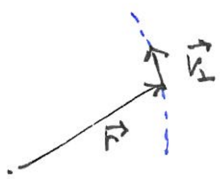
Without the GR term, the equation above is solved by:

$$u = u_N = M\ell^2(1 + e\cos\varphi)$$

where the parameter "e" is the eccentricity fixed by initial conditions.

Consider now the GR term, note that

$$\frac{3Mu^2}{M\ell^{-2}} = 3u^2\ell^2 = 3r^{-3}(r^2\dot{\varphi})^2 \simeq 3\left(r\frac{d\varphi}{dt}\right)^2 \simeq 3\left(\frac{v_\perp}{c}\right)^2 \sim 8\cdot 10^{-8}$$



where $v_\perp$ is Mercury's velocity perpendicular to the radius:

$\Rightarrow$ the GR term can be treated as a perturbation.

Let:

$$u := u_N + v$$

and write an equation for v which is linear in v

$$\underbrace{\ddot{u}_N + u - M\ell^{-2}}_{=0} + \ddot{v} + v = 3M \underbrace{(u_N + v)^2}_{(u_N+v)^2 \simeq u_N^2}$$

$$\ddot{v} + v = 3M u_N^2 = 3M^3 \ell^{-4} \left(1 + 2e\cos\varphi + e^2\cos^2\varphi\right)$$

Equation for a forced oscillator; solution = general solution of homogeneous equation + particular solution of the complete equation. One can verify that the particular solution can be:

$$\bar{v} = 3M^2\ell^{-4}\left[1 + e\varphi\sin\varphi + e^2\left(\frac{1}{2} - \frac{1}{6}\cos 2\varphi\right)\right]$$

$\sim$ constant term + secular term $\propto \varphi\sin\varphi$ + oscillatory term

Consider only the secular term; an approximate solution to the linear equation for v is:

$$u \simeq u_N + v_{secular} = M\ell^{-2}(1 + e\cos\varphi) + 3M^2\ell^{-4} e\varphi\sin\varphi =$$

$$\simeq M\ell^{-2}\left[1 + e\cos(\varphi - 3M^2\ell^{-2}\varphi)\right]$$

where we used: $3M^2\ell^{-2}\varphi \approx \sin(3M^2\ell^{-2}\varphi)$ for small $M^2\ell^{-2}$ and re-express in term of "cos". Hence, if $e \neq 0$, the orbit is not 2π-periodic in φ and it is not an ellipses (only approximately). We get

$\varphi : 0 \to 2\pi$, the perihelion shift of $2\pi\left(1 - 3M^2\ell^{-2}\right)^{-1} \approx 2\pi + 6\pi M^2\ell^{-2}$

or:

$$\Delta\varphi \simeq \frac{6\pi M^2}{\ell^2}.$$

The angular momentum ℓ^2 term can be removed considering a generic equation for an ellipsis:

$$(1 + e\cos\varphi) = u\, a(1 - e^2)$$

with a = semi-major axis, and comparing with u_N: $M\ell^2 = a(1 - e^2)$.

Substituting:

$$\Delta\varphi = \frac{6\pi M}{a(1-e^2)} = \frac{6\pi GM}{a(1-e^2)c^2}$$

☿ data:

$$\frac{GM_\odot}{c^2} \simeq 1.48 \cdot 10^5 \text{ cm}$$

$$a \simeq 5.79 \cdot 10^{12} \text{ cm}$$

$$e \simeq 0.20$$

$$T \simeq 88 \text{ days}$$

$$\Delta\varphi_{☿} \simeq 0.103''/\text{orbit} \simeq 43.0''/100 \text{ yrs}$$

which agrees with the discrepancy measured from Newtonian physics

Note that PSR 1913+16 has a precession of:

$$\Delta\varphi_{PSR} \simeq 4.2^\circ / 100 \text{ yrs}$$

about 270x Mercury. Because the masses of PSR 1913+16 are not known, the measured precession cannot be used to verify GR. It is instead used to estimate the masses themselves.

Exercise : Radially infalling particle

Consider a particle infalling from $r = r_{max}$ on a radial orbit ($\ell = 0$).
How long it takes to reach $r = 2M$?

"How long" $\rightarrow$ proper time or coordinate time.

Consider first proper time.

$$\dot{r}^2 = E^2 - 1 + \frac{2M}{r} \geq 0 \qquad \underline{\ell = 0}$$

$$\Rightarrow \qquad d\tau = - \frac{dr}{\sqrt{E^2 - 1 + \frac{2M}{r}}}$$

infalling

the integral is $\underline{\text{finite } \forall E}$ $\rightarrow$ the particle reaches $2M$ in a finite amount of proper time.

Consider now coordinate time.

$$\frac{dt}{d\tau} = u^0 = g^{00} u_0 = g^{00} \frac{p_0}{m} = - g^{00} E$$
$$= \left(1 - \frac{2M}{r}\right)^{-1} E$$

particle 4-velocity
$\mu = 0$ component

$$dt = E \frac{d\tau}{\left(1 - \frac{2M}{r}\right)} = - \frac{E\, dr}{\left(1 - \frac{2M}{r}\right)\left(E^2 - 1 + \frac{2M}{r}\right)^{1/2}}$$

Let $E = 1$ and $\xi = r - 2M$, then:

$$dt = - \frac{d\xi}{\frac{\xi}{r} \frac{2M^{1/2}}{r^{1/2}}} = - \frac{(\xi + 2M)^{3/2}\, d\xi}{(2M)^{1/2}\ \xi}$$

for $r \to 2M$ $\quad \xi \to 0$ and the integral is $\int \frac{d\xi}{\xi} \sim \ln \xi$

Note the divergent term is $\left(1-\frac{2M}{r}\right)^{-1} \sim \frac{1}{3}$ that does not contain E ...

$\rightarrow$ the coordinate time diverges as $r \rightarrow 2M$.

Summary:

- metric singularity at $r=2M$ ...
- coordinate time divergent for infalling particle ...

... Is there a problem with Schwarzschild coordinates approaching $r \sim 2M$?

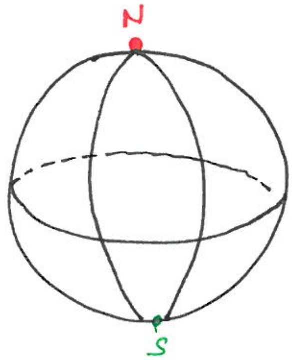
... $r=2M$: Singularity of the geometry or coordinate singularity?

