These semi-private notes are constructed from the following books:

- R.Wald, "General Relativity" University of Chicago Press, 1984
- S.M.Carrol, "Spacetime and Geometry, An Introduction to General Relativity", Addison-Wesley, 2003.
- B.F.Schutz, "A First Course in General Relativity", Cambridge University Press, 1985.

If you decide to use them to study or teach, please
(0) be careful and refer to the original books
(1) cite/refer to my website
(2) let me know and send feedbacks.

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sebastiano.bernuzzi@uni-jena.de

EINSTEIN EQUATIONS

Motivated by EEP and SEP we will assume that:

1. Spacetime is a manifold M ($\dim M = 4$) with a Lorentz metric g
2. Test bodies follow geodesics in M
3. In local Lorentz frames the non-gravitational laws of physics are those of SR
4. Gravity is described by a pure metric theory.

Problem: – Find the equations of motion for particles and fields
– Find the equations of motion for g.

EQUATIONS OF MOTION (EOM) FOR PARTICLES AND FIELDS

Spacetime in SR is $\mathbb{R}^4$ with the Minkowski metric η.
In presence of gravity we could consider the minimal substitution rule:

$$\text{SR} \qquad \begin{array}{ccc} \mathbb{R}^4 & \longrightarrow & M \\ \eta & \longrightarrow & g \\ \partial_a & \longrightarrow & \nabla_a \end{array} \qquad \text{GR}$$

- EOM for a free particle described by a 4-velocity $u^\mu = \frac{dx^\mu}{d\lambda}$

$$\frac{d^2 x^\mu}{d\lambda^2} = 0 \longrightarrow \text{geodesics equation:}$$

$$\frac{d^2 x^\mu}{d\lambda^2} + \Gamma^\mu_{\alpha\beta} \frac{dx^\alpha}{d\lambda} \frac{dx^\beta}{d\lambda} = 0$$

or:

$$u^\mu \nabla_\mu u^\alpha = 0$$

• <u>EOM for a particle under an external force f^α</u> :

$$\underbrace{u^\mu \nabla_\mu u^\alpha}_{\text{acceleration}} = \frac{1}{m} f^\alpha$$

$m \to$ particle's rest mass

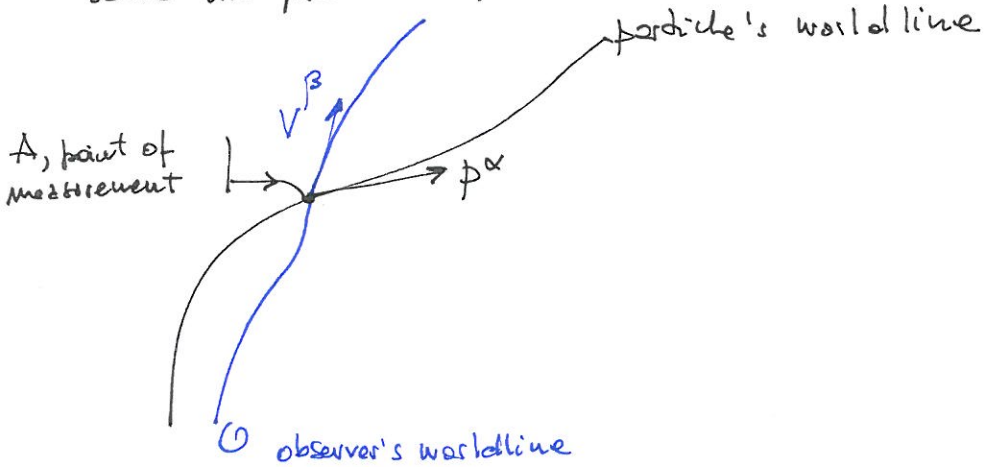
<u>observation</u> :

The 4-momentum of the particle is $p^\alpha = m u^\alpha$

the energy of the particle measured by an observer $\mathcal{O}$ with 4-velocity V^β is, exactly as in SR :

$$E = - p_\alpha V^\alpha ,$$

with the same interpretation:



but with a <u>key difference</u> :

SR) E is the energy measured at point A but <u>also</u> the energy measured by any other distant observer with 4-velocity V^β because we can parallel transport the vectors anywhere in a path independent way.

GR) E is the <u>local</u> energy at point A.
A distance observer <u>cannot</u> in general define the energy of the particle at A
There is <u>no</u> global family of inertial observers!

