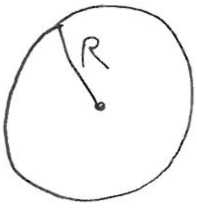
DIFFERENTIAL GEOMETRY (cont.)

Intuitive definitions of <u>curvature</u> can be given considering curves in $\mathbb{R}^2$:

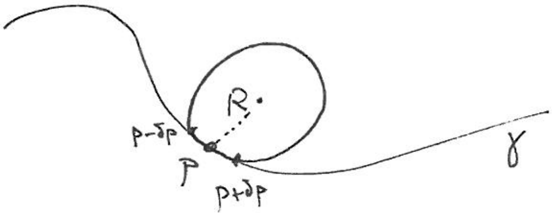
(i) ________ a straight line : curvature $= K \equiv 0$



a circle of radius R : $K = \text{const} \equiv \frac{1}{R}$

a generic curve : – take the osculating circle at p

– assign $K(p) \equiv \frac{1}{R}$

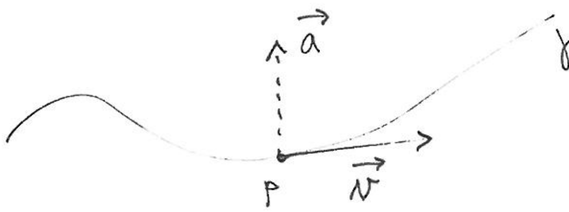


(ii) Physicist way to define K would be:

– consider the tangent vector $\vec{v}$ to the curve

– calculate the acceleration $\vec{a} = \frac{d\vec{v}}{dt}$

– define : $K \equiv |\vec{a}| = \left|\frac{d\vec{v}}{dt}\right|$.



K tells "how fast" the vector $\vec{v}$ rotates from point to point.

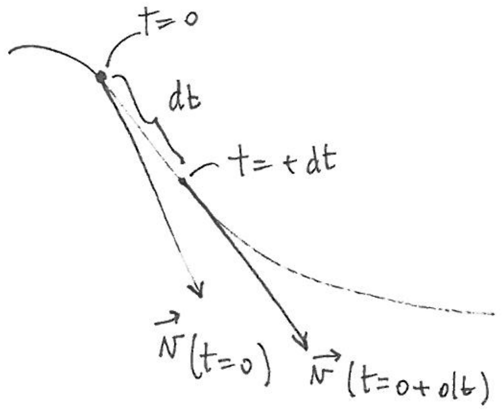
<u>Remark</u>

the two methods are basically the same :

(i) → $K = \frac{1}{R} = \frac{2\pi}{\text{circumference}} = \frac{\text{arc angle}}{\text{length of the curve}}$.

(ii) → $K = \left|\frac{d\vec{v}}{dt}\right| = \frac{d\theta}{dt} = \frac{\text{infinitesimal arc/angle of rotation}}{\text{length of the infinitesimal curve element}}$

for unit vector



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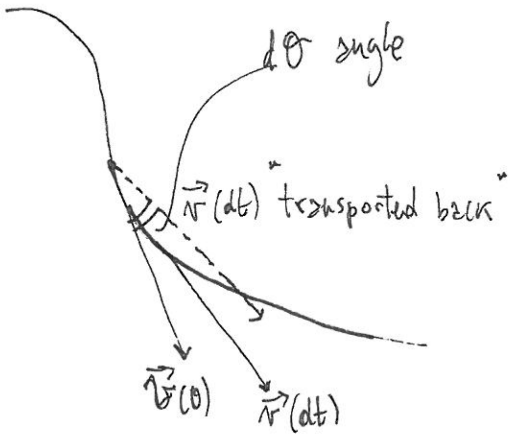
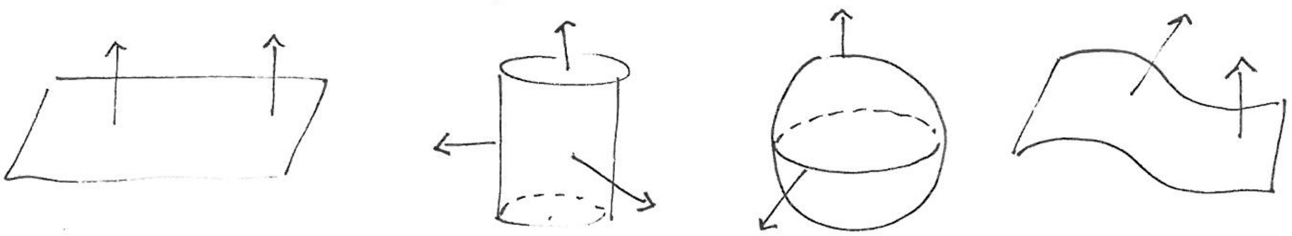


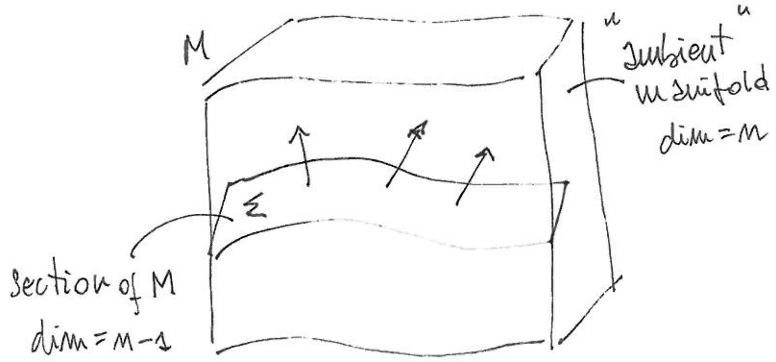
<u>IMPORTANT</u>: to define $d\vec{v}$ we have "transported back" (rigidly moved) the vector $\vec{v}(dt)$ to the point $t=0$.

Let us consider surfaces in 2D:



- One can think of <u>embedding</u> those surfaces in $\mathbb{R}^3$ and study theirs <u>normal vectors</u>. However we do not want to assume a manifold of dimension $m+1$ to study the curvature of a manifold of dim $= m$.

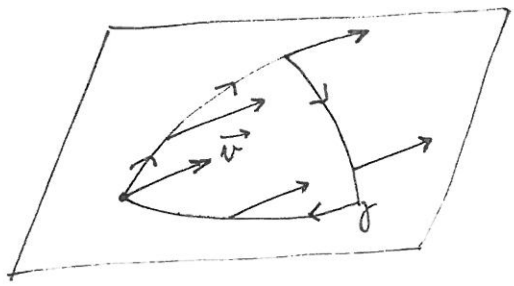
- Given a manifold of dim $= m$, another problem is to define the curvature of a <u>section</u> of dim $= m-1$. Here the embedding would be given, but still one would be caracteriting the curvature of the surface in relation to ambient space. This approach lead to the concept of EXTRINSIC CURVATURE.



EXTRINSIC CURVATURE ≈ Bending of Σ in M.

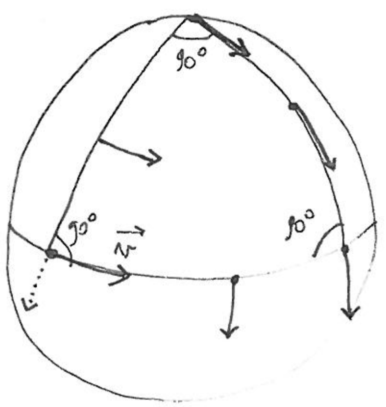
- Yet another problem is the definition of INTRINSIC CURVATURE.
  It can be illustrated with the following example:

  - Take a plane and a closed curve in it:



if one transport vector $\vec{v}$ rigidly around the curve ($\vec{v}$ remains parallel to the previous move), then one obtains the same vector after a loop.

  - Take a sphere and a closed curve, say made of arcs of 90°:



if one transport vector $\vec{v}$ rigidly around now, the "initial" and "final" vectors are rotated (different).

