These semi-private notes are constructed from the following books:

- R.Wald, "General Relativity" University of Chicago Press, 1984
- S.M.Carrol, "Spacetime and Geometry, An Introduction to General Relativity", Addison-Wesley, 2003.
- D.Bauman Cosmology

If you decide to use them to study or teach, please
(0) be careful and refer to the original books
(1) cite/refer to my website
(2) let me know and send feedbacks.

SB 2019
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COSMOLOGY

What spacetime GR predicts for the Universe?

Cosmological scales

GR applies to the description of systems of mass M and size R if:

$$\frac{GM}{c^2 R} \sim 1 .$$

The above condition is met in two cases [Schutz]:

1) R becomes small faster than M
2) M becomes large faster than R

The first case applies to compact objects (isolated systems) like black holes or neutron stars.

The second case applies, for example, to spacetimes filled with matter such that:

$$\rho \sim \text{const} \longrightarrow M \sim \rho R^3 \longrightarrow \frac{M}{R} \sim \rho R^2$$

One expects that at sufficiently large scales the Universe is homogeneous ($\rho \sim$ const), and thus GR should be applied for its description.

Examples

- Galaxy: $R \sim 15$ kpc $\sim 5 \cdot 10^{22}$ cm , $\frac{M}{R} \sim 10^{-6}$ (like solar system)
- Galaxies clusters : $R \sim$ Mpc $= 10^6$ pc (10^3 galaxies)
- Cosmological scales : $R \gtrsim 10^9$ pc $\gtrsim$ Gpc

Main observation : cosmic microwave background (CMB) radiation at $T \sim 3$ K indicates a homogeneous and isotropic Universe at large scales.

HOMOGENEITY & ISOTROPY

The concepts of homogeneity and isotropy can be easily made formal considering symmetry transformations of the metric.

Recall that:

<u>Def</u>: A diffeomorphism $\varphi: M \to M$ is called an <u>isometry</u> iff $\varphi_* g_{ab} = g_{ab}$ where φ_* is the pull-back operation.

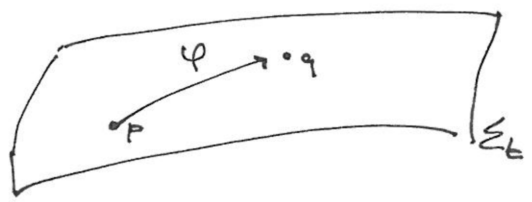
Focus on <u>spatial</u> sections of the spacetime M.

M is:

<u>Def</u> HOMOGENEOUS (spatially) iff $\exists\ \Sigma_t$ (spacelike hypersurface) foliating M such that:

$\forall\ p, q \in \Sigma_t\ \exists$ isometry taking p to q

i.e. $q = \varphi(p)$ with φ isometry

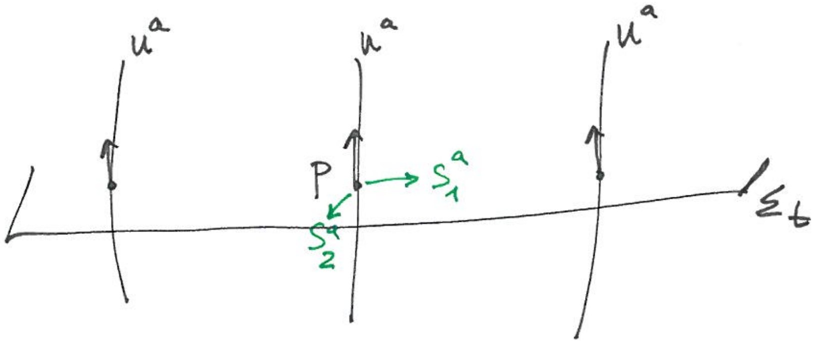


<u>Def</u> ISOTROPIC (spatially) iff Given a family (congruence) of timelike curves with tangent vector u^a,

$\exists$ an isometry that:

i) leaves any point p and u^a invariant

ii) but "rotates" the vectors orthogonal to u^a one into the other.



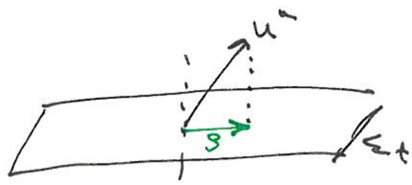
u^a : world line of isotropic observers

Because of the isometry it is impossible to construct a preferred direction on the (spatial) hypersurface orthogonal to u^a, Σ_t.

Observations

- Homogeneity + isotropy $\Rightarrow$ Σ_t must be $\perp$ to u^a.

  Proof (for unique Σ_t)

  Assume: Σ_t is not orthogonal
  Σ_t is unique

  then one can construct a preferred direction s^a and break isotropy. □

- A metric can be homogeneous but nowhere isotropic

  Example: $\Sigma = \mathbb{R} \times S^2$

- A metric can be isotropic around a point, but not homogeneous.

  Example: the cone 

- A metric isotropic <u>every where</u> is homogeneous.

- A metric isotropic around a point and homogeneous is <u>everywhere</u> isotropic.

Exercise:

Discuss and prove last 4 statements above.

ROBERTSON - WALKER METRIC

Metric of a homogeneous and isotropic spacetime.

From the above definitions it should be clear that:

$$M = \mathbb{R} \times \Sigma_t$$

with:

$$g_{ab} = -u_a u_b + \gamma_{ab}(t)$$

where:

- u^a is the isotropic observer velocity;
- γ_{ab} is the metric on the spatial hypersurfaces Σ_t.

The hypothesis of homogeneity and isotropy further constrain γ_{ab}.
One can prove [see e.g. Wald] that Σ_t is a maximally symmetric space:

$${}^{(3)}R_{ijkl} = k\, \gamma_{k[i}\, \gamma_{j]l} \qquad i,j,k,l = 1,2,3$$

where k is a constant proportional to the (constant) Ricci scalar.
From the equation above one has:

$${}^{(3)}R_{jl} = 2k\, \gamma_{jl} \; ; \qquad {}^{(3)}R = 6k .$$

These type of spacetimes are called spacetimes of constant curvature.