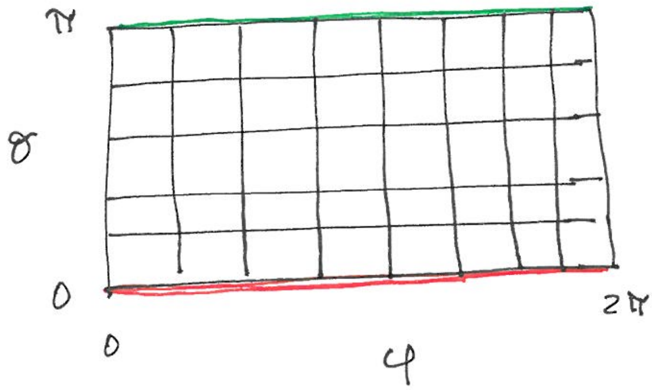
BTW: What is a coordinate singularity?

COORD. SINGULARITY $\rightsquigarrow$ points where the specific coordinates do <u>not</u> describe properly the geometry.

Example: 2-sphere and the poles

Take a 2-sphere and the usual (θ, φ) coordinates:





In the (θ, φ) coordinate diagram it is not obvious that $(\theta = 0, \varphi = \cdot)$ is the north pole, a single point of the manifold. In gen: Are coords "bad" in some points?

A way to discover that is to look at invariant quantities. Specifically:

- calculate the circumference of circles C_θ given by $\theta = \text{const}$;
- because the metric is positive definite, two points are the same point if the distance between them is $= 0$;

Hence:

$$\mathcal{C}_{\bar\theta} = \int_0^{2\pi} d\Omega(\theta = \bar\theta) = \int_0^{2\pi} \sin\bar\theta \, d\varphi = \sin\bar\theta \; 2\pi$$

and

$$\mathcal{C}_{\bar\theta} \underset{\bar\theta \to 0}{\longrightarrow} 0 \quad \wedge \quad \mathcal{C}_{\bar\theta} \underset{\bar\theta \to \pi}{\longrightarrow} 0$$

$\Rightarrow$ The points $(\theta = 0, \cdot)$ are the same one: N
— " — $(\theta = \pi, \cdot)$ — " — — " — : S

$\Rightarrow$ The coordinates (θ, φ) are "bad" for the poles.

Note in GR the metric is not positive definite, so the situation is more complicated

COORDINATES, LIGHT CONES & CAUSALITY

Consider the Schwarzschild metric slightly below $r = 2M$. Let:

$$\xi \equiv 2M - r > 0$$

and

$$g = \frac{\xi}{2M-\xi}\, dt^2 - \frac{2M-\xi}{\xi}\, d\xi^2$$

[henceforth we do not write the "$d\Omega^2$" term in the metric since we know that for (any) $t = \text{const}$, $r = \text{const}$ the metric is the one of 2-spheres]

Now <u>inside</u> $r = 2M$:

$$\xi > 0 \Rightarrow g_{\xi\xi} < 0 \Rightarrow \begin{cases} \xi\ (r) & \text{is a timelike coordinate} \\ t & \text{is a spacelike coordinate} \end{cases}$$

i.e. ∂_ξ is timelike

∂_t is spacelike

Physically, particles <u>must</u> follow timelike paths; but the latter are:

ξ increases $\rightarrow$ r decreases $\rightarrow$ particles must reach $r=0$!

Moreover, if an observer moving with the particle send out a photon, then the photon must move "forward in time" as seen by the observer ...

$\rightarrow$ the photon will also move towards $r=0$!

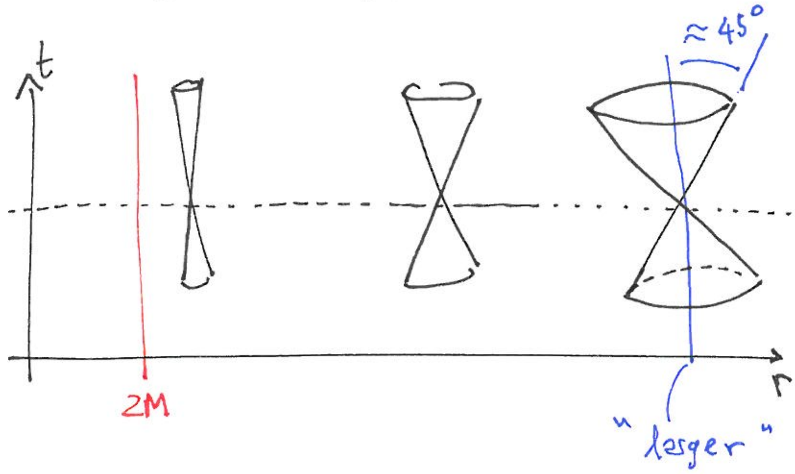
$\Rightarrow$ both particles and photons inside $r=2M$ will move to $r \to 0$, the <u>singularity</u>.

Nothing will get out from the surface $r=2M$...

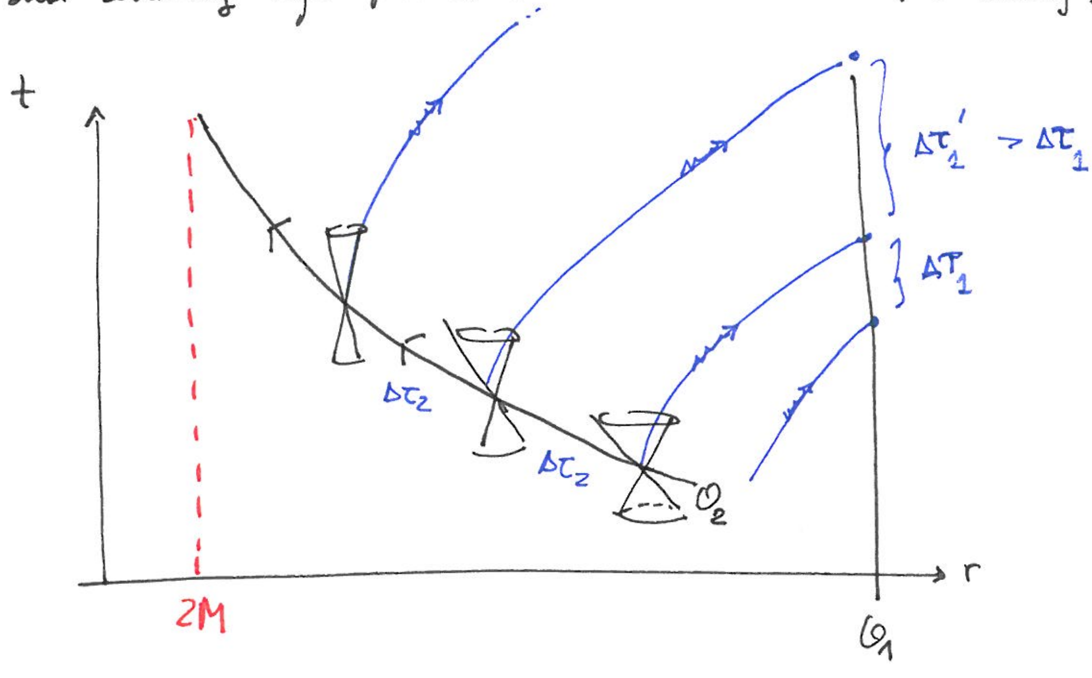
Light cones

Consider radial null curves: $0 = g = -\left(1-\frac{2M}{r}\right)dt^2 + \left(1-\frac{2M}{r}\right)^{-1}dr^2$

$$\frac{dt}{dr} = \pm\left(1-\frac{2M}{r}\right)^{-1/2}$$



The light cones "close up" from large r ($g \sim \eta$) towards $r = 2M$.
At $r=2M$ the cone is ∞-thin and $t \to +\infty$ (see Ex. on radially infalling particle).
As we shall see below $r=2M$ is a coordinate singularity ... but, still, physically "something" is happening. Consider an observer falling towards $r \to 2M$ and sending light pulses to another observer far away:



— O_2 falls towards $r=2M$ (although geodesics in this coordinates do not exist...) ...

— ... but O_1 never see it!

The radial null curve equation is solved by:

$$t = \pm r_* + \text{const}$$

where

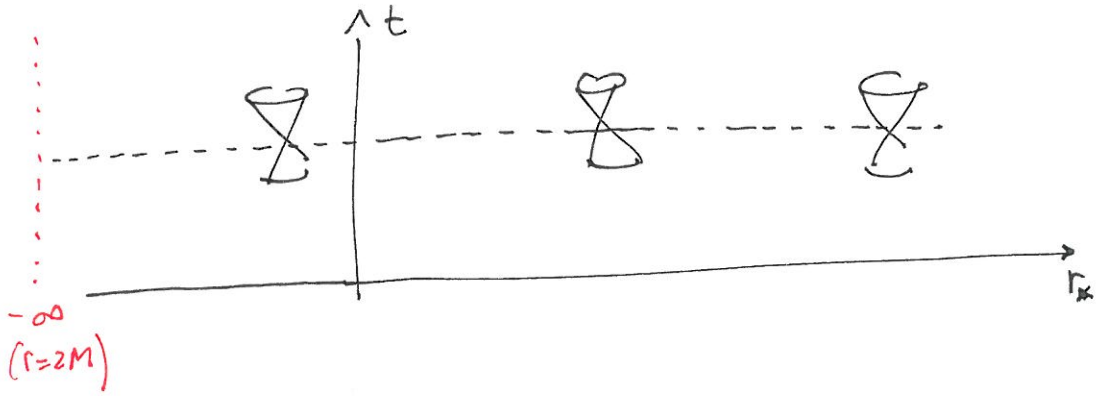
$$r_* = r + 2M \ln\left(\frac{r}{2M} - 1\right) \qquad \text{"tortoise coordinate"}$$

$$r \in [2M, \infty) \Rightarrow r_* \in (-\infty, \infty)$$

In the tortoise coordinate the metric is

$$g = \left(1 - \frac{2M}{r}\right)\left(-dt^2 + dr_*^2\right) \qquad \text{with } r = r(r_*)$$

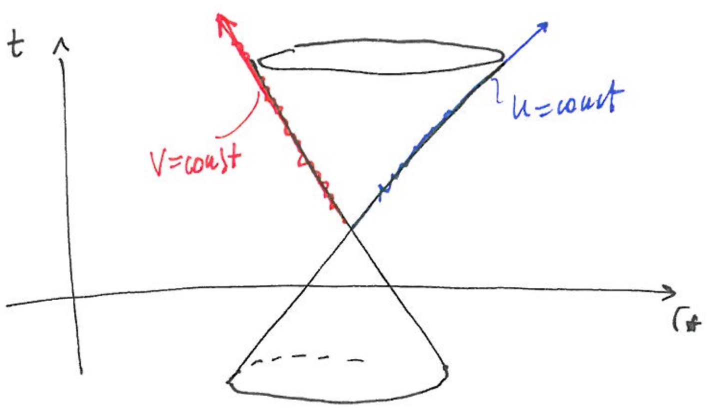
and the cones do not "close up" by approaching $r \to 2M$ $(r_* \to -\infty)$.
In these coordinate "time flows more slowly" and light cones stay $\pm 45°$.
However the surface $r = 2M$ is pushed to $-\infty$.



If one introduces coordinates:

$$u \equiv t - r_*$$
$$v \equiv t + r_* \qquad \text{null coordinates}$$

then ingoing radial null geodesics are given by: $v = \text{const}$
while outgoing radial null geodesics are given by: $u = \text{const}$



Consider now <u>Eddington (1923) - Finkelstein (1958) coordinates</u>

$(t,r) \longmapsto (v,r)$ "ingoing" EF

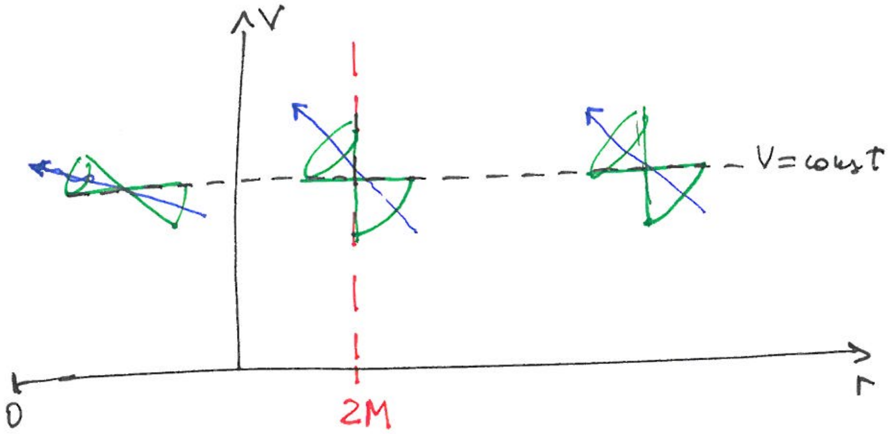
$$g = -\left(1-\frac{2M}{r}\right)dv^2 + (dv dr + dr dv) = -A\, dv^2 + (dv dr + dr dv)$$

the metric is <u>regular</u> ($g_{00}=0$ but $g_{\mu\nu}$ is invertible) and "valid" for $r \in (0,\infty)$* !

the radial null curves are given by :

$$0 = g \Rightarrow A\left(\frac{dv}{dr}\right)^2 = 2\frac{dv}{dr} \Rightarrow \frac{dv}{dr}\left(A\frac{dv}{dr}-2\right) = 0$$

$$\frac{dv}{dr} = \begin{cases} 0 \\ 2A^{-1} = \dfrac{2r}{r-2M} \end{cases}$$



<u>Observations</u>

- light cones remain well behaved at $r=2M$ $\mapsto$ timelike/null geodesics are <u>ok</u> for $r=2M$
- light cones "tilt" $\mapsto$ for $r<2M$ all future directed paths are in the direction of the $r=0$ singularity !

the spacetime has a "special" causal structure : the surface $r=2M$ acts as a "one-way membrane" that shed the interior from the exterior.