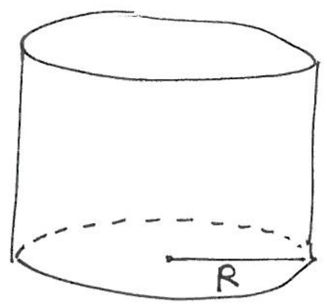
Moreover: a different path would give a different result!

<u>Conclusion</u>: Intrinsic curvature can be "detected" by parallel transporting vectors. In a curved manifold the parallel transport of vectors depends on the path.

<u>Difficulties</u>: on a generic manifold tangent vector spaces at different points are different. One cannot compare $v \in T_pM$ with $u \in T_qM$ if $p \neq q$.
Need to introduce a method to parallel transport vectors on M.

Example : Extrinsic vs. Intrinsic curvature of a cylinder

Take a cylinder immerse in $\mathbb{R}^3$:

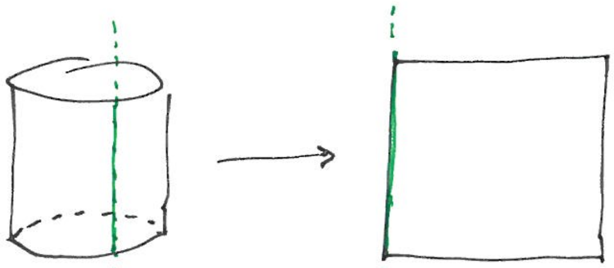


The cylinder is "round" $\Rightarrow$

in relation to the flat $\mathbb{R}^3$ manifold (embedding)

we could assign <u>extrinsic curvature</u> $K = \frac{1}{R}$.

Consider now the cylinder as a 2D surface; it can be "unfolded":



$g = R^2 d\varphi^2 + dz^2$

$g = dy^2 + dz^2$

$y \equiv R\varphi$

- parallel lines remain parallel
- All Euclid axioms are valid
- The only difference w.r.t. $\mathbb{R}^2$ is that "walking" in direction y one would return to the same point (topology).

In this case we assign <u>intrinsic curvature</u> $K = 0$.

<u>Conclusion</u> :

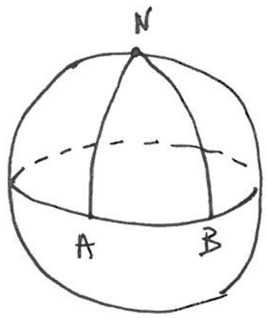
Intrinsic curvature $\longrightarrow$ refers to points and paths that remain in the manifold.

extrinsic curvature $\longrightarrow$ compares paths on the manifold with those in the (higher dimensional) "ambient" manifold.

<u>Remark</u> :

Parallel lines on a sphere meet !

$\rightarrow$ intrinsic curvature is <u>not</u> 0 .



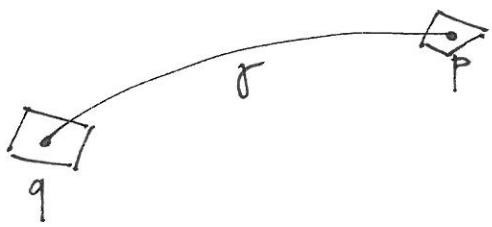
How to define curvature on a manifold?

Observations:

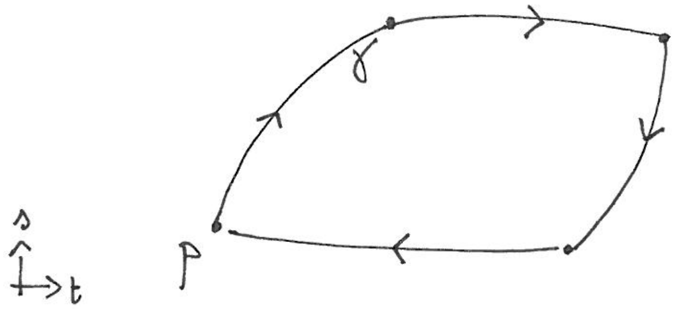
- If we know how to parallel transport a vector v along a curve γ $\Longrightarrow$ Then we can compare vectors of the same T_pM (at the same point) and in particular we can calculate the derivative $\nabla v|_p$

- If we know how to compute $\nabla v|_p$ $\Rightarrow$ Then we can define the parallel transport of v along γ as $\nabla v|_\gamma = 0$

- A derivative operator ∇ and the concept of parallel transport allow us to identify, or CONNECT, the tangent spaces T_pM and T_qM:



- Curvature can be then defined from the parallel transport of a vector along an infinitesimal closed curve:

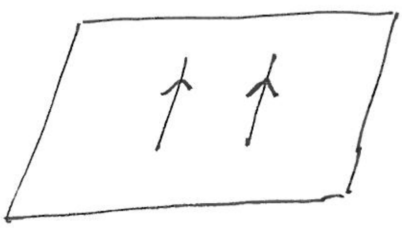


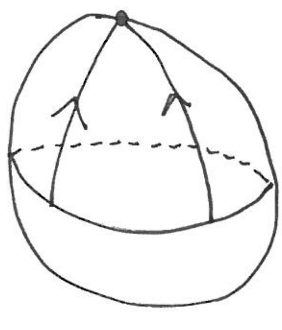
In particular, non-zero intrinsic curvature will correspond to the case where two successive differentiation of v do not commute:

$$[\nabla, \nabla]\, v \neq 0 \quad \Rightarrow \quad K \neq 0$$

Procedure / key points :

- Define ∇ : covariant derivative of tensors ;
- Define parallel transport of a vector v along curve γ with tangent vector t as $t^a \nabla_a v^b = 0$;
- A metric determines a unique ∇, which is called a connection since it defines a map between vectors of T_pM and those at a point infinitesimally close to p.
- the commutator $[\nabla, \nabla]$ defines the Riemann tensor that encodes and quantifies the idea of curvature as "failure of vectors and tensors field to return to their original values after being transported along an infinitesimal loop".
- Geodesics can be also defined as curves whose tangent vector is parallel propagated along itself. They correspond to curves that extremize the length between 2 points.
- The geodesic deviation equation establish that the acceleration between two nearby geodesics is zero if the Riemann tensor (curvature) is zero. This correspond to the intuitive notion that, in absence of curvature, "lines that are straight remain parallel", and, in presence of curvature, "they focus".





COVARIANT DERIVATIVE or CONNECTION

$\nabla : \mathcal{T}(k,l) \to \mathcal{T}(k,l+1)$ is a covariant derivative if :

1. Linear : $\nabla(\alpha A + \beta B) = \alpha \nabla A + \beta \nabla B$ $\quad \alpha,\beta \in \mathbb{R} \quad A,B \in \mathcal{T}(k,l)$
2. Leibnitz : $\nabla(A \otimes B) = \nabla A \otimes B + A \otimes \nabla B$
3. Commute with contraction : $\nabla(CA) = C \nabla A$

4a. Consistent with derivative of functions : $\nabla f \equiv df = \mathrm{grad}(f)$ $\quad f \in \mathcal{F}$

4b. Consistent with concept of tangent vector : $v(f) \equiv v^a \nabla_a f$ $\quad v \in T_pM$

5. Torsion free : $[\nabla, \nabla] f = 0$

Action of ∇ in abstract notation and some properties :