The relevant cases are:

$$\boxed{k = 0, \pm 1}$$

since one can always normalize the constant and what matters is only the sign.

Note that $[k] = L^{-2}$.

Rem:

$$R_{jl} = \gamma^{ik} R_{ijkl} = R^k{}_{jkl}$$

cases: $\quad k: \begin{cases} =0, & \Sigma = \mathbb{R}^3 \qquad \gamma = dx^2 + dy^2 + dz^2 \\ >0, & \Sigma = S^3 \qquad \text{3-sphere} \\ <0, & \Sigma = H^3 \qquad \text{3-hyperboloidals} \end{cases}$

The metric for the cases $k = \pm 1$ can be found by immersion in $\mathbb{R}^4$:

$$\gamma = dx^2 + dy^2 + dz^2 \pm du^2 \qquad \text{with:} \quad x^2 + y^2 + z^2 \pm u^2 = \pm 1 .$$

Differentiate the "conic"-like equation to find:

$$2x\,dx + 2y\,dy + 2z\,dz \pm 2u\,du = 0$$

write:

$$\pm du^2 = \pm \frac{(u\,du)^2}{u^2} = \pm \frac{(x^i dx^i \delta_{ij})^2}{1 \mp (x^2+y^2+z^2)} = \pm \frac{(\delta_{ij} x^i dx^j)^2}{1 \mp \delta_{ij} x^i x^j}$$

with $x^i = (x, y, z)$. Substitute into the metric:

$$\gamma_{ij} = \delta_{ij} + k \frac{x_i x_j}{1 - k x^2} \qquad\qquad x^2 = x_j x^j$$

Note this is valid also for $k=0$.

Using spherical coordinates: $\quad x^i = (r, \theta, \varphi)$

$$dx^2 + dy^2 + dz^2 = dr^2 + r^2 d\Omega^2$$
$$\delta_{ij} x^i dx^j = r\,dr$$

and

$$\gamma = dx^2 + dy^2 + dz^2 + \frac{k}{1-kx^2} (\delta_{ij} x^i dx^j)^2 = dr^2 + r^2 d\Omega^2 + \frac{k}{1-kr^2} r^2 dr^2 =$$

$$= \frac{1 + kr^2 - kr^2}{1 - kr^2} dr^2 + r^2 d\Omega^2 = \frac{dr^2}{1-kr^2} + r^2 d\Omega^2$$

The final form of the metric is obtained considering the radial transformation:

$$d\chi = \frac{dr}{\sqrt{1-Kr}}$$

$$\rightarrow \quad r = S_K(\chi) = \begin{cases} \sin(\chi) & , K = 1 \\ \chi & , K = 0 \\ \sinh(\chi) & , K = -1 \end{cases}$$

Hence:

$$\boxed{\gamma = d\chi^2 + S_K^2(\chi)\, d\Omega^2}$$

The explicit form of the 4-metric can be found as follows.

$$g_{ab} = -u_a u_b + \gamma_{ab}(t)$$

- For a given t, choose coordinate such that $\gamma = d\chi^2 + S_K^2(\chi)\, d\Omega^2$ where (χ, θ, φ) are spherical, Cartesian or hyperboloidal coordinates for $K = 0, \pm 1$

- Transport the coordinates along the isotropic observers, i.e. each isotropic observer is at <u>fixed</u> spatial coordinates.

- Label each surface Σ with the proper time (clocks) of the isotropic observers.

Obtain:

$$\boxed{g = -d\tau^2 + a^2(\tau)\left[d\chi^2 + S_K^2(\chi)\, d\Omega^2\right]}$$

<u>Robertson-Walker metric</u>

$$K = 0, \pm 1$$

$[a(\tau)] = L$ SCALE FACTOR

FRIEDMANN-ROBERTSON-WALKER (FRW) EQUATIONS

The Robertson-Walker metric can be used as ansatz to find solutions of EFE.
The FRW equations determine such solutions for spacetimes:

- Homogeneous
- Isotropic
- Non vacuum

Where the matter content is modeled by a perfect fluid:

$$T_{ab} = (\rho + P) u_a u_b + P g_{ab} .$$

It is natural to choose u^a as the velocity of isotropic observers

$$u^a = (1, 0, 0, 0) ,$$

in such a way the fluid it at <u>rest</u> in coordinates comoving with isotropic obs.
Note that in this case:

$$T_{ab} = \mathrm{diag}(\rho , P \delta_{ij})$$

and:

$$T^a{}_b = \mathrm{diag}(-\rho, P, P, P) \quad ; \quad T = T^a{}_a = -\rho + 3P .$$

EFE read:

$$\begin{cases} G_{\tau\tau} = 8\pi T_{\tau\tau} = 8\pi \rho \\ G_{\perp} = 8\pi T_{\perp} = 8\pi P \end{cases}$$

where "$\perp$", is any of the equations <u>spatially projected</u>: $G_{ab} \hat{s}^a \hat{s}^b \ldots$
They must be (and they are) all equivalent because of isotropy!

Another way of writing EFE is to consider the Trace-reverse equation:

$$R_{\mu\nu} = 8\pi \left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T\right) .$$

For the $k=0$ case one has:

$$g = -d\tau^2 + a^2\,(dx^2 + dy^2 + dz^2)$$

christoffel:

$$\Gamma^{\tau}_{ii} = a\dot{a} \qquad\qquad i = 1,2,3$$

$$\Gamma^{i}_{i\tau} = \frac{\dot{a}}{a}$$

Ricci:

$$R_{\tau\tau} = -\,3\,\frac{\ddot{a}}{a}$$

$$R_{11} = a\ddot{a} + 2\dot{e}^2$$

Spatial projection:

$$R_\perp = R_{ab}\,\hat{S}^a\hat{S}^b = a^{-2}\,R_{11}$$

because:

$$S^a = (0,1,0,0)$$

$$S^2 = S_a S^a = g_{ab}\,S^a S^b = a^2$$

$$S^a \to a^{-2} S^a \;, \text{ normalized}, \quad \hat{S}^a = a^{-2} S^a .$$

Ricci scalar:

$$R = -R_{\tau\tau} + 3R_\perp = 6\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2}\right)$$