$r=2M$ is called an <u>EVENT HORIZON</u> (E.H.)

Note the E.H. is a <u>null surface</u> : if one considers the surfaces $r=$ const their normal is $n_a = \mathrm{grad}(r) = (dr)_a$. Thus : $n^2 = g^{ab} n_a n_b \stackrel{?}{=} 0$

---

* Note that initially the metric is defined only for $r>2M$, which is the domain of validity of the transformation for v. <u>Afterwards</u> we can analytically continue to $r \leq 2M$.

$$n^2 = g^{\mu\nu} n_\mu n_\nu = g^{\mu\nu} \partial_\mu r \, \partial_\nu r = g^{rr} = \left(1 - \frac{2M}{r}\right)$$

since $\quad g = \begin{pmatrix} -A & 1 \\ 1 & 0 \end{pmatrix} \Rightarrow g^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & A \end{pmatrix}$.

$n^2 = 0$ iff $r = 2M$: in the family of hypersurfaces $r = \text{const}$, the one $r = 2M$ is null. Its normal vector is

$$n = \left(g^{ab} \partial_b r\right) \partial_a \Big|_{r=2M}$$

$$= \left(g^{rr} \underbrace{\frac{\partial r}{\partial r}}_{=1} \partial_r + \underbrace{g^{vr}}_{=1} \underbrace{\frac{\partial r}{\partial r}}_{=1} \partial_v + \underbrace{g^{rv}}_{=1} \underbrace{\frac{\partial r}{\partial v}}_{=0} \partial_r \right)\Bigg|_{r=2M} =$$

$$= \left(g^{rr} \partial_r + \partial_v\right)\Big|_{r=2M} =$$

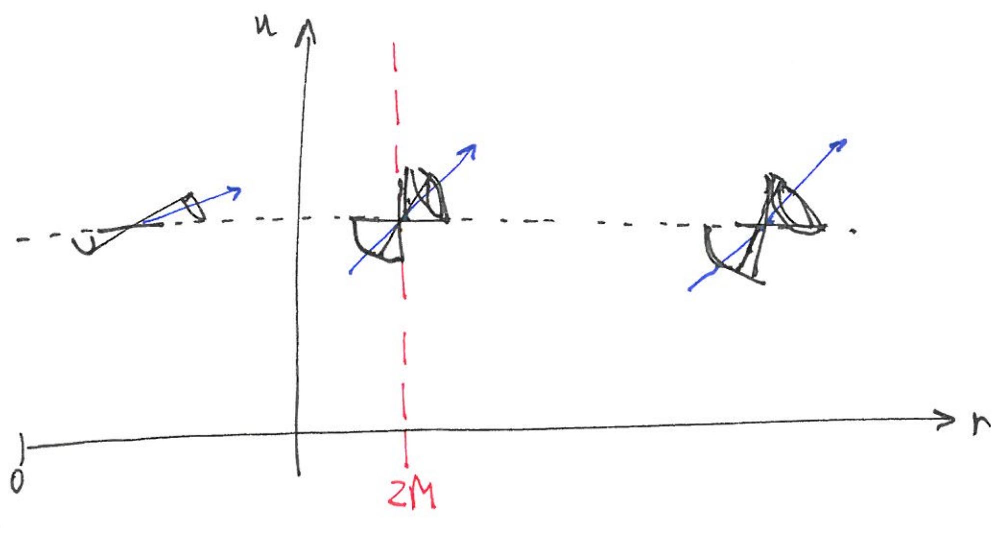
$$= \left(A \partial_r + \partial_v\right)\Big|_{r=2M} \underset{\substack{\uparrow \\ A=0}}{=} \partial_v$$

Note: normals to null surfaces cannot be normalized.

Consider now "outgoing" EF : (u, r)

$$g = -A\, du^2 - (du\, dr + dr\, du)$$

one can repeat analogous considerations and arrive to the following pic:



this is correct because:

$v = t + r_* = \text{const} \quad \wedge \quad r_* \to -\infty \quad \Rightarrow \quad t \to +\infty$

$u = t - r_* = \text{const} \quad \wedge \quad r_* \to +\infty \quad \Rightarrow \quad t \to -\infty$

but physically it indicates that only past-directed curves can cross the horizon !?
(or, alternatively, that the horizon is a one-way membrane ... where everything gets out)

Summary (so far):

- Schwarzschild coordinates (t, r) do not seem good for $r \leq 2M$
- Solving the radial null geodesic equation we have derived the tortoise coordinate: $r_* \equiv r + 2M \ln\left(\frac{r}{2M} - 1\right)$
- From r_* one can define null coordinates (adapted to null geodesics):

$$u \equiv t - r_*$$
$$v \equiv t + r_*$$

- Eddington-Finkelstein coordinates are:

(v, r) "ingoing"
(u, r) "outgoing"

In EF coords we can analitically continue (extend) the metric to $r \in (0, \infty)$ because it is regular. Schwarzschild coordinate singularity "disappears".
The null surface $r = 2M$, however, characterize a particular causal structure of the spacetime:

- ingoing coordinates: $r = 2M$ is an event horizon from which no particles or photons can escape.
  All the future-oriented curves (timelike or null) stay inside $r = 2M$ if start from inside.

- outgoing coordinates: $r = 2M$ has similar property but "time reversal"
  All the future-oriented curves must go out from the $r = 2M$ ...

The analysis in EF coords suggests that the two pairs of coordinates are exploring different parts (regions) of the spacetime.

Problem: Is it possible to define/find coordinates that describe the whole spacetime?

KRUSKALL-SZEKERES (MAXIMAL) EXTENSION (1960)

In null coordinates the metric is:

$$g = -\frac{1}{2}\left(1-\frac{2M}{r}\right)\left(dv\,du + du\,dv\right)$$

where $r = r(u,v)$: $\frac{1}{2}(v-u) = r_* = r + 2M\ln\left(\frac{r}{2M}-1\right)$

and the metric is singular at $r=2M$.

For $r>2M$ define:

$$\bar{V} \equiv e^{\frac{v}{4M}} = \left(\frac{r}{2M}-1\right)^{1/2} e^{\frac{r+t}{4M}}$$

$$\bar{U} \equiv -e^{\frac{u}{4M}} = -\left(\frac{r}{2M}-1\right)^{1/2} e^{\frac{r-t}{4M}}$$

$(\bar{V},\bar{U})$ are null coordinates ($\partial_{\bar{V}}, \partial_{\bar{U}}$ are null vectors)

The metric reads:

$$g = -\frac{16M^3}{r} e^{-\frac{r}{2M}} \left(d\bar{V}\,d\bar{U} + d\bar{U}\,d\bar{V}\right)$$

it is again regular and can be analytically extended to

$$r>0$$

Note: $r=2M \leftrightarrow \bar{V}=0=\bar{U}$.