- $T^{a_1 \cdots a_k}{}_{b_1 \cdots b_l} \longrightarrow \nabla_c T^{a_1 \cdots a_k}{}_{b_1 \cdots b_l}$
- (2.) : $\nabla_c \left(A^{a_1 \cdots a_k}{}_{b_1 \cdots b_l} B^{a_1 \cdots a_{k'}}{}_{b_1 \cdots b_{l'}} \right) = \nabla_c A^{a_1 \cdots a_k}{}_{b_1 \cdots b_l} \cdot B^{a_1 \cdots a_{k'}}{}_{b_1 \cdots b_{l'}} + A^{a_1 \cdots a_k}{}_{b_1 \cdots b_l} \nabla_c B^{a_1 \cdots a_{k'}}{}_{b_1 \cdots b_{l'}}$
- (3.) : $\nabla_d \left(A^{a_1 \cdots c \cdots a_k}{}_{b_1 \cdots c \cdots b_l} \right) = \nabla_d A^{a_1 \cdots c \cdots a_k}{}_{b_1 \cdots c \cdots b_l}$
- Note that ∇_c is not a dual vector!
- (5.) $\Rightarrow$ the commutator of two vectors can be expressed as :

$v, u \in T_pM$

$$
\begin{aligned}
[v,u](f) &= v(u(f)) - u(v(f)) = \\
&= v(u^a \nabla_a f) - u(v^b \nabla_b f) = \\
&= v^d \nabla_d (u^a \nabla_a f) - u^c \nabla_c (v^b \nabla_b f) = \\
&= v^d \nabla_d u^a \nabla_a f + v^d u^a \nabla_d \nabla_a f - u^c \nabla_c v^b \nabla_b f - u^c v^b \nabla_c \nabla_b f = \\
&= \underbrace{v^d \nabla_d}_{\substack{d \text{ is "mute"} \\ a \to c}} u^a \nabla_a f - u^c \nabla_c \underbrace{v^b \nabla_b}_{\substack{b \text{ is "mute"} \\ b \to a}} f + v^d u^a \nabla_d \nabla_a f - u^c v^b \nabla_c \nabla_b f =
\end{aligned}
$$

$$= \left(v^b \nabla_b u^a - u^b \nabla_b v^a \right) \nabla_a f - \underbrace{[\nabla, \nabla] f}_{= 0 \ (5.)}$$

Main steps to define a unique connection :

(i) In a coordinate system the partial derivative ∂_a satisfy prop. 1.–5. but the resulting object ∂T is not a tensor (coordinate dependent).

(ii) The difference between two cov. derivatives $(\nabla - \tilde{\nabla}) \in \mathcal{T}(1,2)$ is a tensor.

(iii) Given a metric, there is a unique cov. derivative compatible with the metric, i.e. that satisfies $\nabla g = 0$.

(iv) The unique, metric compatible, covariant derivative can be constructed from the partial derivatives and the Christoffel symbols (or connection coefficients).

(i) Partial derivatives in coord basis

Introduce a coordinate system and $\{\partial_\mu\}$ as basis of T_pM.

Call $T^{\mu_1 \cdots \mu_k}{}_{\nu_1 \cdots \nu_\ell}$ the components of $T^{a_1 \cdots a_k}{}_{b_1 \cdots b_\ell} \in \mathcal{T}(k,\ell)$.

Consider:

$$\partial_\sigma T^{\mu_1 \cdots \mu_k}{}_{\nu_1 \cdots \nu_\ell}$$

– Properties 1.–5. follow from the properties of ∂ ...

– ... but they are not tensor components since they do not transform as such.

Example:

$$\partial_\mu v^\sigma \longrightarrow \partial_{\mu'} v^{\sigma'} = \frac{\partial x^\mu}{\partial x^{\mu'}} \partial_\mu \left(\frac{\partial x^{\sigma'}}{\partial x^\sigma} v^\sigma \right) = \underbrace{\frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^{\sigma'}}{\partial x^\sigma} \partial_\mu v^\sigma}_{\text{this term alone would be ok ...}} + \underbrace{\frac{\partial x^\mu}{\partial x^{\mu'}} v^\sigma \frac{\partial x^{\sigma'}}{\partial x^\mu \partial x^\sigma}}_{\text{... this is an "extra" term.}}$$

A possible approach to define ∇ is to introduce at this point a symbol with the 3 indexes $\Gamma^{\sigma'}_{\mu'\sigma}$ and search for an expression of the Γ's such that

$$\nabla_\mu v^\sigma = \partial_\mu v^\sigma + \Gamma^\sigma_{\mu\lambda} v^\lambda$$ transforms as a tensor and is unique.

We will basically follow this route but instead of proceeding with "direct" components calculations we will show (ii) and (iii) and get the result from those theorems [Wald].

(ii) Difference of covariant derivatives

Take 2 cov. derivatives ∇ and $\tilde{\nabla}$.

Property 4. $\Rightarrow$ on functions they are the same operator $\nabla f = \tilde{\nabla} f = df$.

Consider their action on dual vectors and specifically:

$$\tilde{\nabla}_a (f\omega_b) - \nabla_a (f\omega_b) = \tilde{\nabla}_a f \cdot \omega_b + f\,\tilde{\nabla}_a \omega_b - \nabla_a f \cdot \omega_b - f\,\nabla_a \omega_b =$$

$$= \cancel{(df)_a \omega_b} + f\,\tilde{\nabla}_a \omega_b - \cancel{(df)_a \omega_b} - f\,\nabla_a \omega_b =$$

$$= f\left(\tilde{\nabla}_a - \nabla_a\right)\omega_b .$$

the difference is

- linear in f
- depends only on objects at point p

$\Rightarrow$ it is a tensor of type $(1,2)$

$$\tilde{\nabla}_a \omega_b - \nabla_a \omega_b \equiv C^c{}_{ab}\,\omega_c \quad \text{with} \quad C^a{}_{ab} \in \mathcal{T}(1,2)$$

Note also that prop. 5. $\Rightarrow$

$$\underbrace{\nabla_a \nabla_b f}_{\text{symmetric}} = \underbrace{\tilde{\nabla}_a \tilde{\nabla}_b f}_{\text{symmetric}} - C^c{}_{ab}\,\nabla_c f \qquad (\omega_b = \nabla_b f)$$

$\Rightarrow$ $C^c{}_{ab} = C^c{}_{ba}$, must be symmetric in (ab).

Consider now the action of the difference on vectors:

$$\underset{\substack{\uparrow \\ w_b v^b \text{ is a function}}}{0 =} (\tilde{\nabla}_a - \nabla_a)(w_b v^b) \underset{\substack{\uparrow \\ 1., 2.}}{=} (\tilde{\nabla}_a - \nabla_a)(w_b)\, v^b + w_b\, (\tilde{\nabla}_a - \nabla_a)\, v^b =$$

$$= \underbrace{C^c{}_{ab}\, w_c v^b}_{\substack{c \to b \\ b \to d \\ = C^b{}_{ad} w_b v^d}} + w_b\, (\tilde{\nabla}_a - \nabla_a) v^b =$$

$$= w_b\, \left(C^b{}_{ad} v^b + (\tilde{\nabla}_a - \nabla_a) v^b \right) \qquad \forall\, w_b$$

$$\Rightarrow \qquad \nabla_a v^b = \tilde{\nabla}_a v^b + C^b{}_{ad} v^d \qquad \text{or} \qquad (\nabla_a - \tilde{\nabla}_a) v^b = C^b{}_{ad} v^d$$

For generic tensors one obtains:

$$\boxed{\nabla_c T^{a_1 \dots a_k}{}_{b_1 \dots b_l} = \tilde{\nabla}_c T^{a_1 \dots a_k}{}_{b_1 \dots b_l} + \sum_i C^{a_i}{}_{cd} T^{a_1 \dots d \dots a_k}{}_{b_1 \dots b_l} - \sum_j C^d{}_{c b_j} T^{a_1 \dots a_k}{}_{b_1 \dots d \dots b_l}}$$

- The difference between 2 cov. derivative is completely characterized by the tensor $C^a{}_{bc}$;
- Conversely, given $\tilde{\nabla}$ and a tensor field $C^a{}_{(bc)}$, the equation above defines a cov. der. ∇;
- Cov. derivative is in general <u>not</u> unique because it requires to specify $C^a{}_{(bc)}$ i.e. $\frac{n^2(n+1)}{2}$ independent components.
- More in general ∇ can be defined by specifying $\tilde{\nabla}$ and $C^a{}_{(bc)}$ and using the equation. For example, one could use $\tilde{\nabla} = \partial$ in a coordinate basis and appropriate choice for C. However one expect in that case that the "symbols with 3 indexes" appearing in the equation do not change as tensor components under coordinate transformation. The reason is that by changing the basis one changes also the choice for tensor $C^a{}_{(bc)}$:

$$\partial_\mu \to \partial_{\mu'} \qquad \underline{\text{and}} \qquad C^a{}_{(bc)} \to C'^a{}_{(bc)}$$

(iii) Levi-Civita connection

Given a metric g_{ab}, there is a unique covariant derivative that satisfies:

$$\nabla_a g_{bc} = 0 .$$

∇ is said compatible with g.