Hence:

$$G_{\tau\tau} = R_{\tau\tau} + \frac{1}{2}R = 3\left(\frac{\dot{a}}{a}\right)^2 = 8\pi\rho$$

$$G_\perp = R_\perp - \frac{1}{2}R = -2\frac{\ddot{a}}{a} - \left(\frac{\dot{a}}{a}\right)^2 = 8\pi P$$

combine the term $\left(\frac{\dot{a}}{a}\right)^2$ and re-write the second equation:

$$\begin{cases} 3\left(\dfrac{\dot{a}}{a}\right)^2 = 8\pi\rho \\[2ex] 3\,\dfrac{\ddot{a}}{a} = -4\pi(\rho + 3P) \end{cases}$$

The calculation for generic k gives :

$$\begin{cases} \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3}\rho - \frac{k}{a^2} \\ \frac{\ddot{a}}{a} = -\frac{4\pi}{3}(\rho + 3P) \end{cases} \qquad \underline{\underline{\text{FRW}}}$$

Evolution equations for the scale factor ("velocity" and "acceleration").
Equation for the matter can be found from

$$0 = \nabla_\mu T^\mu{}_0 = \partial_\mu T^\mu{}_0 + \Gamma^\mu_{\mu\alpha} T^\alpha{}_0 - \Gamma^\alpha_{\mu 0} T^\mu{}_\alpha = -\partial_\tau \rho - 3\frac{\dot{a}}{a}(\rho + P) \Rightarrow$$

$$\dot{\rho} + 3(\rho + P)\frac{\dot{a}}{a} = 0 .$$

The system of equations must be closed by specifying the <u>equation of state</u> of the matter in the form of a relation $P = f(\rho)$ (see below).

Let us discuss basic consequences of the FRW equations.

<u>Dynamical Universe</u>

FRW predict that if

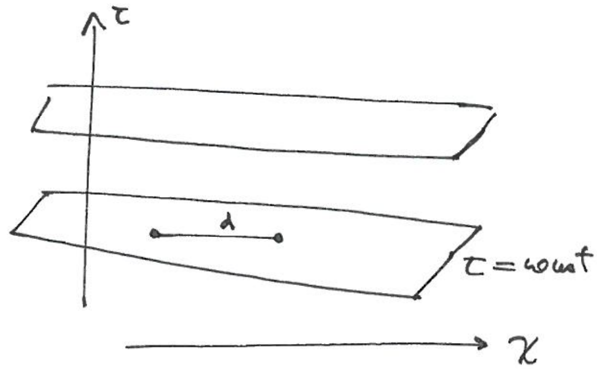
$$\left.\begin{matrix} \rho > 0 \\ P \geq 0 \end{matrix}\right\} \Rightarrow \ddot{a} < 0 \Rightarrow \dot{a} \gtrless 0 ,$$

the universe cannot be static, i.e. it must expand or contract.
It is important to stress that what "expands" (or "contracts") here is the distance scale between two isotropic observers <u>at the same τ</u>.
Given the RW metric :

$$g = -d\tau^2 + a^2(\tau)\left[d\chi^2 + S_k^2(\chi)\, d^2\Omega\right] ,$$

the comoving radial coordinate distance at given τ is:

$$d(\tau) = a(\tau)\,\chi$$



the expansion velocity is:

$$v = \dot{d} = \dot{a}\,\chi = \dot{a}\,\frac{d}{a} = \frac{\dot{a}}{a}\,d$$

the quantity:

$$\boxed{H(\tau) \equiv \frac{\dot{a}}{a}}$$ Hubble parameter

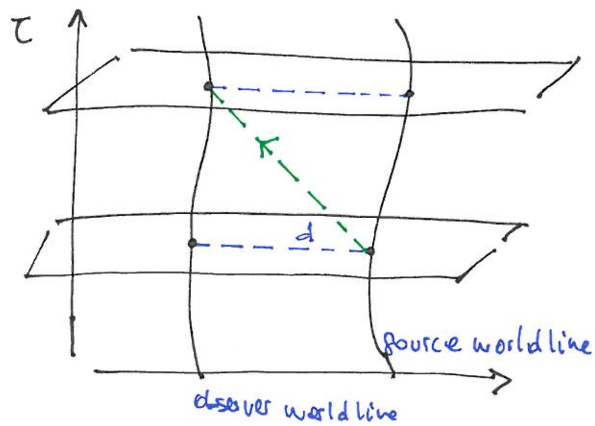
but it is not a constant. The relation: $v = H d$ is the Hubble law.

Note that v and d can be interpreted as "physical" velocities and distances only for sufficiently close objects:

$$d \ll H_0^{-1} \quad \text{Hubble radius}$$

or small red-shifts (see below).

If the distances are larger the distance (velocity) is meaningless because needs to consider the following situation:



and, in fact, v can be larger than the speed of light for distant events.

Big-Bang*

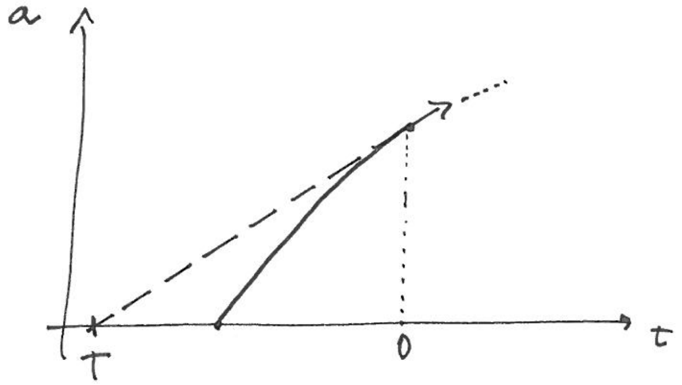
Hubble has observed that close Galaxies are moving apart from eachother, i.e.

$$H_0 > 0 \Rightarrow \dot{a}_0 > 0,$$

where $H_0 = H(\tau=0)$ and $\dot{a}_0 = a(\tau=0)$, $\tau=0$ is "now". However:

$$\left.\begin{array}{l} \ddot{a} < 0 \\ \dot{a}_0 > 0 \end{array}\right\} \Rightarrow \text{the Universe was expanding } \underline{\text{faster}} \text{ at earlier times} \ldots$$

$\Rightarrow$ one can set a lower limit to the time at which $a(\tau) = 0$, by simply extrapolating back the <u>current</u> rate of expansion:



$$T_H \equiv \frac{a_0}{\dot{a}_0} = H_0^{-1} \qquad \underline{\text{Hubble time}}$$

At any time $\tau > T_H$ the Universe was in a state in which:

i) distance between all points of space is zero

ii) curvature is infinite $(R \sim a^{-2})$

iii) matter had infinite density

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* Hoyle, 1949 on BBC radio.