- EOM for a particle in a static and weak gravity field

Let us check the Newtonian limit of our "geodesic on curve spacetime" prescription for the motion in a gravitational field.

Problem: How do the Newton equations:

$$\frac{d^2x^i}{dt^2} + \partial_i \phi = 0$$

ϕ — grav. potential e.g. $-\frac{GM}{r}$

relate to the geodesic equation:

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} = 0$$

in case of:

- static gravitational field (ϕ, g are time-independent);
- low velocity of the particle $\frac{v}{c} \ll 1$;
- weak gravitational field.

?

- Newton limit: small velocity

$$u^\mu = \dot{x}^\mu = \left(\frac{dt}{d\tau}, c^{-1}\frac{dx^i}{d\tau}\right) \quad \text{particle 4-velocity}$$

want: $c^{-1}\frac{dx^i}{d\tau} \ll 1$

Note that $\frac{dt}{d\tau} = \gamma \geq 1$, small velocities $\Rightarrow$ $\gamma \sim 1$ and $\frac{dt}{d\tau} \gg c^{-1}\frac{dx^i}{d\tau}$.

- Newton limit: weak field

$g = \eta + h$ with η Minkowski metric

h "small perturbation"

How can we quantify "small" ?

Note g is <u>not</u> positive definite ...

→ Take <u>global coordinates</u> corresponding to Cartesian for $h=0$ (SR).

Require that the components are small with respect to those of η:

$$|h_{\alpha\beta}| \ll 1 \qquad \forall \alpha,\beta .$$

Linearize equations in $h_{\alpha\beta}$.

- Geodesic equation in small-velocity and weak field limit :

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{00}\frac{dx^0}{d\tau}\frac{dx^0}{d\tau} + \underbrace{\Gamma^\mu_{0i}\frac{dx^0}{d\tau}\frac{dx^i}{d\tau}}_{\sim \frac{1}{c}} + \Gamma^\mu_{ij}\underbrace{\frac{dx^i}{d\tau}\frac{dx^j}{d\tau}}_{\sim \frac{1}{c^2}} = 0$$

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{00}\left(\frac{dx^0}{d\tau}\right)^2 \approx 0$$

with:

$$\Gamma^\mu_{00} = \frac{1}{2} g^{\mu\lambda}\left(\partial_0 g_{0\lambda} + \partial_0 g_{\lambda 0} - \partial_\lambda g_{00}\right)$$

linearize in h the term $\partial_\lambda g_{00}$:

$$\frac{1}{2} g^{\mu\lambda}(\partial_\lambda g_{00}) = \frac{1}{2}\left(\eta^{\mu\lambda} + h^{\mu\lambda}\right)\left[\partial_\lambda\left(\eta_{00} + h_{00}\right)\right] =$$

$$\simeq \frac{1}{2}\left(\eta^{\mu\lambda} + O(h)\right)\left[\partial_\lambda h_{00}\right]$$

$$\simeq \frac{1}{2}\eta^{\mu\lambda}\partial_\lambda h_{00}$$

Lesson: linearize in $h_{\alpha\beta}$ $\Rightarrow$ raise indices with <u>flat metric</u> $\eta_{\alpha\beta}$.

- Impose <u>static</u> assumption

$$\partial_0 g_{\mu\nu} = 0 \quad \Rightarrow \quad \Gamma^\mu_{00} = -\frac{1}{2} g^{\mu\nu} \partial_\nu g_{00} \cong -\frac{1}{2} \eta^{\mu\nu} \partial_\nu h_{00}$$

Geodesic equation is:

$$\frac{d^2 x^\mu}{d\tau^2} - \frac{1}{2} \eta^{\mu\nu} \partial_\nu h_{00} \left(\frac{dx^0}{d\tau}\right)^2 = 0$$

$$\partial_0 h_{00} = 0 \quad (\text{static field}) \quad \Rightarrow \quad \frac{d^2 x^0}{d\tau^2} = \frac{d^2 t}{d\tau} = 0 \quad \Rightarrow \quad t = \alpha\tau + \beta .$$

The remaining spatial equations are:

$$\frac{d^2 x^i}{d\tau^2} - \frac{1}{2} \eta^{ij} \partial_j h_{00} \left(\frac{dx^0}{d\tau}\right)^2 = 0$$

$$\frac{d^2 x^i}{d\tau^2} - \frac{1}{2} \delta^{ij} \partial_j h_{00} \left(\frac{dx^0}{d\tau}\right)^2 = 0$$