The Kruskall-Szekeres (K.S.) coordinates are defined as the timelike and spacelike coordinates related to $(\bar{V},\bar{U})$:

$$T \equiv \frac{1}{2}\left(\bar{V}+\bar{U}\right)$$

$$R \equiv \frac{1}{2}\left(\bar{V}-\bar{U}\right)$$

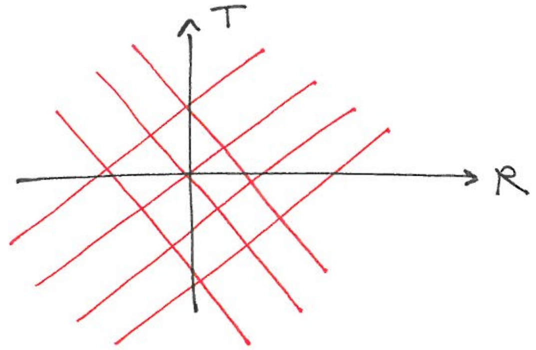
The metric reads:

$$g = \frac{32M^3}{r} e^{-\frac{r}{2M}} \left(-dT^2 + dR^2\right)$$

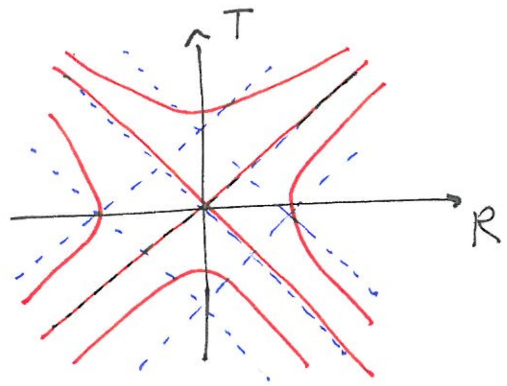
where $r = r(T, R)$: $T^2 - R^2 = \left(1 - \frac{r}{2M}\right) e^{\frac{r}{2M}}$.

Properties :

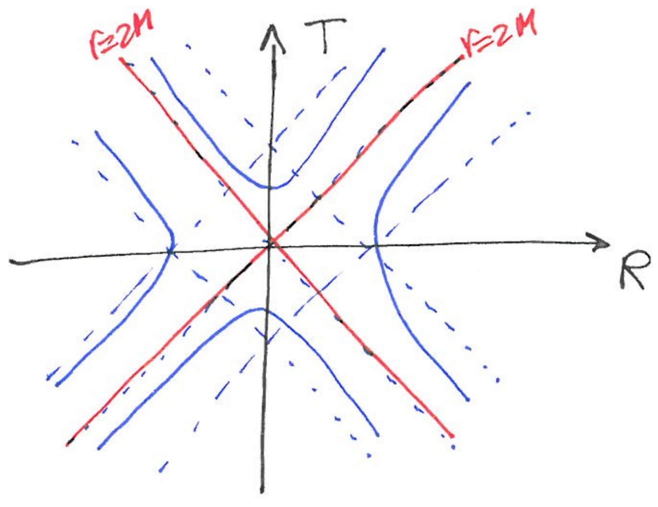
— Radial null curves ; $g = 0 \rightarrow \left(\frac{dT}{dR}\right)^2 = 0 \rightarrow T = \pm R + \text{const}$,
Straight lines like in Minkowski :



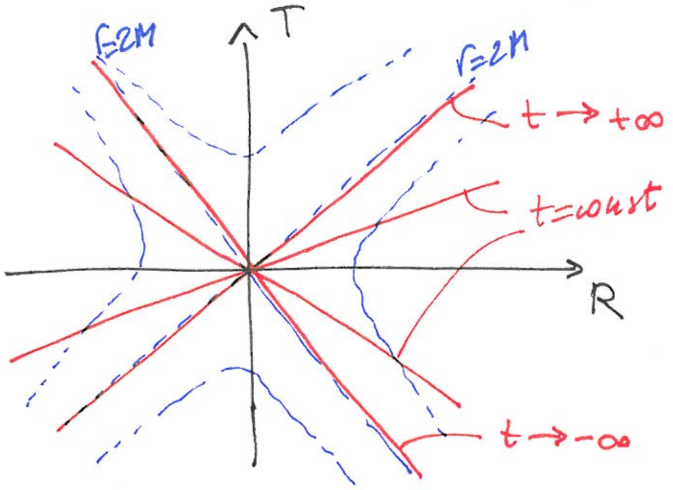
— $r = \text{const}$ surfaces : $T^2 - R^2 = \text{const}$ $\rightarrow$ hyperbolae



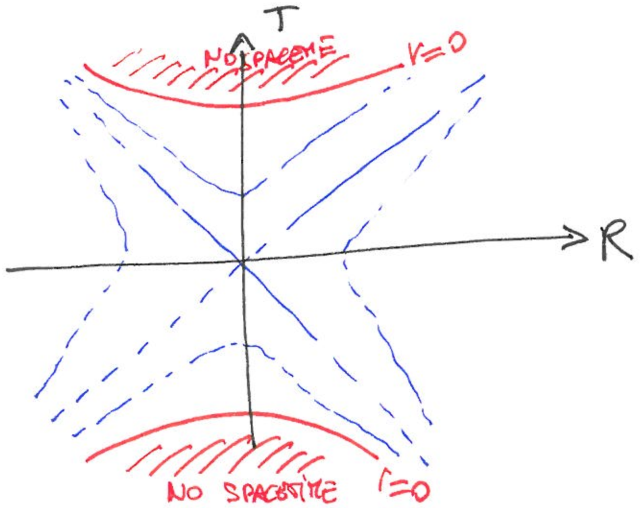
— Event horizon : special hyperbolae $r = 2M$ i.e. $T^2 - R^2 = 0 \Rightarrow T = \pm R$