Evolution of the scale factor

Consider the equation:

$$\left(\frac{\dot a}{a}\right)^2 = H^2 = \frac{8\pi}{3}\rho - \frac{k}{a^2}$$

Define a critical density:

$$\rho_c \equiv \frac{3H^2}{8\pi}$$

and a density parameter:

$$\Omega \equiv \frac{\rho}{\rho_c} = \frac{8\pi}{3}\rho H^{-2}.$$

Note that, as the Hubble parameter, the density parameter is time dependent.

The equation above can be re-written as:

$$\boxed{\Omega - 1 = \frac{k}{H^2 a^2}}$$

and shows that:

ρ	Ω	k	*
$< \rho_c$	< 1	< 0	
$= \rho_c$	$= 1$	$= 0$	flat
$> \rho_c$	> 1	> 0	closed

$$\boxed{\dot a^2 = \frac{8\pi}{3}\rho a^2 - k}$$

$k=0$: $\rho a^2 \xrightarrow[a\to+\infty]{} 0$: $\dot a \to 0$

$k=-1$: $\rho a^2 \xrightarrow[a\to+\infty]{} 0$: $\dot a \to +1$

$k=+1$: $\dot a^2 = \rho a^2 - 1 > 0$

- If $k = 0, -1$ and $\rho > 0$, then $\dot a > 0$ and expansion must continue;
  - $k = 0$ and $\rho a^2 \xrightarrow[a\to\infty]{} 0 \Rightarrow \dot a \xrightarrow[a\to\infty]{} 0$
  - $k = -1$ and $\rho a^2 \xrightarrow[a\to\infty]{} 0 \Rightarrow \dot a \xrightarrow[a\to\infty]{} 1$

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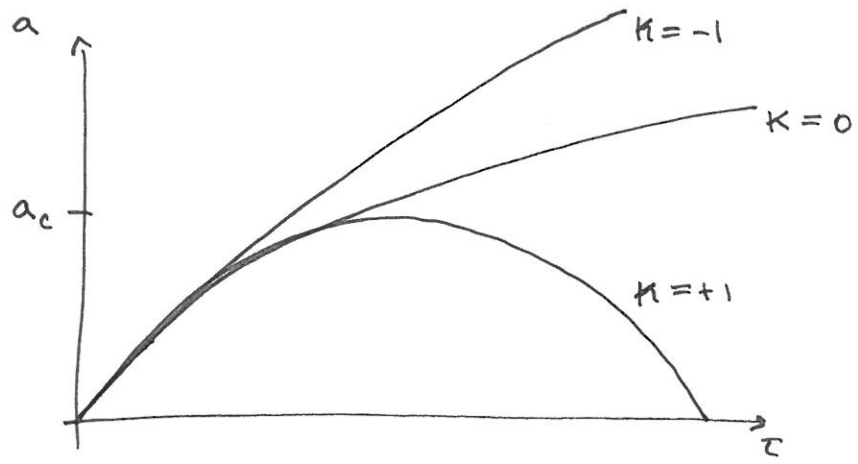
* And "standard" matter, no cosmological constant.

- If $K=+1$ and $\rho>0$, then $\dot{a}^2 = \rho a^2 - 1 > 0 \Rightarrow \exists\, a_c : \quad a(\tau) < a_c \;\forall \tau$

  a_c marks a critical value, a "maximum size", after which the Universe must contract again $\left(\ddot{a} \geq -\frac{4\pi}{3}(\rho+3P)a_c < 0\right)$



Note that current observations indicate $\Omega \sim 1$.

Matter and radiation

Consider the matter equation:

$$\dot{\rho} + 3(\rho+P)\frac{\dot{a}}{a} = 0$$

with an equation of state of the form:

$$P = w\rho$$

where:

$$w = \begin{cases} 1/3 & \text{radiation} \\ 0 & \text{dust (matter)} \\ -1 & \text{vacuum energy } (\Lambda) \end{cases}.$$

the equation writes:

$$\frac{\dot{\rho}}{\rho} = -3(1+w)\frac{\dot{a}}{a}$$

with solution:

$$\rho \propto a^{-3(1+w)} \quad \text{or} \quad \rho a^{+3(1+w)} = \text{const}.$$

For matter (dust) : $\rho_M a^3 = \text{const}$

→ conservation of rest-mass : the number density of particles must decrease as the Universe expands.

For radiation : $\rho_R a^4 = \text{const}$

→ the energy density of photons decreases more rapidly than the increase in volume because photons lose energy due to redshift.

The solutions above indicate that :

- Radiation is the dominant contribution to the total ρ at early τ ;
- As the Universe expand, the matter contribution decay slower ;
- Matter is the dominant contribution to the total ρ at late τ .

Both matter and radiation decay faster than a^{-2}; hence in the FRW equation

$$\rho a^2 \to 0 \quad , \text{ for matter and radiation}$$

as we have assumed above.

Note that for vacuum energy : $\rho_V \propto a^0$

→ (if present) vacuum energy dominate over matter and radiation at late times.

The current observations indicate that

$$\frac{\rho_M}{\rho_R} \sim 10^3 \quad \text{today} ,$$

but current cosmological model require to assume vacuum energy.