Substitute $\tau \to t$ (or multiply by $\frac{d\tau^2}{dt^2}$ ...):

$$\frac{d^2 x^i}{dt} - \frac{1}{2} \delta^{ij} \partial_j h_{00} = 0 .$$

Compare with Newton law:

$$\boxed{h_{00} = -2\phi}$$

- EOM for scalar field

$$\Box\phi - m^2\phi = 0 \quad : \quad \Box_\eta \equiv \eta^{ab}\partial_a\partial_b \longrightarrow \Box_g = g^{ab}\partial_a\partial_b$$

Note however that there are other possible generalizations consistent with the "minimal substitution" rule, e.g.

$$\Box\phi - m^2\phi - \alpha R\phi = 0 .$$

There is no general rule to decide. Often the minimal coupling "principle" is simply impose: matter fields do not couple to Riemann tensor and its contractions (cf. hypothesis ~~#~~4.).

- Maxwell equations

$$\begin{cases} \nabla_a F^{ba} = j^b \\ \nabla_{[a}F_{cb]} = 0 \end{cases}$$

$$\nabla_{[a}F_{cb]} = 0 \Rightarrow F_{ab} = \partial_a A_b - \partial_b A_b = \nabla_a A_b - \nabla_b A_b$$
$$= \partial_a A_b - \Gamma^c_{ab}A_c - \partial_b A_a + \Gamma^c_{ba}A_c$$
$$= \partial_a A_b - \cancel{\Gamma^c_{ab}A_c} - \partial_b A_a + \cancel{\Gamma^c_{ab}A_c}$$

As in SR we can introduce a vector potential.

F_{ab} is the antisymm. derivative of the vector potential.

EOM for the vector potential ?

SR $\quad \begin{cases} \partial_\alpha A^\alpha = 0, \text{ Lorentz gauge} \\ \Box_\eta A_\alpha = - j_\alpha \end{cases}$

Minimal substitution guess:

GR $\quad \begin{cases} \nabla_\alpha A^\alpha = 0, \quad \text{Lorentz gauge in GR} \\ \Box_g A_\alpha = - j_\alpha \end{cases}$

verify:

$$\nabla^a F_{ab} = - j_b$$

$$\nabla^a (\nabla_a A_b - \nabla_b A_a) = - j_b$$

$$\nabla^a \nabla_a A_b - \nabla^a \nabla_b A_a = - j_b$$

$$\Box_g A_b - \nabla_a \nabla_b A^a = - j_b$$

$$\Box_g A_b - \nabla_b (\nabla_a A^a) - R^c{}_b A_c = - j_b$$

if we impose $\nabla_a A^a = 0$ "GR" Lorentz gauge

then the EOM are:

$$\Box_g A_b - R^c{}_b A_c = - j_b$$

and they differ from the minimal substitution guess by a curvature term!

However the equation above is the correct one because, in particular, is consistent with current conservation:

$$\nabla_b j^b = 0 .$$

Different ways of proving current conservation on generic spacetimes:

i) For any antisymm. tensor (0,2) one has: $\nabla_\nu F^{\mu\nu} = \frac{1}{\sqrt{-g}} \partial_\nu \left(\sqrt{-g} F^{\mu\nu} \right)$

For any vector one has: $\nabla_\mu j^\mu = \frac{1}{\sqrt{-g}} \partial_\mu (-g j^\mu)$

where $g = \det g_{ab}$.

Hence Maxwell equation and current conservation can be written as

$$\frac{1}{\sqrt{-g}} \partial_\nu \left(\sqrt{-g} F^{\mu\nu} \right) = j^\mu$$

$$\partial_\mu \left(\sqrt{-g}\, j^\mu \right) = 0$$

by derivation w.r.t. ∂_μ of the first equation:

$$\underbrace{\underbrace{\partial_\mu \left[\partial_\nu \right.}_{\text{symm.}} \left(\sqrt{-g}\, \underbrace{F^{\mu\nu}}_{\text{antisymm.}} \right) \Big]}_{(\text{scalar}) \quad = 0} = \partial_\mu \left(\sqrt{-g}\, j^\mu \right)$$

ii) $\nabla_b \nabla_a F^{ba} \equiv 0$

Because: $[\nabla_a, \nabla_b] F^{ab} = -R_{abc}{}^{a} F^{cb} - R_{abc}{}^{b} F^{ac} =$