PROOF.

For any derivative operator $\tilde{\nabla}$ we construct the connection compatible with g.

$$0 = \nabla_a g_{bc} \underset{\underset{(ii)}{\uparrow}}{=} \tilde{\nabla}_a g_{bc} - C^d{}_{ab}\, g_{dc} - C^d{}_{ac}\, g_{bd}$$

$$\Rightarrow \quad \tilde{\nabla}_a g_{bc} = C^{d}{}_{ab}\, g_{dc} + C^{d}{}_{ac}\, g_{bd} \underset{\underset{\sum_d}{\uparrow}}{=} C_{cab} + C_{bac}$$

indexes substitution:

$$\begin{cases} \tilde{\nabla}_a g_{bc} = C_{cab} + C_{bac} & (abc) \\ \tilde{\nabla}_b g_{ac} = C_{cba} + C_{abc} & (bac) \\ \tilde{\nabla}_c g_{ab} = C_{bca} + C_{acb} & (cab) \end{cases}$$

$$(abc) + (bac) - (cab) = \tilde{\nabla}_a g_{bc} + \tilde{\nabla}_b g_{ac} - \tilde{\nabla}_c g_{ab} =$$

$$= C_{cab} + C_{bac} + \underbrace{C_{cba}}_{C_{cab}} + C_{abc} - \underbrace{C_{bca}}_{=C_{bac}} - \underbrace{C_{acb}}_{C_{abc}} =$$

$$= C_{cab} + \cancel{C_{bac}} + C_{cab} + \cancel{C_{abc}} - \cancel{C_{bac}} - \cancel{C_{abc}}$$

$$= 2\, C_{cab}$$

$$\Rightarrow \boxed{C^c{}_{ab} = g^{cd}\, C_{dab} = \frac{1}{2} g^{cd} \left(\tilde{\nabla}_a g_{bd} + \tilde{\nabla}_b g_{ad} - \tilde{\nabla}_d g_{ab} \right)}$$

This choice of $C^c{}_{ab}$ guarantees that $\nabla_a g_{bc} = 0$ for a given $\tilde{\nabla}$. □

(iv) Levi-Civita connection and Christoffel symbols

Given a metric on the manifold (M, g) the cov. derivative is defined by taking $\tilde{\nabla} = \partial$ as given by local coordinates and computing $C^a{}_{bc}$ from that derivative operator.

$$\Gamma^{\sigma}_{\mu\nu} \equiv \frac{1}{2} g^{\rho\sigma} \left(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu} \right)$$

are called <u>Christoffel symbols</u> and the cov. derivative of vectors, dual, tensor are:

$$\nabla_\mu v^\nu = \partial_\mu v^\nu + \Gamma^{\nu}_{\mu\sigma} v^\sigma$$

$$\nabla_\mu w_\nu = \partial_\mu w_\nu - \Gamma^{\lambda}_{\mu\nu} w_\lambda$$

$$\nabla_\sigma T^{\mu_1 \cdots \mu_k}{}_{\nu_1 \cdots \nu_l} = \partial_\sigma T^{\mu_1 \cdots \mu_k}{}_{\nu_1 \cdots \nu_l} + \sum_j \Gamma^{\mu_j}_{\sigma\lambda} T^{\mu_1 \cdots \lambda \cdots \mu_k}{}_{\nu_1 \cdots \nu_l} - \sum_i \Gamma^{\lambda}_{\sigma\nu_i} T^{\mu_1 \cdots \mu_k}{}_{\nu_1 \cdots \lambda \cdots \nu_l}$$

<u>Observations</u>

- Γ's are defined by assuming a particular derivative operator and coordinate basis. They do <u>not</u> transform as a tensor if one changes basis.

- Given the basis $e_{\mu_1} \otimes \cdots \otimes e_{\mu_k} \otimes e^{*\nu_1} \otimes \cdots \otimes e^{*\nu_l}$ of $\tau(k,l)$, the cov. derivative is expressed as:

$$\nabla T = \underbrace{\nabla_{\sigma} T^{\mu_1 \cdots \mu_k}{}_{\nu_1 \cdots \nu_l}}_{\text{components}} \underbrace{e_{\mu_1} \otimes \cdots \otimes e_{\mu_k} \otimes e^{*\nu_1} \otimes \cdots \otimes e^{*\nu_l} \underbrace{\otimes e^{*\sigma}}_{\text{last position!}}}_{\text{basis}}$$

Alternative notation for the components:

$$\nabla_\sigma T^{\mu_1 \cdots \mu_k}{}_{\nu_1 \cdots \nu_l} \longleftrightarrow T^{\mu_1 \cdots \mu_k}{}_{\nu_1 \cdots \nu_l ; \sigma}$$

PARALLEL TRANSPORT

Consider a cov. derivative ∇ and a curve γ with tangent vector t^a,

Def: the vector field v^a is parallel transported along γ $\iff$ $\boxed{t^a \nabla_a v^b = 0}$

Similarly, the parallel transport along γ of any tensor is

$$t^a \nabla_a T^{b_1 \dots b_k}{}_{c_1 \dots c_l} = 0 .$$

Observations

- Parallel transport depends only on the values of v on γ (do not need a "full" field)
- In a coord. system :

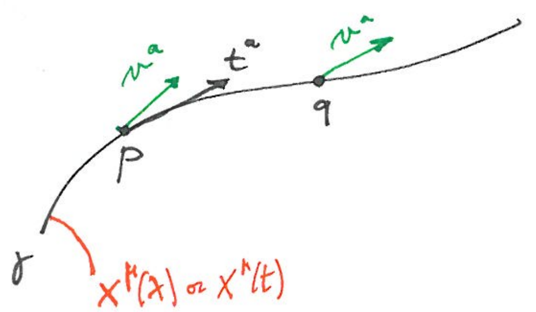
$$0 = t^a \nabla_a v^b \rightarrow \quad t^\mu \partial_\mu v^\alpha + t^\mu \Gamma^\alpha_{\mu\sigma} v^\sigma = 0$$

$$\frac{dv^\alpha}{dt} + t^\mu \Gamma^\alpha_{\mu\sigma} v^\sigma = 0$$

the latter equation is a ODE system $\Rightarrow$ $\exists!$ solution locally given a value $v^\mu(t=0)$.

$\Rightarrow$ a vector at a point p $(t=0)$ defines uniquely the parallel transported vector along γ.

$\Rightarrow$ given γ and ∇, the tangent space T_pM can be connected to T_qM with $q \in \gamma$



Geometrical meaning of connection compatible with metric

If we additionally have a metric g, the intuitive notion of "vector rigidly transported" or "kept parallel" can be implemented by requiring :

$$g(v,u) = g_{ab} v^a u^b = \text{constant along } \gamma \qquad \left(g(v,u) = \text{"angle between } v \text{ and } u\text{"} \right)$$

for v^a and u^a parallel transported.