Exercise: Equation of state for radiation

Radiation can be described by a gas of relativistic particles using the perfect fluid model:

$$T_{ab} = (\rho + P) u_a u_b + P g_{ab} .$$

At the same time electromagnetic radiation has the stress-energy tensor:

$$T_{ab} = F_{ac} F_b{}^c - \frac{1}{4} g_{ab} F^{cd} F_{cd} .$$

Take the trace of both:

$$T^a{}_a = -\rho + 3P \qquad \text{(P.F.)}$$

$$T^a{}_a = 0 \qquad \text{(E.M.)}$$

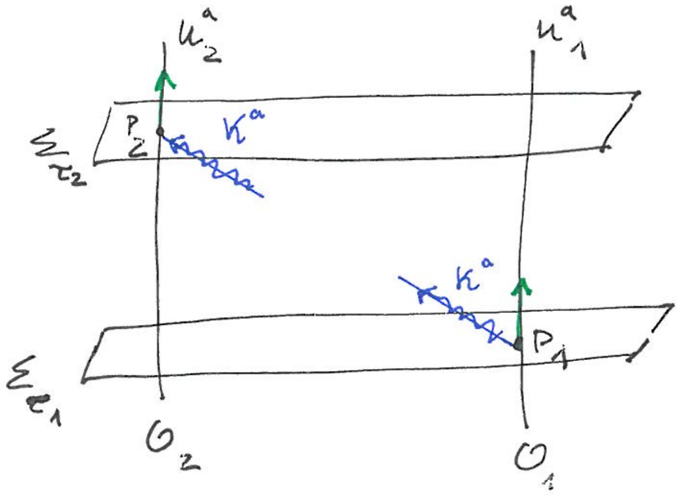
Since they have to be equal, one finds that:

$$p = \frac{1}{3} \rho .$$

REDSHIFT

The Hubble observation on the expansion of the Universe is based on the redshift of spectral lines of distant (but not too much...) Galaxies.

Consider two isotropic observers and the following situation:



A photon is emitted in P_1 and absorbed in P_2.
The photon momentum is k^a and the photon frequency measured by an observer of 4-velocity u^a is:

$$\omega = -k_a u^a$$

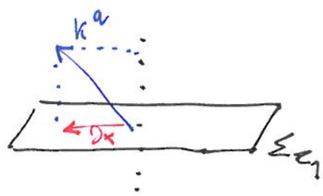
The redshift can be calculated from symmetry considerations.

(i) there exist a killing vector (K.V.) that points to the direction of the projection of K^a on Σ_{τ_1} <u>and</u> k^a on Σ_{τ_2}.

For example:

$k=0$ K.V. : ∂_x, ∂_y, ∂_z

Take : ∂_x is the projection of k^a on Σ_{τ_1}



then: $K^a(\partial_y)_a = 0 = K^a(\partial_z)_a$ on Σ_{τ_1}.

Because ∂_y, ∂_z are K.V. the product $K^a(\partial.)_a$ must be constant (K^a is tangent to null geodesic) $\Rightarrow$ $K^a(\partial_y)_a = 0 = K^a(\partial_z)_a$ on Σ_{τ_2}.

(ii) Take ξ^a the K.V. determined as per point (i), e.g. $\xi^a = (\partial_x)^a$, $\kappa = 0$.

The magnitude $|\xi| = (\xi^a \xi_a)^{1/2}$ is proportional to the scale factor:

$$\frac{\sqrt{\xi^a \xi_a}\,\big|_{p_1}}{\sqrt{\xi^a \xi_a}\,\big|_{p_2}} = \frac{a(\tau_1)}{a(\tau_2)}$$

Immediately obvious for $\xi^a = (\partial_x)^a$ $\kappa = 0$, but holds in general.

(iii) Since k^a is null the projection against u^a must be

$$0 = k^2 \quad \Rightarrow \quad k_a u^a = -k_a \frac{\xi^a}{|\xi|} .$$

Putting together things:

$$\frac{\omega_2}{\omega_1} = \frac{k_a u^a|_{p_2}}{k_a u^a|_{p_1}} \underset{\text{(iii)}}{=} \frac{k_a \xi^a}{|\xi|}\bigg|_{p_2} \cdot \frac{|\xi|}{k^a \xi_a}\bigg|_{p_1} \underset{\text{(i)}}{=} \frac{|\xi|_{p_1}}{|\xi|_{p_2}} \underset{\text{(ii)}}{=} \frac{a(\tau_1)}{a(\tau_2)}$$

As the Universe expands, the wavelenght of the photon increases proportionally to the scale factor.

Redshift:

$$z = \frac{\lambda_2 - \lambda_1}{\lambda_1} = \frac{\omega_1}{\omega_2} - 1 = \frac{a(\tau_2)}{a(\tau_1)} - 1 .$$

For nearby galaxies, the light emitted travels

$$\Delta\tau = \tau_2 - \tau_1 \approx d$$

and

$$a(\tau_2) \cong a(\tau_1) + \Delta\tau\, \dot{a} .$$

Hence:

$$z \simeq \frac{\dot{a}}{a} d = H d .$$

$\Delta\tau = \tau_2 - \tau_1 \simeq d$

$$a_2 \simeq a_1 + \Delta\tau\, \dot{a}$$

$$\frac{a_2}{a_1} \simeq 1 + \Delta\tau \frac{\dot{a}}{a}$$

$$\underbrace{\frac{a_2}{a_1} - 1}_{z} \simeq \Delta\tau \frac{\dot{a}}{a} = d H$$

Distance measurement

Consider the measurement of the distance between O_1 and O_2.

Light signals emitted by O_1 will be measured by O_2 at later time $\tau_2 > \tau_1$.

In a flat and static spacetime, O_2 measures a <u>flux</u>:

$$F = \frac{\text{intrinsic luminosity of the source}}{4\pi \, \chi^2} = \frac{L}{4\pi\chi^2} = \frac{\dot{E}}{4\pi\chi^2}$$

In FRW spacetime the formula needs to be changed to account for different effects:

- The area of the "spherical" wave front is:

$$\chi^2 \longmapsto S_k^2(\chi)$$

- Each photon will be redshifted : $E_{rec} = \dfrac{E_{em}}{(1+z)}$

- The delay in the photons time arrivals due to expansion. Two photons emitted at time interval dt will arrive at time interval $dt\,(1+z)$

Hence:

$$F = \frac{L}{4\pi \, S_k^2(\chi)\,(1+z)^2} = \frac{L}{4\pi \, D_L}$$

the quantity D_L is called <u>luminosity distance</u>.