$$= + R_{bac}{}^{a} F^{cb} - R_{abc}{}^{b} F^{ac} =$$

$$= + R_{bc} F^{cb} - R_{ac} F^{ac} = 2 R_{bc} F^{cb}$$

$$= -2 R_{cb} F^{bc}$$

$$\left. \begin{array}{l} \rightarrow [\nabla, \nabla] F = 0 \\ F \text{ antisymm.} \end{array} \right\} \Rightarrow \nabla_b \nabla_a F^{ba} = 0.$$

iii) Write Maxwell equations in terms of the exterior derivative and codifferential:

$$\begin{cases} dF = 0 \\ d * F = * j \end{cases}$$

Levi-Civita symbol

$\epsilon_{a_1 \dots a_n} = \sqrt{-g}\, \tilde{\epsilon}_{a_1 \dots a_n}$

Levi-Civita tensor

$*$: Hodge operator $\Lambda_p \rightarrow \Lambda_{n-p}$, $\quad (* \omega)_{a_1 \dots a_{n-p}} \equiv \frac{1}{p!} \epsilon^{b_1 \dots b_p}{}_{a_1 \dots a_{n-p}} \omega_{b_1 \dots b_p}$.

$$d^2 = 0 \quad \Rightarrow \quad d * j = 0$$

The above equation corresponds to the divergence of the vector $j^a = g^{ab} j_b$.

[Carrol, Wald books in Appendix]

• EOM for the stress-energy tensor and perfect fluids

The definition of T_{ab} we gave in terms of energy, momentum and stress tensor of a continuum distribution of matter as measured by an observer $\mathcal{O}$ carries over to GR.

For example for a perfect fluid in GR one has:

$$T_{ab} = (\rho + P)\, u_a u_b + g_{ab} P .$$

(SR) EOM

$$\partial^a T_{ab} = 0 \overset{*}{\Longrightarrow} \begin{cases} u^a \partial_a \rho + (\rho+P)\,\partial^a u_a = 0 \\ \\ (P+\rho)\, u^a \partial_a u_b + (\eta_{ab} + u_a u_b)\,\partial^a P = 0 \end{cases}$$

If one considers a family of inertial observers with 4-velocity v^a such that $\partial_a v^b = 0$ (all parallel velocities), Then the mass energy current density 4-velocity of the fluid measured by those observers is

$$J_a \equiv - T_{ab} v^b .$$

The EOM implies:

$$\partial^a J_a = - \partial^a (T_{ab} v^b) = - \partial^a T_{ab} \cdot v^b - T_{ab}\, \partial^a v^b \doteq 0 .$$

which, in turn, implies energy conservation if intergrated over a volume:

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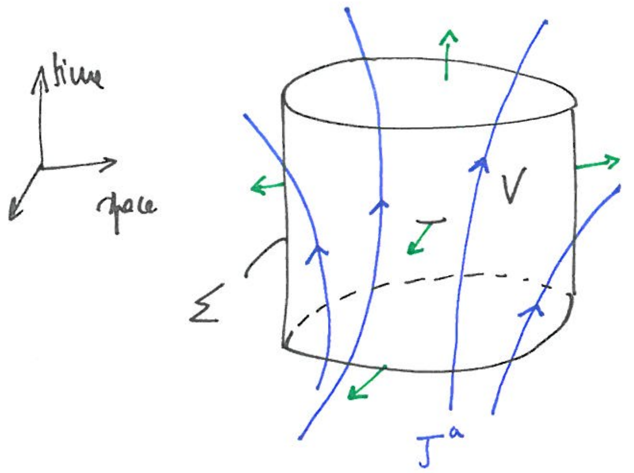
* The first equation can be found by projecting $\partial^a T_{ab}$ along u^a: $u^b \partial^a T_{ab} = 0$.
The second equation can be found by projecting $\perp$ to u^a: $\underbrace{(\delta^c{}_b + u^c u_b)}_{\text{"projector"}} \partial_a T^{ab} = 0$.

Exercise: compute the projections.

Exercise: Compute the Newtonian limit.



$$0 = \int_V \partial_a J^a = \int_\Sigma J^a n_a$$

This result is not restricted to perfect fluids but holds for any matter distribution described by a symmetric T_{ab} and satisfying $\partial^a T_{ab} = 0$.

GR $\boxed{\nabla^a T_{ab} = 0}$ "minimal substitution" rule.