We have then :

$$0 = t^a \nabla_a (g_{cb} v^c u^b) = t^a v^c u^b \nabla_a g_{cb} + g_{cb} \underbrace{t^a \nabla_a v^c}_{=0} \cdot u^b + g_{cb} \underbrace{t^a \nabla_a u^b}_{=0} \cdot v^c$$

$$= t^a v^c u^b \nabla_a g_{cb} \qquad \forall\, v,u \text{ parallel transported}, \forall \gamma$$

$\rightarrow$ $\nabla_a g_{cb} = 0$ is a natural condition to impose.

"Measure" of non-commutation of covariant derivatives

Given: ∇ cov. derivative
ω dual vector field
f function

Consider:
$$\nabla_a\nabla_b(f\omega_c) = \nabla_a(f\nabla_b\omega_c) + \nabla_a(\omega_c\nabla_b f) =$$
$$= \nabla_a\omega_c\nabla_b f + \omega_c\nabla_a\nabla_b f + \nabla_a f\nabla_b\omega_c + f\nabla_a\nabla_b\omega_c$$

and the difference: $(\nabla_a\nabla_b - \nabla_b\nabla_a)(f\omega_c) = f(\nabla_a\nabla_b - \nabla_b\nabla_a)\omega_c$

$\Rightarrow$ $\nabla_a\nabla_b - \nabla_b\nabla_a$ maps T_p^*M to tensor $\tau(0,3)$ linearly, i.e.

$\exists$ a tensor $(1,3)$: $(\nabla_a\nabla_b - \nabla_b\nabla_a)\omega_c \equiv R_{abc}{}^d\omega_d$ Riemann tensor

With a similar calculation as the one for the cov. derivative one finds:

$$0 = (\nabla_a\nabla_b - \nabla_b\nabla_a)(t^c\omega_c) \Rightarrow \nabla_a\nabla_b t^c - \nabla_b\nabla_a t^c = -R_{abd}{}^c t^d$$

and in general:

$$\boxed{(\nabla_a\nabla_b - \nabla_b\nabla_a)T^{c_1\dots c_k}{}_{d_1\dots d_l} = -\sum_{i=1}^{k} R_{abe}{}^{c_i} T^{c_1\dots e\dots c_k}{}_{d_1\dots d_l} + \sum_{j=1}^{l} R_{abd_j}{}^{e} T^{c_1\dots c_k}{}_{d_1\dots e\dots d_l}}$$

Properties of Riemann

1.– $R_{abc}{}^d = -R_{bac}{}^d$ antisymmetric in index 1 and 2.

Follows from definition: $[\nabla_a,\nabla_b]\omega_c = R_{abc}{}^d\omega_d$

2.– $R_{[abc]}{}^d = 0$

Direct proof: $R_{[abc]}{}^d\omega_d = \nabla_{[a}\nabla_b\omega_{c]} - \nabla_{[b}\nabla_a\omega_{c]} =$

$$= 2\nabla_{[a}\nabla_b\omega_{c]} = 2\partial_{[a}\partial_b\omega_{c]} = 2d^2\omega = 0 \quad \square$$

3.- $R_{abcd} = - R_{abdc}$ (if ∇ is compatible with g)

Direct proof: $\underset{\uparrow \atop \nabla g = 0}{0} = (\nabla_a \nabla_b - \nabla_b \nabla_a)\, g_{cd} \underset{\uparrow \atop \text{def. of } R_{abc}{}^{d}}{=} R_{abc}{}^{e} g_{ed} + R_{abd}{}^{e} g_{ce}$ ▫

4.- Bianchi identities $\boxed{\nabla_{[a} R_{bc]d}{}^{e} = 0}$

Proof:

$$(\nabla_a \nabla_b - \nabla_b \nabla_a)\, \underbrace{\nabla_c w_d}_{\text{consider this as object with 2 indexes ...}} = R_{abc}{}^{e} \nabla_e w_d + R_{abd}{}^{f} \nabla_c w_f$$

$$\nabla_a (\nabla_b \nabla_c w_d - \nabla_c \nabla_d w_d) = \nabla_a (R_{bcd}{}^{e} w_e) = w_e \nabla_a R_{bcd}{}^{e} + R_{bcd}{}^{e} \nabla_a w_e$$

by antisymmetrizing the 2 above equations in $[abc]$ → the LHS become equal; RHS :

$$\underbrace{R_{[abc]}{}^{e}}_{=0 \atop 2.} \nabla_e w_d + R_{[ab|d}{}^{f} \nabla_{c]} w_f = w_e \nabla_{[a} R_{bc]d}{}^{e} + R_{[bc|d}{}^{e} \nabla_{a]} w_e$$

these two cancel ▫

5.- $R_{abcd} = R_{cdab}$ symmetric in pairs of down-indexes [Exercise]

6.- In coordinate basis: [Exercise]

$$R_{\mu\nu\rho}{}^{\sigma} = \partial_\nu \Gamma^{\sigma}_{\mu\rho} - \partial_\mu \Gamma^{\sigma}_{\nu\rho} + \sum_\alpha \left(\Gamma^{\alpha}_{\mu\rho} \Gamma^{\sigma}_{\alpha\nu} - \Gamma^{\alpha}_{\nu\rho} \Gamma^{\sigma}_{\alpha\mu} \right)$$

i.e. " $R \sim \partial\Gamma - \partial\Gamma + \Gamma\Gamma - \Gamma\Gamma \sim \partial^2 g$ "

7. — Contractions of the Riemann

7a. — $R_{ac} \equiv R_{abc}{}^{b}$ <u>Ricci tensor</u> (0,2)

- $R_{ac} = R_{ca}$, symmetric

7b. — $R \equiv g^{ac} R_{ac} = R^{a}{}_{a}$ <u>Ricci scalar</u>

7c. — <u>Contracted Bianchi identities</u>

4. → contract "a" with "e" and use symmetries: [Exercise]

$$\nabla_a R_{bcd}{}^{a} + \nabla_b R_{cd} - \nabla_c R_{bd} = 0$$

raise "d" with metric:

$$\nabla_a R_{bc}{}^{da} + \nabla_b R_c{}^{d} - \nabla_c R_b{}^{d} = 0$$

contract "b" with "d":

$$\nabla_a R_c{}^{a} + \nabla_b R_c{}^{b} - \nabla_c R_b{}^{b} = 0$$

$$2 \nabla_a R_c{}^{a} - \nabla_c R = 0$$

$$\nabla_a R_c^{a} - \frac{1}{2} \nabla_c R = 0$$

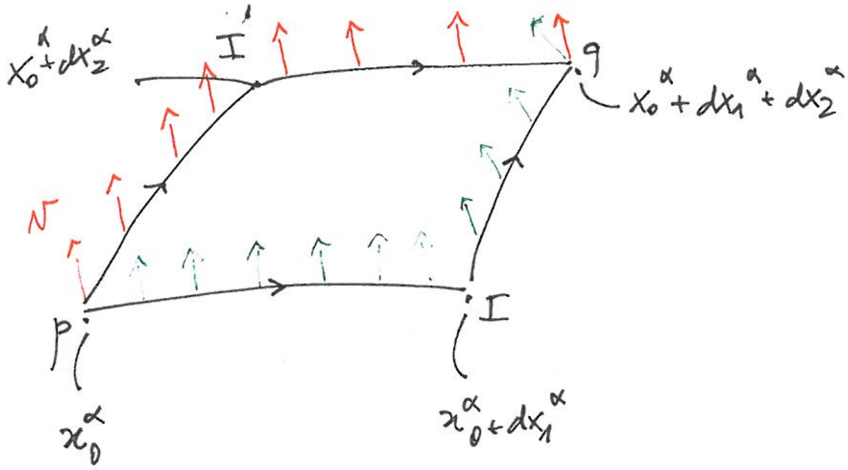
$$\nabla^a \underbrace{\left(R_{ca} - \frac{1}{2} g_{ca} R \right)}_{\equiv G_{ca}} = 0$$