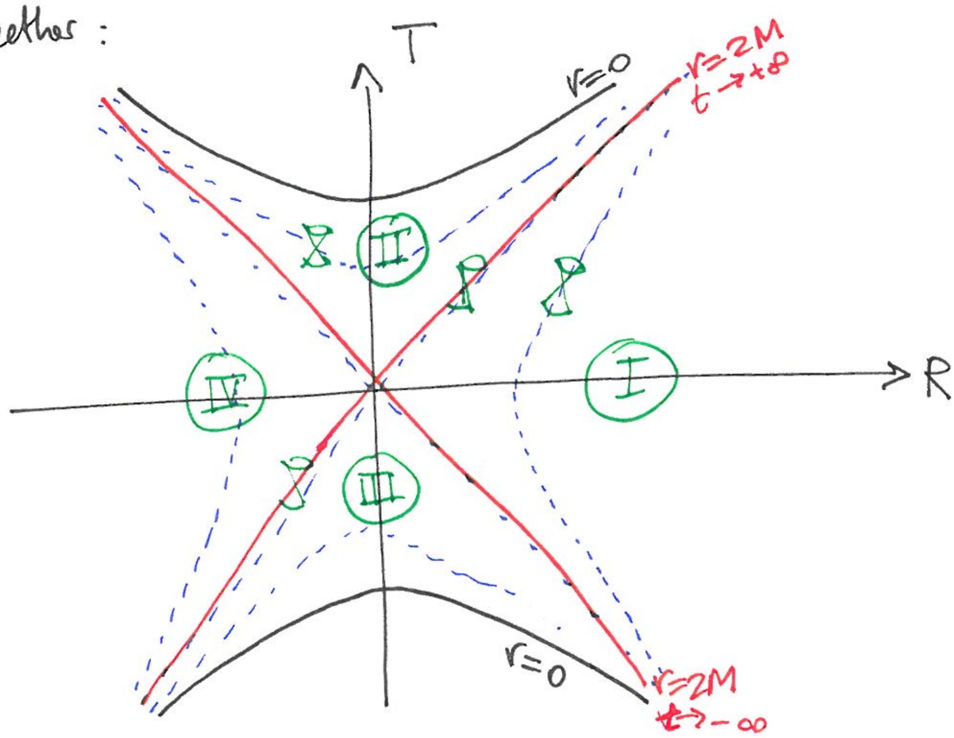
— $t = \text{const}$ surfaces : $\dfrac{T}{R} = \tanh\left(\dfrac{t}{4M}\right)$, lines with slope $\sim \tanh(t)$



— Singularity $r=0$; $r>0 \leftrightarrow T^2 - R^2 > 1$



Put things together :



Explore Schwarzschild spacetime:

- 4 regions

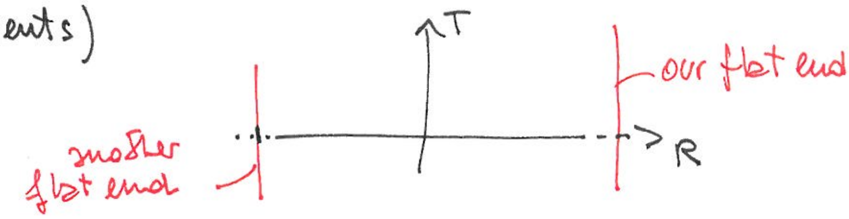
- Region I : "our ordinary" exterior solution where (t,r) coordinates are well-behaved.

Following future-directed null rays one can go from I to II
Following past-directed null rays one can go from I to III

- Region II : BLACK HOLE, we travel in and not out
once in we reach the singularity $r=0$

- Region III : time-reversal of II, we cannot go there
Things escape from the past singularity $r=0$ and cross the past horizon $r=2M$ towards the future
WHITE HOLE

- Region IV : Asymptotically flat region disconnected from I (spacelike events)



Consider the radial coordinate:

$$x: \quad r = x\left(1+\frac{M}{2x}\right)^2 = x\,\psi^2(x)$$

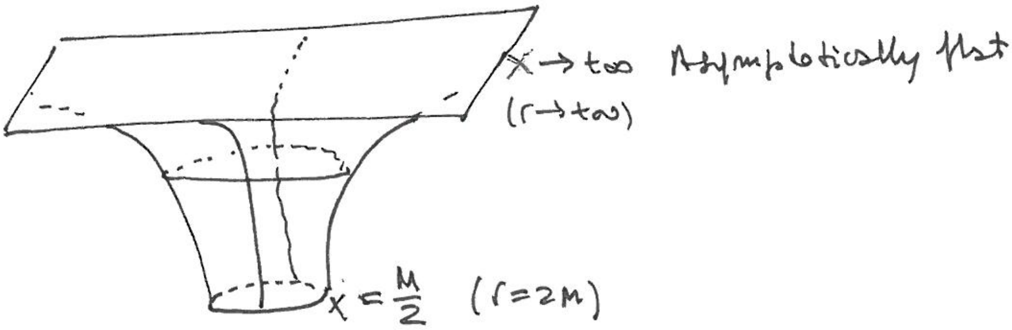
for $x > \frac{M}{2}$: $$x = \frac{1}{2}\left[r + \sqrt{r(r-2M)} - \frac{M}{2}\right], \quad r > 2M$$

$$g = -\left(1-\frac{M}{2x}\right)^2 \psi^2(x)\,dt^2 + \psi^4(x)\left[dx^2 + x^2 d\Omega^2\right]$$

For $t = \text{const}$

$$g = \psi^4 ds^2 = \underbrace{\psi^4}_{\text{conformal factor}} \underbrace{\left(dx^2 + x^2 d\Omega^2 \right)}_{\text{Euclidean line element}}$$

Visualize g by suppressing the θ-dimension:

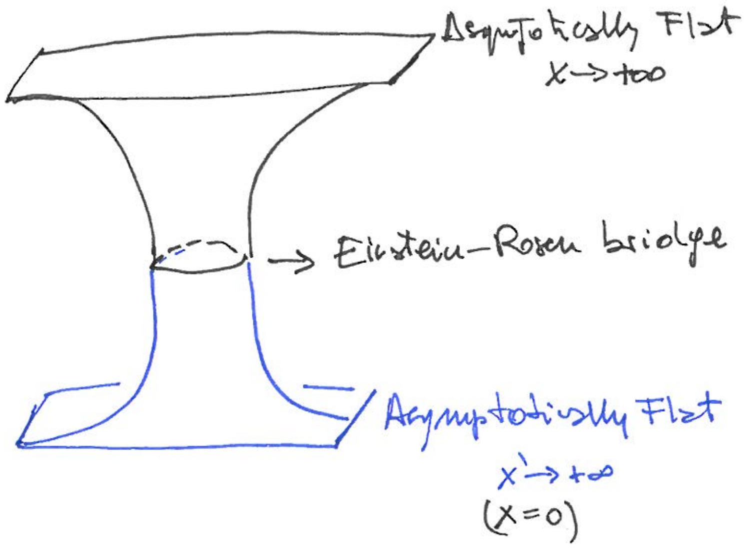


Observe that the transformation:

$$x \longmapsto x' = \frac{M^2}{4x}$$

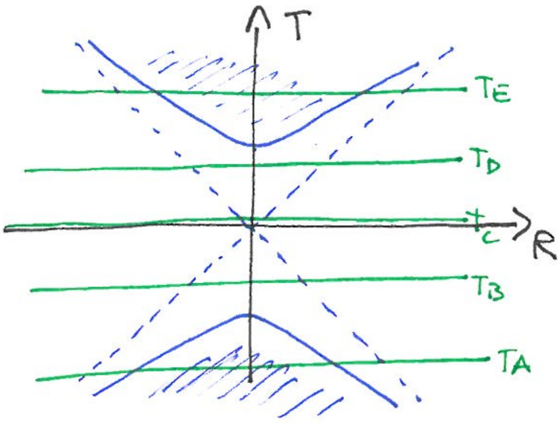
- leaves invariant the spheres $x = x' = \frac{M}{2}$
- leaves the g metric in the same form: $g = \psi^4(x')\left(dx'^2 + x'^2 d\Omega^2\right)$
- maps $x = 0$ to $x' \to \infty$, as well as all the other points