E.g. For a perfect fluid :
$$\begin{cases} u^a \nabla_a \rho + (\rho + P)\nabla^a u_a = 0 \\ (\rho+P)\, u^a \nabla_a u_b + (g_{ab} + u_a u_b)\nabla^a P = 0 \end{cases}$$

In GR $J_a \equiv -T_{ab} V^b$ has still the meaning of energy-momentum 4-vector as measured by observers with 4-velocity V^b $(V^b V_b = -1)$. That simply follows from the general definition of T_{ab}.

However, the equation $\nabla^a T_{ab} = 0$ cannot be interpreted as "energy conservation" anymore.

In order to have "strict" energy conservation one would need to find observers such that:

$$\begin{cases} V_a V^a = -1 \\ \nabla_{(b} V_{a)} = 0 \quad \text{or} \quad \nabla_b V_a = 0 \end{cases}$$

Then one would have: $\nabla^a J_a = -\nabla^a (T_{ab} V^b) = -\underbrace{\nabla^a T_{ab}}_{=0} V^b - T_{ab} \underbrace{\nabla^a V^b}_{=0}$.

Note: $T_{ab} \nabla^{(b} V^{a)} = T_{ab} \frac{1}{2}(\nabla^b V^a + \nabla^a V^b) = \frac{1}{2} T_{ab} \nabla^b V^a + \frac{1}{2} T_{ba} \nabla^a V^b = T_{ab} \nabla^b V^a$.

But in curve spacetime, in general, one cannot find such observers.
Physically, there exist no global inertial observers!
If one considers the perfect fluid, one can think that tidal forces do work on the fluid and increase or decrease the local measure of energy.

On sufficient small region $R \ll (\text{curvature})^{-1}$ one can find observers such that $\nabla_b V^a \approx 0$. The equation $\nabla^a T_{ab} = 0$ could be thus considered as a <u>local</u> conservation equation.

<u>Killing vectors</u>

From the discussion above one can understand that vector:

$$\nabla_{(a} K_{b)} = 0$$

play some special role.
Indeed the equation above is called <u>Killing equation</u> and its solutions are called <u>Killing vector</u>. Notably:

if a killing vector exist, then one can define the current:

$$J_K{}^a \equiv T^{ab} K_b$$

and it will be automatically <u>conserved</u>: $\nabla_a J_K{}^a = 0$!

Since conserved quantities are associated to <u>symmetries</u> (Nöther theorem) one can easily understand that Killing vector are associated to symmetries;
Technically, <u>Killing vectors generate isometries</u>.

Imagine the metric has some symmetry. Then in some "adapted coordinates" the metric components will be independent on certain coordinates, example:

- $g = dx^2 + f(y)\,dy^2$ is invariant under translations: $x \to x+a$.
- $g = A(r)\,dr^2 + B(r)\,d^2\Omega$ is a spherically symmetric metric (φ-independent).

In general, given a symmetry, there will be a given coordinate, say σ^*, such that:

$$\partial_{\sigma^*} g_{\mu\nu} = 0 \qquad \forall\, \mu,\nu.$$

and the Killing vector is simply given by $K = \partial_{\sigma^*}$, i.e. with components:

$$K^\mu = (\partial_{\sigma^*})^\mu = \delta^\mu_{\sigma^*}.$$

- How do geodesics look in presence of symmetries?

$$u^\alpha \nabla_\alpha u^\mu = 0$$

$$0 = u^\alpha \nabla_\alpha u^\mu \underset{\nabla g = 0}{=} u^\alpha \nabla_\alpha u_\mu = u^\alpha \partial_\alpha u_\mu - \Gamma^\sigma_{\alpha\mu} u^\alpha u_\sigma =$$

raise that

$$= \frac{dx^\alpha}{d\lambda} \partial_\alpha u_\mu - \frac{1}{2} g^{\sigma\nu} \left(\partial_\alpha g_{\mu\nu} + \partial_\mu g_{\nu\alpha} - \partial_\nu g_{\alpha\mu}\right) u^\alpha u_\sigma$$

$$= \frac{du_\mu}{d\lambda} - \frac{1}{2}\left(\underbrace{\partial_\alpha g_{\mu\nu}}_{\text{symm.}} + \partial_\mu g_{\nu\alpha} - \underbrace{\partial_\nu g_{\alpha\mu}}_{\text{symm.}}\right) \underbrace{u^\alpha u^\nu}_{\text{symmetric}}$$

The 2 symm. combinations cancel!

$$= \frac{du_\mu}{d\lambda} - \frac{1}{2} \partial_\mu g_{\nu\alpha} u^\alpha u^\sigma$$

If $\mu = \sigma^*$, then $\partial_{\sigma^*} g_{\nu\alpha} = 0 \Rightarrow \boxed{\dfrac{du_{\sigma^*}}{d\lambda} = 0}$ Conserved quantity of the geodesic motion.