The formula above is used to determine cosmological parameters given

- observables $(F, z, \dots)$
- standard candles : sources for which L is known or can be estimated accurately.

Note that the formula contains χ ... which is not observable.

χ can be eliminated in favour of cosmological parameters.

For example, if one considers null radial geodesics:

$$0 = g = -d\tau^2 + a^2 d\chi^2 \quad \Longrightarrow$$

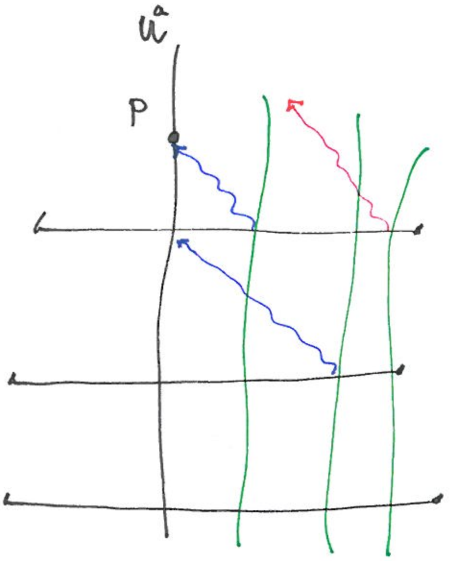
$$\chi = \int \frac{d\tau}{a(\tau)} \underset{H = \frac{\dot{a}}{a}}{=} \int \frac{da}{a^2 H} \underset{a = (1+z)^{-1}}{=} \int \frac{dz}{H}$$

See [Carroll book] for a more sophisticated calculation.

HORIZONS

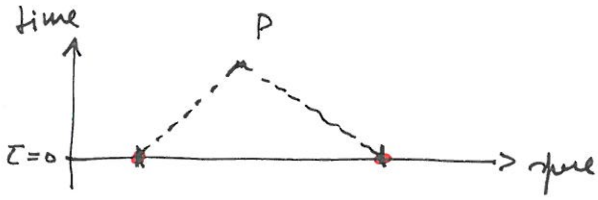
Sitting at a point of the spacetime, how much Universe can be observed?

Given an event p, which isotropic observers could have sent a signal that reached an isotropic observer at or before p?



Def: PARTICLE HORIZON AT P

Boundary of the region that contains world lines of particles (observers) that intersect the past light cone of p.



Note:

- Not a trivial question since the universe "started" at finite time
- One would expect that, since $a \to 0$ at Big Bang, all isotropic observers could communicate with each others...

Consider the flat Universe $k=0$:

$$g = -d\tau^2 + a^2(\tau)\,(dx^2 + dy^2 + dz^2)$$

and make the coordinate transformation:

$$\tau \longmapsto \tilde{t} = \int \frac{d\tau}{a(\tau)}$$ <u>conformal time coordinate</u>

obtain:

$$g = a^2(\tilde{t})\left(-d\tilde{t}^2 + dx^2 + dy^2 + dz^2\right)$$

The metric is said <u>conformally flat</u>, and it has the property that:

a vector is timelike/null/spacelike iff it has the same property in the flat metric.

$\Rightarrow$ causal structure of the Universe $k=0$ is the same as the Minkowski ...

... as far as the conformal time transformation is valid.

In particular:

- if $\tilde{t} = \int \frac{d\tau}{a(\tau)}$ <u>diverges</u> as one approaches the Big Bang singularity

$$a(\tau \to 0) \to 0$$

  then, the RW metric will be related to Minkowski all the way down to

$$t \to -\infty$$

  and <u>no</u> particle horizons will exist.

- else, if $\tilde{t} = \int \frac{d\tau}{a(\tau)}$ <u>converges</u> as $\tau \to 0$, then

  particle horizons will occur.

The integral diverges if : $a(\tau) \sim \alpha \tau$ for $\tau \to 0$ and some constant $\alpha \in \mathbb{R}$.

The presence of horizons depends on particular solutions of the FRW eqs.

A summary of solutions for $k = 0, \pm 1$ and standard matter content can be found in e.g. [T26.51 of Wald].

Those solutions show that in most of the cases particle horizon <u>are</u> present.

Moreover, for the case $k=+1$ (closed Universe):

- Particle horizon ceases to exist at the maximum expansion a_c for <u>dust</u>
- but continue to exist also afterwards for <u>radiation</u>.

COSMOLOGICAL CONSTANT

Before Hubble observations Einstein looked for a <u>static</u> solution for the Universe. The only way to obtain such a solution is to modify the field equations by a term proportional to the metric:

$$G_{ab} + \Lambda g_{ab} = 8\pi T_{ab} \qquad (1917)$$

where Λ is the <u>cosmological constant</u>.

Observations

- The EFE with cosmological constant are the most general equations for a $(0,2)$ symmetric tensor which has zero divergence and it is build with metric's 2^{nd} derivatives.

- The presence of Λ makes EFE incompatible with Newton unless $\Lambda \equiv 0$.

- The Λ constant is often "moved" to the r.h.s. and interpreted as <u>vacuum energy</u>:

$$G_{ab} = 8\pi \left(T_{ab} - \frac{\Lambda}{8\pi} g_{ab} \right) = 8\pi \left(T_{ab}^{matter} + T_{ab}^{vacuum} \right)$$

  with: $T_{ab}^{vacuum} = -\frac{\Lambda}{8\pi} g_{ab} \equiv \rho_{vacuum} g_{ab}$.

  The vacuum energy has an unclear origin:

  - constant of nature
  - energy of quantum fields in vacuum state
  - energy of classical scalar fields (dark energy)

  ...

- Λ is currently necessary in the most accepted cosmological models, but its physical origin and interpretation remains an open problem.