$\equiv G_{ca}$ <u>Einstein tensor</u>

Riemann tensor as "measure" of the deviation for parallel transported vectors

Consider the parallel transport of a vector from p to $q = p + \delta p$ along two paths:

$$p \to I \to q$$
$$p \to I' \to q$$



Parallel transport equation:
$$t^\mu \partial_\mu v^\alpha + t^\mu \Gamma^\alpha_{\mu\beta} v^\beta = 0$$
$$\to \quad dx^\mu \partial_\mu v^\alpha + dx^\mu \Gamma^\alpha_{\mu\beta} v^\beta = 0$$
$$\boxed{dx^\mu \partial_\mu v^\alpha = - \Gamma^\alpha_{\mu\beta}\, dx^\mu v^\beta}$$

For the infinitesimal step we write:

$$v^\alpha(I) = v^\alpha(p) - \Gamma^\alpha_{\mu\beta} v^\beta(p)\, dx_1^\mu \qquad \text{with} \quad \Gamma^\alpha_{\mu\beta} = \Gamma^\alpha_{\mu\beta}(p)$$

and similarly:

$$v^\alpha(q) = v^\alpha(I) - \Gamma^\alpha_{\mu\beta} v^\beta(I)\, dx_2^\beta \qquad \text{with} \quad \Gamma^\alpha_{\mu\beta} = \Gamma^\alpha_{\mu\beta}(I)$$

where:

$$\Gamma^\alpha_{\mu\beta}(I) \simeq \Gamma^\alpha_{\mu\beta}(p) + \frac{\partial \Gamma^\alpha_{\mu\beta}}{\partial x^\nu}(p)\, dx_1^\nu + \mathcal{O}[(dx_1)^2].$$

For the path $p \to I \to q$ (omit indexes ...):

$$v(q) \simeq v(p) - \Gamma(p) v(p)\, dx_1 - \Gamma(I)\left[v(p) - \Gamma(p) v(p)\, dx_1\right] dx_2 \qquad \leftarrow v(I) = v(p) - \Gamma v\, dx_1$$

$$= v(p) - \Gamma(p) v(p)\, dx_1 \underbrace{- \Gamma(I) v(p)}\, dx_2 + \underbrace{\Gamma(I)}\Gamma(p) v(p)\, dx_1 dx_2 \qquad \leftarrow \Gamma(I) = \Gamma(p) + \partial\Gamma\, dx_1$$

$$= v(p) - \Gamma(p) v(p)\, dx_1 - \left(\Gamma(p) + \partial\Gamma(p)\, dx_1\right) v(p)\, dx_2 + \left(\Gamma(p) + \partial\Gamma(p)\, dx_1\right)\Gamma(p) v(p)\, dx_1 dx_2 =$$

= ... discarding terms $\mathcal{O}(dx_1^2)$ and putting the indexes ... =

$$v^\alpha(q) = v^\alpha(p) - \Gamma^\alpha_{\beta\mu} v^\beta(p)\, dx_1^\mu - \Gamma^\alpha_{\beta\nu} v^\beta(p)\, dx_2^\nu + \Gamma^\alpha_{\beta\nu} \Gamma^\beta_{\sigma\boxed{\mu}} v^\sigma(p)\, dx_1^{\boxed{\mu}} dx_2^{\nu} - \frac{\partial \Gamma^\alpha_{\beta\nu}}{\partial x^{\boxed{\mu}}} v^\beta(p)\, dx_1^{\boxed{\mu}} dx_2^\nu$$

and similarly for path $p \to I' \to q$:

$$v'^\alpha(q) = v^\alpha(p) - \Gamma^\alpha_{\beta\mu} v^\beta(p)\, dx_2^\mu - \Gamma^\alpha_{\beta\nu} v^\beta(p)\, dx_2^\nu + \Gamma^\alpha_{\beta\boxed{\mu}} \Gamma^\beta_{\sigma\nu} v^\sigma(p)\, dx_1^{\boxed{\mu}} dx_2^{\nu} - \frac{\partial \Gamma^\alpha_{\beta\mu}}{\partial x^{\boxed{\nu}}} v^\beta(p)\, dx_1^\mu dx_2^{\boxed{\nu}}$$

Taking the difference:

$v'^\alpha(q) - v^\alpha(q) =$ ... terms $\propto dx_1^\mu$ or dx_2^ν cancel each others!

... terms $\propto dx_1^\mu dx_2^\nu$ do not ! $\to$ different indexes are contracted :

$$= \underbrace{\left[\frac{\partial \Gamma^\alpha_{\beta\nu}}{\partial x^\mu} v^\beta(p) - \frac{\partial \Gamma^\alpha_{\beta\mu}}{\partial x^\nu} v^\beta(p) + \Gamma^\alpha_{\beta\mu} \Gamma^\beta_{\sigma\nu} v^\sigma(p) - \Gamma^\alpha_{\beta\nu} \Gamma^\beta_{\sigma\mu} v^\sigma(p) \right]}_{\partial\Gamma - \partial\Gamma + \Gamma\Gamma - \Gamma\Gamma \;\to\; \text{Riemann !!!}} dx_1^\mu dx_2^\nu$$

- Parallel transport is path independent at first order in dx

- Curvature emerge at second order in dx :

$$v'^\alpha(q) - v^\alpha(q) = R^\alpha_{\;\mu\nu\sigma} v^\sigma dx_1^\mu dx_2^\nu$$

Flat spacetime $\longleftrightarrow$ $R_{abc}^{\;\;\;\;d} = 0$

GEODESICS

Intuitive idea : "lines as straight as possible"
"lines of minimal length"

Def (1): curves whose tangent vector maintains its direction

$$\gamma : \quad t^a \nabla_a t^b = \alpha\, t^b$$

If a curve is parametrized by λ, one can always change the parametrization in order to have $\alpha = 0$. (AFFINE PARAMETRIZATION)

Using an affine parameter :

Def (2): geodesic = curve whose tangent vector is parallel propagated along itself

$$\gamma : \quad \boxed{t^a \nabla_a t^b = 0}$$

In a coord. basis :

$$\frac{dt^\mu}{d\lambda} + \Gamma^\mu_{\sigma\nu} t^\sigma t^\nu \underset{\uparrow \atop t^\mu = \frac{dx^\mu}{d\lambda}}{=} \frac{d^2 x^\mu}{d\lambda^2} + \Gamma^\mu_{\sigma\nu} \frac{dx^\sigma}{d\lambda}\frac{dx^\nu}{d\lambda} = 0$$

Observation :

- System of 2^{nd} order ODEs $\Rightarrow$ exist locally unique solution given initial data
  $\Rightarrow \exists !$ (locally) a geodesic passing by p with tangent t^a

- Reparametrization : $\lambda = \lambda(s)$

$$\frac{d}{d\lambda} = \frac{ds}{d\lambda}\frac{d}{ds} \equiv f \frac{d}{ds}$$

$$\frac{dx^\mu}{d\lambda} \equiv \dot{x}^\mu = f \frac{dx^\mu}{ds} \equiv x^{\mu\prime}$$

$$\frac{d}{d\lambda}\dot{x} + \Gamma \dot{x}\dot{x} = 0 \qquad \text{(omitt indexes)}$$

$$f(fx')' + \Gamma f^2 x' x' = 0$$

$$f^2 x'' + f' x' + \Gamma f^2 x' x' = 0$$

$$\left.\begin{aligned} x'' + \Gamma x' x' = \underbrace{-\frac{f'}{f^2} x'}_{= \alpha x'} \end{aligned}\right\} \Rightarrow \alpha = -\frac{f'}{f^2}$$

$$\alpha = 0 \Rightarrow f' = \frac{d^2 s}{d\lambda^2} = 0 \Rightarrow s = a\lambda + b \qquad a, b \in \mathbb{R}$$

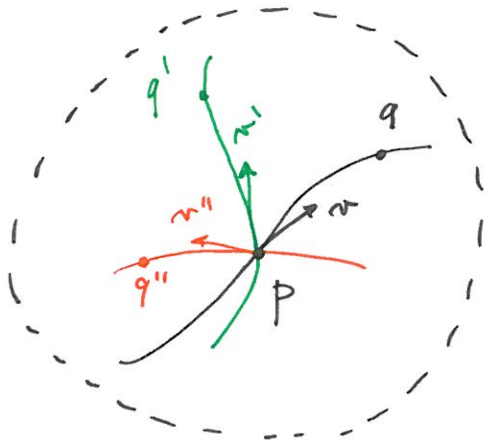
Riemann normal coordinates

Uniqueness of geodesics allow us to define special coordinates at a point p such that all geodesics by p are mapped to straight lines in $\mathbb{R}^m$.