But the conserved quantity can be written as the contraction between p_μ and the Killing vector:

$$u_{\sigma*} = K^\mu u_\mu = u^\mu K_\mu = u_\mu \delta^\mu_{\sigma*} .$$

which means:

$$\frac{du_{\sigma*}}{d\lambda} = 0 \quad \Rightarrow \quad u^\mu \nabla_\mu (K_\nu u^\nu) = 0$$

$$= u^\mu K_\nu \nabla_\mu u^\nu + u^\mu u^\nu \nabla_\mu K_\nu =$$

$$= K_\nu \underbrace{u^\mu \nabla_\mu u^\nu}_{=0} + u^\mu u^\nu \nabla_\mu K_\nu =$$

$$= u^\mu u^\nu \nabla_{(\mu} K_{\nu)}$$

<u>Killing equation</u> $\Rightarrow$ <u>conserved quantity along geodesics</u>

$$\nabla_{(\mu} K_{\nu)} = 0 \qquad \Rightarrow \qquad u^\mu \nabla_\mu (K_\nu u^\nu) = 0 .$$

i.e.

$$\nabla_\mu K_\nu = - \nabla_\nu K_\mu$$

antisymmetric.

Another way to express the Killing equation is using the Lie derivative:

$$\mathcal{L}_K g_{ab} = 2\nabla_{(a} K_{b)} = 0 .$$

- Killing vectors and Riemann tensor

One can prove [exercise] that the derivatives of a Killing vector is related to the Riemann tensor according to:

$$\nabla_a \nabla_b K^c = R_{bad}{}^c K^d .$$

Contracting a-c one obtains:

$$\nabla_a \nabla_b K^a = R_{bd} K^d .$$

Using Bianchi identities and the antisymmetry of $\nabla_a K_b$ one obtains also:

$$K^a \nabla_a R = 0 ,$$

i.e. the directional derivative of the Ricci scalar along the Killing vector is zero. Intuitively "geometry is not changing along the Killing vector".

PROOF of above equation. Main ingredients [calculation "rules"]:

i) $\nabla^a G_{ab} = 0$

ii) $\nabla_a K_b$ antisymm $\Rightarrow S^{ab} \nabla_a K_b = 0$ for any symmetric tensor S^{ab}

iii) $[\nabla_a, \nabla_b] T^{ab} = R_{ab}(T^{ab} - T^{ba}) = 0$ for any tensor T_{ab}

iv)
$$[\nabla_a, \nabla_b] A^{ab} = \nabla_a \nabla_b A^{ab} - \nabla_b \nabla_a A^{ab}$$
$$= \nabla_a \nabla_b A^{ab} - \underbrace{\nabla_a \nabla_b A^{ba}}_{\text{just rename indexes}}$$
$$= \nabla_a \nabla_b A^{ab} + \nabla_a \nabla_b A^{ab}$$
$$= 2 \nabla_a \nabla_b A^{ab}$$

v) iii) + iv) $\Rightarrow \nabla_a \nabla_b A^{ab} = \frac{1}{2} [\nabla_a, \nabla_b] A^{ab} = 0$

Using the above properties:

$$0 = (\nabla^a G_{ab}) K^b = (\nabla^a R_{ab} - \tfrac{1}{2} \underbrace{g_{ab} \nabla^a}_{\nabla_b} R) K^b \Rightarrow$$

$$2 K^b \nabla^a R_{ab} = K^b \nabla_b R$$

$$2 \nabla^a (K^b R_{ab}) - 2 \underbrace{R_{ab} \nabla^a K^b}_{=0 \ \ iii)} = K^b \nabla_b R$$

$$2 \nabla^a \nabla^c \nabla_a K_c = K^b \nabla_b R$$

$$\overset{iv)}{\parallel}$$

$$2 [\nabla^a, \nabla^c] \nabla_a K_c = K^b \nabla_b R$$

$$\overset{v)}{\parallel}$$

$$0 = K^b \nabla_b R \qquad \square$$

PROPERTIES USED ALL TIMES

T_{ab} : generic $(0,2)$ tensor
A_{ab} : antisymm $(0,2)$ tensor
S_{ab} : symmm $(0,2)$ tensor