FWR equations with Λ

$$\begin{cases} \left(\frac{\dot a}{a}\right)^2 = \frac{8\pi}{3}\rho + \frac{\Lambda}{3} - \frac{k}{a^2} \\ \frac{\ddot a}{a} = -\frac{4\pi}{3}(\rho + 3P) + \frac{\Lambda}{3} \end{cases}$$

QUICK CALCULATION:

$$\ddot a = 0 \Rightarrow \frac{\Lambda}{3} = \frac{4\pi}{3}\rho \Rightarrow \boxed{\Lambda > 0}$$

$$\dot a = 0 \Rightarrow a^2 = \frac{k}{4\pi\rho} \Rightarrow \boxed{k > 0}$$

$$a = +\sqrt{\frac{k}{4\pi\rho}}$$

- Λ must be positive
- Universe must be spherical

Static solution for dust

Want: $\dot a = 0 = \ddot a$ $\quad (P = 0)$

$$\ddot e = 0 \Rightarrow \frac{\Lambda}{3} = \frac{4\pi}{3}\rho$$

Substitute in first eq: $\frac{8\pi}{3}\rho + \frac{4\pi}{3}\rho - \frac{k}{a^2} = \frac{\dot a}{a}$

$$\dot a = 0 \Rightarrow 4\pi\rho - \frac{k}{a^2} = 0 \qquad (*)$$

for dust: $\rho a^3 = \text{const} = \rho_0 a_0^3 = \rho_0$

Temporarily set:
- $a_0 = a(\tau = 0) = 1$ "now"
- $\rho_0 = \rho(\tau = 0)$

Substitute $\rho = \rho_0 a^{-3}$ in $(*)$: $\quad 4\pi\rho_0 - k a = 0 \qquad (*)'$.

From $(*)'$ observe that:

$$\left.\begin{matrix}\rho_0 > 0 \\ a_0 > 0\end{matrix}\right\} \Rightarrow \boxed{k > 0}$$ spherical universe (de Sitter)

and obtain:

$$\bar a = \frac{4\pi\rho_0}{k} = \left(\frac{4\pi\rho_0}{\Lambda}\right)^{1/3}$$ static scale factor

Note also that:

$$\Lambda = 4\pi\rho_0 \bar a^{-3} = \frac{k}{16\pi^2\rho_0^2} > 0 .$$

ΛCDM MODEL

Realistic cosmology models are based on FRW equations and include:

- Radiation
- Matter (dust)
- Λ term.

The first FRW equation can be written:

$$H^2 = \frac{8\pi}{3}\left(\rho_R + \rho_M + \rho_\Lambda\right) - \frac{K}{a^2}$$

Divide by H^2 and introduce:

$$\rho_c \equiv \frac{3}{8\pi} H^2$$

$$\Omega_i \equiv \frac{\rho_i}{\rho_c} \qquad i = R, M, \Lambda$$

one gets:

$$1 = \Omega_R + \Omega_M + \Omega_\Lambda - \frac{K}{a^2 H^2}$$

which suggest to further write:

$$\Omega_K \equiv -\frac{K}{a^2 H^2} = 1 - \Omega_R + \Omega_M + \Omega_\Lambda = 1 - \sum_i \Omega_i$$

Consider "today" $\tau = 0$:

$$a(0) \equiv a_0 = 1$$

$$H(0) \equiv H_0$$

$$\rho_c(0) \equiv \rho_{c0} = \frac{3}{8\pi} H_0^2$$

$$\Omega_i(0) \equiv \Omega_{i0} = \frac{\rho_i(0)}{\rho_{c0}} = \frac{\rho_i(0)}{\frac{3}{8\pi} H_0^2}$$

$$\Omega_K(0) \equiv \Omega_K(0) = -\frac{K}{a_0^2 H_0^2} = 1 - \sum_i \Omega_{i0}$$

Re-express the ρ's in the Friedmann equation as:

$$\frac{8\pi}{3}\rho_R = \frac{8\pi}{3}\rho_R(0)\frac{a^4(0)}{a^4(t)} = \frac{8\pi}{3}\rho_{R0}\frac{a_0^4}{a^4} = H_0^2\,\Omega_{R0}\,a^{-4} \qquad (a_0=1)$$

and similarly:

$$\frac{8\pi}{3}\rho_M = H_0^2\,\Omega_{M0}\,a^{-3}$$

$$\frac{8\pi}{3}\rho_\Lambda = H_0^2\,\Omega_{\Lambda 0}$$

$$-\frac{k}{H^2} = H_0^2\,\Omega_k\,a^{-2}$$

To obtain:

$$\boxed{\frac{H^2}{H_0^2} = \Omega_{R0}\,a^{-4} + \Omega_{M0}\,a^{-3} + \Omega_k\,a^{-2} + \Omega_{\Lambda 0}}$$

Current observations indicate:

$\Omega_{R0} \sim 10^{-4}$

$\Omega_{M0} \sim 0.3$ — $\Omega_{M0}\,(\text{baryon}) \sim 0.03$; $\Omega_{M0}\,(\text{Dark matter}) \sim 0.27$

$\Omega_{\Lambda 0} \sim 0.7$

$\Omega_k \lesssim 0.01$

Note there're large errors on each number!

The value Ω_{M0} is mostly estimated by considering gravitational effects but it is incompatible with the estimate of the contribution of baryonic matter solely $\rightarrow$ hypothesis of "Dark Matter" (non-baryonic matter).

Note the equation above clearly indicate the contribution to the Universe's history by each density parameter.

The history of the Universe is determined by the relative influence of the density Ω's :

$$\Omega_\Lambda \propto \Omega_K a^2 \propto \Omega_M a^3 \propto \Omega_R a^4$$

Where the curvature parameter is fixed by :

$$\Omega_K = 1 - \Omega_R - \Omega_M - \Omega_\Lambda \ .$$

The <u>future</u>, in particular, is determined by Ω_Λ.

if $\Omega_\Lambda < 0$ (vacuum energy is negative) : deceleration and collapse

if $\Omega_\Lambda \geq 0$: expansion, unless the value of Ω_M is sufficiently large to halt the expansion before the Ω_Λ term takes over.