Given $v \in T_pM \Rightarrow \exists!\ \gamma_v$ geodesic by p with tangent v

$\Rightarrow$ We can associate each point around p with a vector $v \in T_pM$ by considering the curve γ_v that connects p with the given point and the map:

$$\exp_p : T_pM \longrightarrow M$$
$$v \longrightarrow \exp_p(v) \equiv \gamma_v(\lambda = 1)$$



The Riemann coordinates of point q are <u>defined</u> as the components of the vector:

$$q = (v^1, v^2, \ldots, v^m) .$$

Observations

- In general, existence of $\exp_p$ is guaranteed only <u>locally</u> (local existence of geodesic)
- It can be proven that $\exp_p$ is locally 1-to-1 map.
- Point p has normal coordinates $p = (0, 0, \ldots, 0)$
- In normal coordinates the curve γ_v is represented as

$$\gamma_v : \quad x^\mu(\lambda) = (\lambda v^1, \lambda v^2, \ldots, \lambda v^m)$$

$\rightarrow$ <u>straight lines</u> in $\mathbb{R}^m$!

– Because those are straight lines, the geodesic equations must be:

$$0 = \frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu_{\rho\sigma}\frac{dx^\sigma}{d\lambda}\frac{dx^\rho}{d\lambda} = \frac{d^2x^\mu}{d\lambda^2} \Rightarrow \boxed{\Gamma^\mu_{\rho\sigma} = 0} \quad \underline{\underline{\text{at } p}}$$

(velocity $\frac{dx^\mu}{d\lambda}$ is arbitrary)

– Hence, in these coords the cov. derivative $\nabla = \partial$ and the metric compatibility implies:

$$\left.\begin{aligned} \nabla g &= \partial g + \Gamma g = 0 \\ \Gamma g &= 0 \end{aligned}\right\} \Rightarrow \boxed{\partial g_{\mu\nu} = 0} \quad \underline{\underline{\text{at } p}}$$

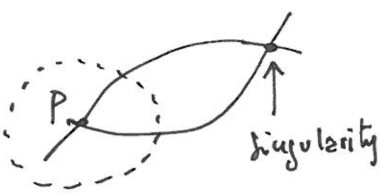
– In view of GR this suggests that one can always construct coordinates such that

$$g_{\mu\nu} = \eta_{\mu\nu} \, , \quad \underline{\underline{\text{at } p}}$$

and also:

$$g_{\mu\nu} \simeq \eta_{\mu\nu} + \underbrace{\frac{\partial^2 g}{\partial x \partial x}}_{\approx \text{ Riemann}} \delta x \, \delta x \quad , \quad \text{around } p$$

– Finally note, in general, far from p geodesics can cross:



in this case the manifold is said *geodesically incomplete*.

Geodesics extremise the length of a curve

g_{ab} metric with signature $- + + +$ $\quad (n=4)$

t^a vector tangent to γ

γ	$\|t\|^2 \equiv g(t,t)$	
spacelike	>0	$\ell \equiv \int \left(g_{ab} t^a t^b\right)^{1/2} d\lambda$, length
null	$=0$	$\ell = 0$
timelike	<0	$\tau \equiv \int \left(-g_{ab} t^a t^b\right)^{1/2} dt$, proper time

Observations:

- ℓ is not defined if γ changes "character" from spacelike to timelike.
- Geodesics in a Lorentz manifold cannot change from timelike to other types because, from the definition of parallel transport, the norm must be constant:

$$t^c \nabla_c \left(|t|^2\right) = t^c \nabla_c \left(g_{ab} t^a t^b\right) = 2 g_{ab} \underbrace{t^c \nabla_c t^b}_{=0} = 0 \quad \Rightarrow \quad |t|^2 = \text{const}.$$

- Proper length or time do not depend on the curve parametrization.

~

Consider a spacelike curve parametrized by $\lambda \in [a,b] \subset \mathbb{R}$: $|t|^2 = 1$.

Let us choose a chart x^μ and perform the variation of ℓ :

$$\ell = \int_a^b \left(g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu\right)^{1/2} d\lambda \quad , \qquad x^\mu \to x^\mu + \delta x^\mu$$

$$\delta l = \int_a^b \underbrace{\left(g_{\mu\nu} t^\mu t^\nu\right)}_{=|t|^2=1}^{-1/2} \cdot \left[\underbrace{g_{\alpha\beta} \dot{x}^\alpha \frac{d\,\delta x^\beta}{d\lambda}}_{\text{P.P.}} + \underbrace{\frac{1}{2} \frac{\partial g_{\alpha\beta}}{\partial x^\sigma} \delta x^\sigma \dot{x}^\alpha \dot{x}^\beta}_{\sigma\to\beta\ ;\ \beta\to\nu \ \text{(rename)}} \right] d\lambda =$$

$$= \int_a^b \left[\frac{d}{d\lambda}\left(g_{\alpha\beta} \dot{x}^\alpha \delta x^\beta\right) - \frac{d}{d\lambda}\left(g_{\alpha\beta}\dot{x}^\alpha\right)\delta x^\beta + \frac{1}{2}\frac{\partial g_{\alpha\nu}}{\partial x^\beta} \dot{x}^\alpha \dot{x}^\nu \delta x^\beta \right] d\lambda =$$

$$= \underbrace{\left[g_{\alpha\beta}\dot{x}^\alpha \delta x^\beta \right]_a^b}_{=0 \Leftarrow \delta x^\beta(a)=0=\delta x^\beta} - \int_a^b \left[\frac{d}{d\lambda}\left(g_{\alpha\beta}\dot{x}^\alpha\right)\delta x^\beta \ \frac{1}{2}\frac{\partial g_{\alpha\nu}}{\partial x^\beta} \dot{x}^\alpha \dot{x}^\nu \delta x^\beta \right] d\lambda =$$

$$= \int_a^b \left[-\frac{d}{d\lambda}\left(g_{\alpha\beta}\dot{x}^\alpha\right) + \frac{1}{2}\frac{\partial g_{\alpha\nu}}{\partial x^\beta}\dot{x}^\alpha \dot{x}^\nu \right] \delta x^\beta \, d\lambda$$

$$\delta l = 0 \quad \forall \ \delta x^\beta \quad \Rightarrow$$

$$\underbrace{- g_{\alpha\beta}\frac{d^2 x^\alpha}{d\lambda^2} - \frac{\partial g_{\alpha\beta}}{\partial x^\nu}\frac{dx^\nu}{d\lambda}\frac{dx^\alpha}{d\lambda} + \frac{1}{2}\frac{\partial g_{\alpha\nu}}{\partial x^\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\nu}{d\lambda}}_{\text{just another way of re-writing the geodesic eq!}} = 0$$

Conclusion: geodesics extremize l.