- $S^{ab} A_{ab} = 0$
- $[\nabla_a, \nabla_b] T^{ab} = \underbrace{R_{ab}}_{\text{Ricci, symmm}} (T^{ab} - T^{ba}) = 0$
- $0 = [\nabla_a, \nabla_b] A^{ab} = 2 \nabla_a \nabla_b A^{ab}$

Lie derivative

Derivative operators studied so far:

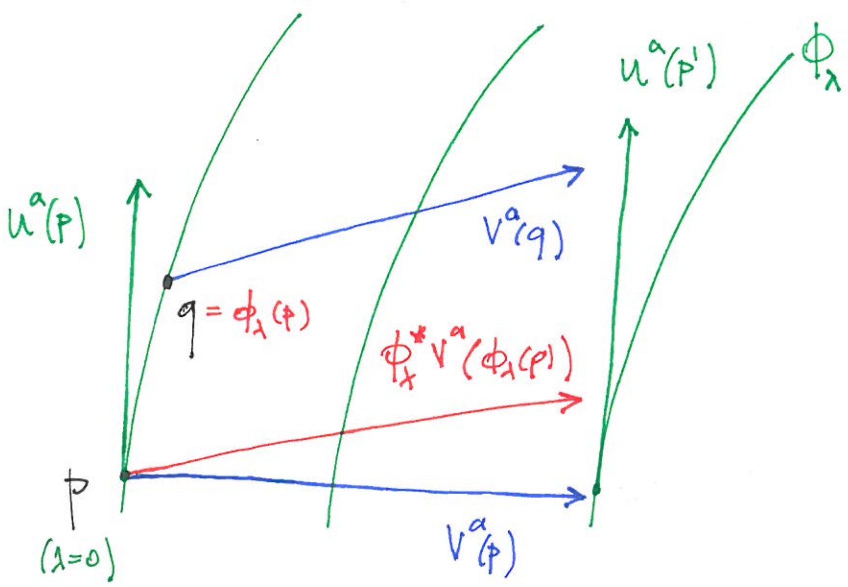
d Exterior derivative
- specific for p-forms
- no metric required
- key relevance: Stokes theorem.

∇ Covariant derivative
- for any tensor
- Levi-Civita connection compatible with the metric g
- key relevance: parallel transport, curvature.

Let us introduce:

$\mathcal{L}$ Lie derivative
- for any tensor
- "directional derivative" along a vector u^a
- key relevance: symmetries.

Consider a vector field u^a and its associated field lines ϕ_λ (integral curves of u^a). The latter constitute a one-parameter family of diffeomorphism.



INTEGRAL CURVES : $\frac{dx^\mu}{d\lambda} = u^\mu$

$\exists!$ locally a solution.

Consider a vector V^a and ask : what is its variation along u^a ?

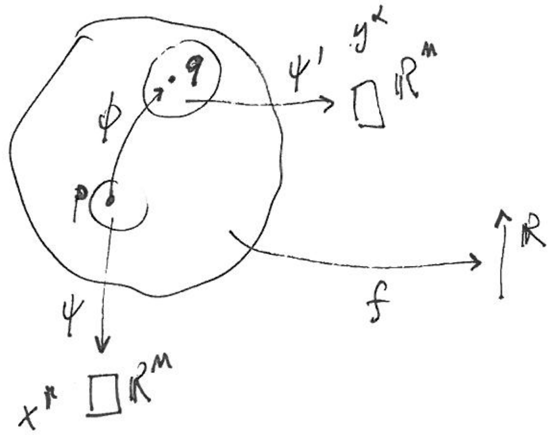
We cannot compare vectors at different points because they belong to different vector spaces. We need to parallel transport them ... or transport them in other ways :

Take a point p, where one has $u^a(p), V^a(p) \in T_pM$.