The various possibilities can be studied using the Friedmann equation :

- Neglect Ω_R
- Study the equation at turn around point $H=0$, which represent the collapse threshold $a(\text{collapse}) = a_\cdot$:

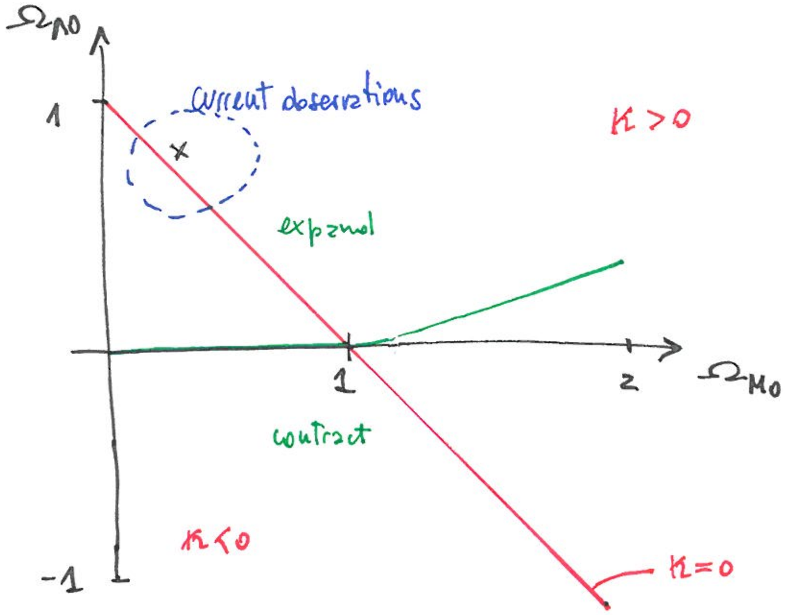
$$H^2 = 0 = \Omega_{\Lambda 0} + \Omega_{K0}\, a_\cdot^{-2} + \Omega_{M0}\, a_\cdot^{-3}$$

or:

$$\Omega_{\Lambda 0}\, a_\cdot^3 + (1 - \Omega_{M0} - \Omega_{\Lambda 0})\, a_\cdot + \Omega_{M0} = 0$$

Cubic equation for $a_\cdot$ → look for <u>real</u> solutions given values of Ω_{M0} and $\Omega_{\Lambda 0}$.
One obtains that such solutions are allowed for :

$$\Omega_{\Lambda 0} \geq \begin{cases} 0 & 0 \leq \Omega_{M0} \leq 1 \\ 4\Omega_{M0} \cos^3\left[\frac{1}{3}\arccos\left(\frac{1-\Omega_{M0}}{\Omega_{M0}}\right) + \frac{4\pi}{3}\right] & \Omega_{M0} > 1 \end{cases}$$



- Above the green line Universe expands ;
- Below the green line Universe contract ;
- The red line marks $k=0$; at the right / left one has $k>0$ / $k<0$.

INFLATION (ideas)

Two main problems arise confronting FRW models with observations:

1) We observe $\Omega - 1 = \frac{K}{H_0^2 a_0^2} \sim 0$ but expect $\Omega \gg 1$.

Considering :

$$\rho_M \sim a^{-3}$$
$$\rho_R \sim a^{-4}$$

and no vacuum energy ($\rho_V = 0$), one would expect that the Friedmann equation r.h.s.

$$H^2 = \frac{8\pi}{3}(\rho_M + \rho_R) - \frac{K}{a^2}$$

to be dominated by the curvature term :

$$\frac{K a^{-2}}{\frac{8\pi}{3}(\rho_M + \rho_R)} \sim \frac{K a^{-2}}{a^{-3}} \gg 1 \quad \text{for} \quad \begin{cases} K \neq 0 \\ a \text{ expanding} \end{cases}$$

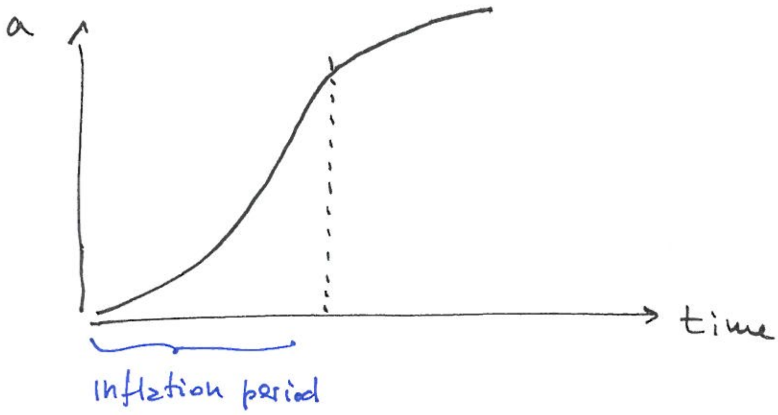
Alternatively, $\Omega \sim 1$ is an unstable point in matter-radiation dominated Universes : any deviation (small) will grow further.

2) CMB data are isotropic to a very high degree of accuracy.
The natural explanation is that the radiation in the whole universe had the possibility of interact and thermalize during the recombination era.
However, this is incompatible with FRW and the presence of particle horizon during the recombination era.

These problems are know as
- FLATNESS PROBLEM
- HORIZON PROBLEM

A solution to both problems is provided by the INFLATION HYPOTHESIS and inflationary models:

Assume the early times dynamics of the Universe was characterized by a fast expansion ($\ddot{a} > 0$):



[Guth 1980, + Linde, Albrecht, Steinhardt]

Inflation must be given by some field other than matter and radiation that operated at early time (→ scalar fields, vacuum energy, ... etc)

1) An "early" source of matter that dominate the expansion: $\rho_\phi \sim a$ that grows faster than $-Ka^{-2}$ drives $\Omega \to 1$, i.e. to the current energy level ...

2) A rapid expansion would allow regions initially close to "spread out" at large distance by keeping the same conditions of matter ...

Note that inflation would also provide a mechanism to "seed" the structures observed today in the Universe.
Quantum fluctuations during inflation would generate and be consistent with CMB anisotropies.