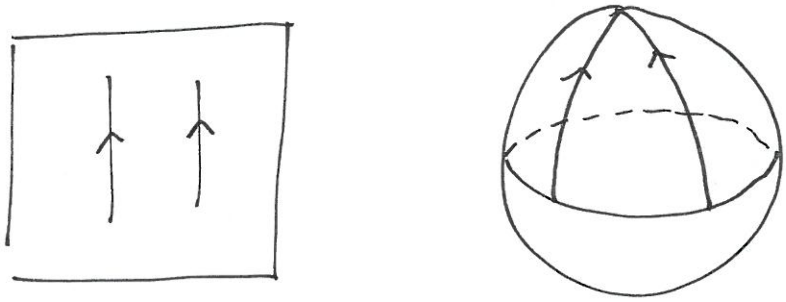
Observations:

- A similar calculation holds for timelike γ: geodesics $\leftrightarrow$ $\delta\tau = 0$
- Geodesic equation can be derived from the Lagrangian

$$L^2 \equiv g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu$$

Geodesic deviation

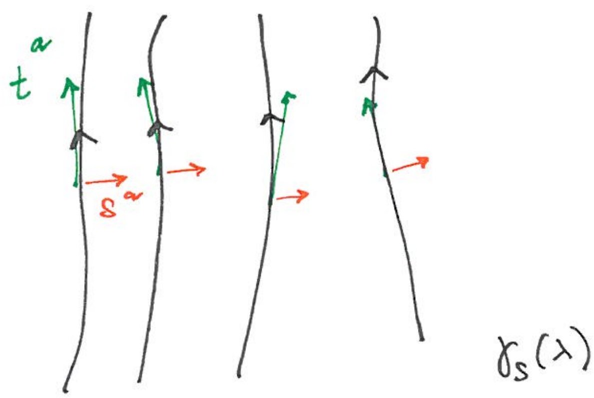
An intuitive effect of curvature is that geodesics focus:



Is this effect described by our formalism and captured by the Riemann tensor?

Consider a 1-parameter family of geodesics $\gamma_s(\lambda)$, where λ is the affine par. and $s \in \mathbb{R}$ control a <u>smooth</u> variation from one geodesic to the next.

Assume $\gamma_s(\lambda)$ do <u>not</u> cross. $\gamma_s(\lambda)$ is a 2D surface in an n-dimensional manifold:



Define the vector fields:

$$t^a \equiv \left(\frac{\partial}{\partial \lambda}\right)^a \quad \text{and} \quad s^a \equiv \left(\frac{\partial}{\partial s}\right)^a$$

such that

t^a: $\quad t^a \nabla_a t^b = 0 \quad$ is the tangent to γ

s^a: $\quad$ DEVIATION VECTOR, describes the displacement to an infinitesimally close curve $\gamma_{s+\delta s}$

t^a and S^a commute:

$$[t,s] = T^\mu \partial_\mu (S^\nu \partial_\nu) - S^\mu \partial_\mu (T^\nu \partial_\nu) =$$

$$= \frac{\partial x^\mu}{\partial \lambda} \partial_\mu \left(\frac{\partial x^\nu}{\partial s} \partial_\nu \right) - \frac{\partial x^\mu}{\partial s} \partial_\mu \left(\frac{\partial x^\nu}{\partial \lambda} \partial_\nu \right) =$$

$$= \cancel{\frac{\partial x^\mu}{\partial \lambda} \frac{\partial x^\nu}{\partial s} \partial_\mu \partial_\nu} + \frac{\partial x^\mu}{\partial \lambda} \underbrace{\frac{\partial}{\partial s}(\delta^\nu_\mu)}_{=0} \partial_\nu - \cancel{\frac{\partial x^\mu}{\partial s} \frac{\partial x^\nu}{\partial \lambda} \partial_\mu \partial_\nu} - \frac{\partial x^\mu}{\partial s} \underbrace{\frac{\partial}{\partial \lambda}(\delta^\nu_\mu)}_{=0} \partial_\nu = 0$$

but the commutator can be written:

$$0 = [t,s] = t^a \nabla_a s^b - s^a \nabla_a t^b$$

$$\rightarrow t^c \nabla_c (t^a s_a) = \underbrace{t^c \nabla_c t^a}_{=0} s_a + t^c t^a \nabla_c s_a$$

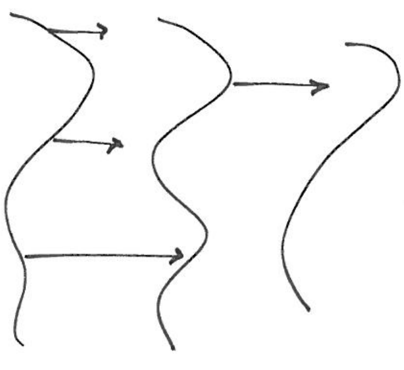
$$= t_a t^c \nabla_c s^a \underset{\substack{\uparrow \\ [t,s]=0}}{=} t_a s^c \nabla_c t^a = \frac{1}{2} s^c \nabla_c (\underbrace{t_a t^a}_{=-1}) = 0$$

$\rightarrow$ The quantity $t^a s_a$ = constant along the geodesics, and by suitable parametrization one can have $t^a s_a = 0$.

Def: $V^a \equiv t^b \nabla_b s^a$ "geodesics relative velocity"

rate of change of the deviation vector along γ_s

Def: $a^a \equiv t^b \nabla_b V^a$ "geodesics acceleration"



$$a^a = t^c \nabla_c V^a = t^c \nabla_c (t^b \nabla_b S^a) =$$

$$\underset{[t,s]=0}{=} t^c \nabla_c (S^b \nabla_b t^a) = \underbrace{t^c \nabla_c S^b}_{= S^c \nabla_c t^c} \cdot \nabla_b t^a + t^c s^b \underbrace{\nabla_c \nabla_b t^a}_{= \nabla_b \nabla_c t^a - R_{cbd}{}^a t^d} =$$

$$= \underbrace{S^c \nabla_c t \cdot \nabla_b t^a + t^c s^b \nabla_b \nabla_c t^a} - t^c s^b R_{cbd}{}^a t^d =$$

$$= S^c \nabla_c (\underbrace{t^b \nabla_b t^a}_{=0}) - R_{cbd}{}^a t^c s^b t^d$$

→ the acceleration of geodesics (between γ_s and $\gamma_{s+\delta s}$) is governed by the Riemann tensor and the <u>geodesic deviation equation</u>:

$$a^a = -R_{cbd}{}^a t^c s^b t^d$$

if Riemann $= 0$, then the acceleration is 0

<u>the Riemann tensor measures how geodesics bends and accelerate toward each other</u>.

Note the above equation can be written:

$$\underbrace{t^c \nabla_c (t^d \nabla_d S^a)} = - t^d t^c R_{cbd}{}^a S^b$$

$$= \nabla_t (\nabla_t S^a) =$$

$$= \nabla_t^2 S^a$$

Example : Tidal forces in Newtonian gravity

Newton equations of motion for two test bodies with trajectories nearby :

$$x^i(t)$$
$$x^i(t) + s^i(t)$$

are :

$$\ddot{x}^i = -(\partial_i \phi)\big|_{x(t)}$$

$$\ddot{x}^i + \ddot{s}^i = -(\partial_i \phi)\big|_{x(t)+s(t)} \simeq -(\partial_i \phi)_{x(t)} - \left(\frac{\partial}{\partial x^j}(\partial_i \phi)\right)_{x(t)} s^j + O(s^2)$$

The relative acceleration between the two bodies is given by :

$$\ddot{s}^i = - \underbrace{\frac{\partial \phi}{\partial x^j \partial x^i}}_{\text{Def: tidal tensor}} s^j$$

This equation is more than an analogy with the geodesic deviation :

in GR it is precisely the Newtonian limit !

We have thus the correspondence :

$$\text{Riemann} \longleftrightarrow \text{Tidal tensor}$$

Relative acceleration (tidal forces) cannot be transformed away in GR...