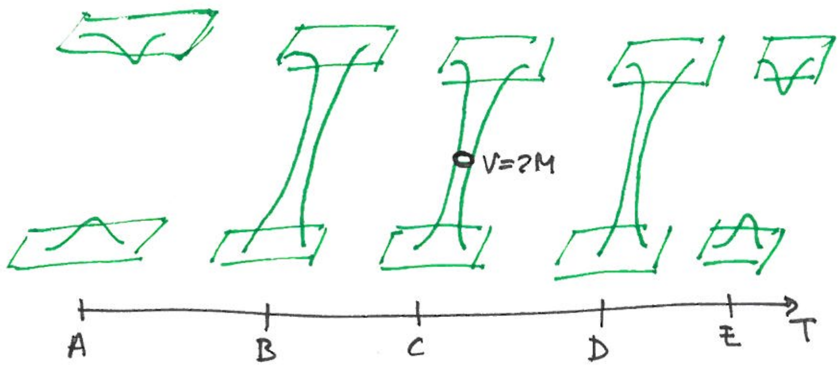
Hence one can "glue" another copy thought the minimal surface $x = \frac{M}{2}$:



WORMHOLE.

In the Kruskall diagram:





the wormhole join for some time region I with region IV, but timelike observers cannot cross it.

CONFORMAL COMPACTIFICATION

Consider Minkowski spacetime: $\eta = -dt^2 + dr^2$, with:

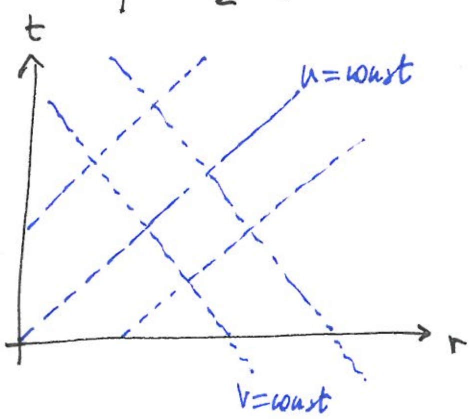
$$-\infty < t < +\infty$$
$$0 \leq r < +\infty$$

Q: Is it possible to describe the spacetime with coordinates with compact support?

Null coordinates:

$$\begin{cases} u = t-r \\ v = t+r \end{cases} \quad \text{with} \quad u,v \in (-\infty,\infty),\ u \leq v$$

$$\eta = -\frac{1}{2}(du\,dv + dv\,du)$$



Compactify null coordinates with "arctan":

$$\begin{cases} U \equiv \arctan u \\ V \equiv \arctan v \end{cases} \qquad U, V \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \qquad U \leq V$$

$$\eta = W^{-2}(U,V)\left[-2\,(dU\,dV + dV\,dU)\right]$$

$$W \equiv 2 \cos U \cos V \qquad : \underline{\text{conformal factor}}$$

Compactified timelike/spacelike coords. are now given by:

$$\begin{cases} T = V + U \\ R = V - U \end{cases} \qquad \begin{array}{l} 0 \leq R < \pi \\ |T| + R < \pi \end{array}$$

$$\eta = W^{-2}\left(-dT^2 + dR^2\right)$$

$$W = \cos T + \cos R$$

The above expressions show that the original (physical) metric is related to a <u>conformal metric</u>:

$$\tilde{\eta} \equiv W^2 \eta = -dT^2 + dR^2 + \sin^2 R \, d\Omega^2$$

where we have restored the 2-spheres.

The conformal metric describes a manifold

$$M = \mathbb{R} \times S_3$$

[The same as the Einstein static universe...]

Suppressing the (θ, φ) coordinates the manifold can be drawn as a <u>cylinder</u> in the R and T coordinates:

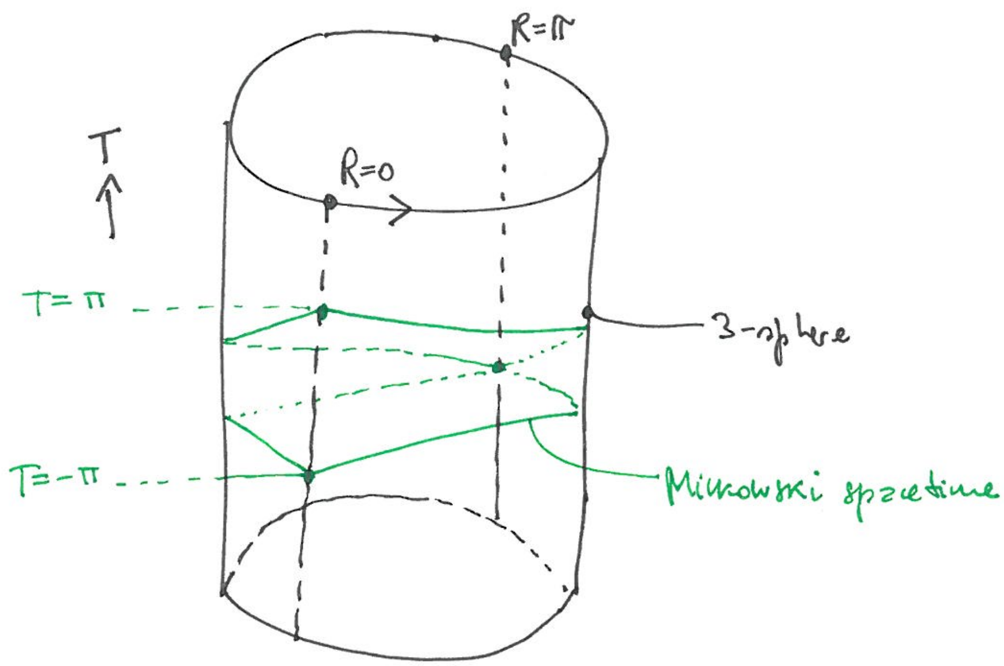
Properties

- light cones are at $45°$
- Minkowski spacetime is the interior + $R=0$
- Boundaries of the diagram are called conformal infinity :

$$\left.\begin{array}{ll} i^0 & \text{(point)} \\ i^{\pm} & \text{(point)} \end{array}\right\} \text{N/S poles of } S_3$$

$$\mathscr{I}^{+} \text{ (null surfaces)}$$

- Timelike geodesics start at i^- and end at i^+
- Null geodesics start at $\mathscr{I}^-$ and end at $\mathscr{I}^+$
- Spacelike geodesics start and end at i^0



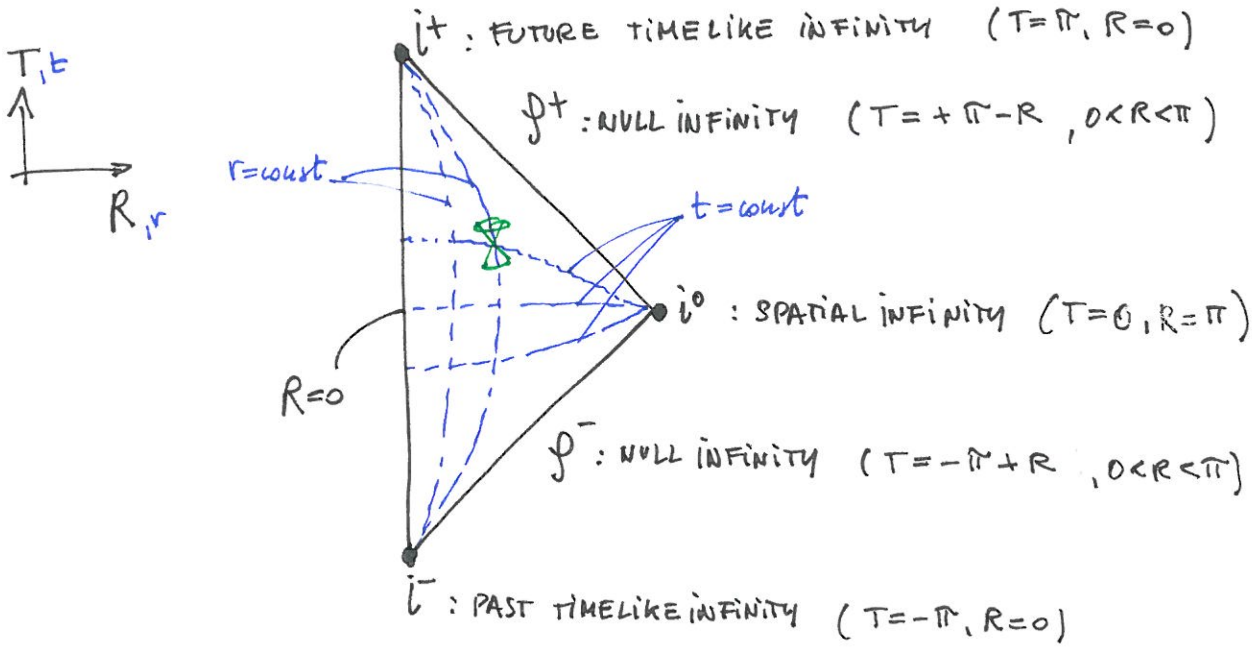
where each point of the cylinder is a 3-sphere.

The Minkowski spacetime is a <u>portion</u> of the manifold defined by

$$|T| + R < \pi$$

and it is drawn in green.

If one now opens the cylinder and consider only the Minkowski region:



Then one can describe Minkowski spacetime in a compact diagram.

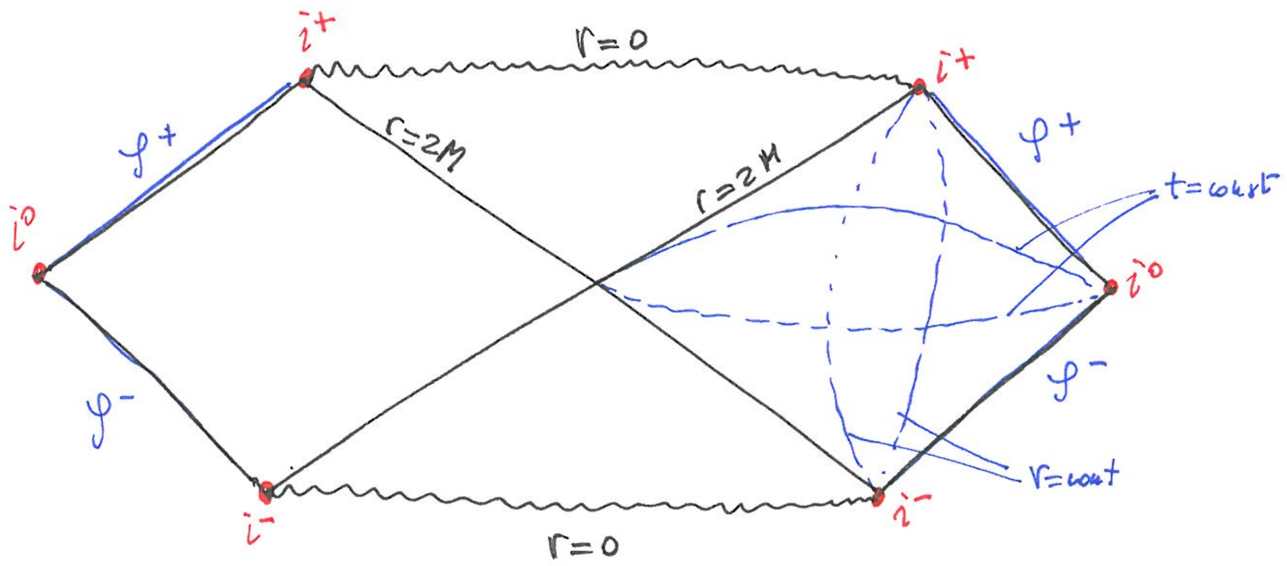
Conformal diagram for Schwarzschild spacetime

Conformal coordinates can be constructed from the null coordinates $(\bar{u}, \bar{v})$ using the "arctan" compactification:

$$\begin{cases} q \equiv \arctan \dfrac{\bar{v}}{\sqrt{2M}} \\ p \equiv \arctan \dfrac{\bar{u}}{\sqrt{2M}} \end{cases}$$

that maps to : $\quad q, p \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and $-\frac{\pi}{2} < p + q < \frac{\pi}{2}$.

At constant angular coordinates, the metric in (p, q) coordinates is conformally related to Minkowski. The diagram looks like:



where one recognize the 4-regions.

Observations

- light cones are at $45°$;
- $i^\pm$ are distinct from $r=0$;
- conformal infinity is the same as in Minkowski because Schwarzschild is asymptotically flat.