Take a second point q infinitesimally close to p and along the integral curve of u^a

$$q = \phi_\lambda(p) \quad \text{with } \lambda \text{ small}$$

the integral curves on the manifold can be considered as <u>active coordinate transformations</u>



$$\phi_* V(f) = (\phi V)^\alpha \partial_\alpha f$$
$$= V^\mu \partial_\mu (f \circ \phi) =$$
$$= V^\mu \frac{\partial y^\alpha}{\partial x^\mu} \partial_\alpha f$$

"push forward" of $V \in T_pM$ to $\phi_* V \in T_{\phi(p)} M$

and used to "move" tensors with "push forward" and "pull back" operation (if ϕ^{-1} exists). Using the pull back, we then transport $V^a(q)$ back to p and define the Lie derivative as :

$$\mathcal{L}_u V \equiv \lim_{\lambda \to 0} \frac{\phi^* V - V}{\lambda} \qquad \text{at } p .$$

Introduce now coordinates adapted to the vector u^a, i.e. y^μ : $u^\mu = (1, 0, 0, 0, ..)$.
In this coordinates :

$$\phi_\lambda(p) = (y^0 + \lambda, y^1, y^2, ...)$$

hence :

$$\mathcal{L}_u V^\mu = \frac{\partial V^\mu}{\partial y^0} \quad .$$

While the expression is clearly not covariant, one can observe that in the same coords:

$$[u,v]^\mu = u^\nu \partial_\nu v^\mu - v^\nu \partial_\nu u^\mu = \frac{\partial v^\mu}{\partial y^0}$$

In arbitrary coordinates one generalizes to:

$$\mathcal{L}_u v^\mu = u^\nu \partial_\nu v^\mu - v^\nu \partial_\nu u^\mu = [u,v]^\mu$$

and thus

$$\mathcal{L}_u V \equiv [u,v] .$$

<u>Observations</u>

- The commutator is sometimes called Lie bracket
- $\mathcal{L}_u V = - \mathcal{L}_V u$
- For generic forms and tensors it generalizes to:

$$\mathcal{L}_u T^{a_1 \dots a_k}{}_{b_1 \dots b_l} = u^c \nabla_c T^{a_1 \dots a_k}{}_{b_1 \dots b_l} - \sum_j (\nabla_c u^{a_j}) T^{a_1 \dots c \dots a_k}{}_{b_1 \dots b_l} + \sum_i (\nabla_{b_i} u^c) T^{a_1 \dots a_k}{}_{b_1 \dots c \dots b_l}$$

with ∇ <u>any</u> torsion-free derivative (not necessarily Levi-Civita).

- Lie derivative of the metric:

$$\mathcal{L}_u g_{ab} = u^c \underbrace{\nabla_c g_{ab}}_{=0} + \nabla_a u^c \cdot g_{cb} + \nabla_b u^c \cdot g_{ac} = 2 \nabla_{(a} u_{b)}$$

with ∇ Levi-Civita connection.

For a killing vector one has: $\mathcal{L}_u g_{ab} = 0$

- <u>Diffeomorphism invariant</u>.

If $\phi_\lambda^* T = T \quad \forall \lambda \in \mathbb{R}$, then the tensor $T \in \tau(k,l)$ is <u>invariant</u> with respect to the one-parameter group of diffeomorphism generated by u^a.

- Diffeomorphism invariant $\Longleftrightarrow$ $\mathcal{L}_u T = 0$.

  $\phi_\lambda^* T = T$

EOM FOR THE METRIC

__Want__ : equations for g.

- tensorial equations
- Newton limit.

Newton EOM : $\Delta \phi = 4\pi G \rho$

$\Delta\phi$: 2nd derivatives of grav. potential

ρ: matter distribution

change notation : $\Delta h_{00} = 4\pi G T_{00}$

Δh_{00}: 2nd derivatives of the metric

T_{00}: stress-energy tensor

idea : $$\underbrace{\text{"} \partial^2 g_{ab} \text{"}} = K G T_{ab}$$

Here we need a tensor symmetric in (ab). What to use?

__opt. 0__ : $\Box_g g_{ab} = \nabla_a(\nabla^a g_{bc}) \equiv 0$:(

__opt. 1__ : $R_{ab}[g]$ Ricci tensor, contains $\partial^2 g_{ab}$ ... but:

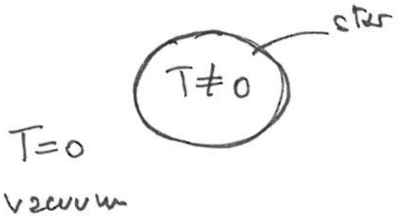
- Bianchi identities $\nabla^a R_{ab} = \frac{1}{2}\nabla_b R$ are __incompatible__ with energy-momentum local conservation $\nabla^a T_{ab} = 0$ since in general $\nabla_b R \neq 0$ ...

- Moreover, if we had to assume $\nabla_b R = 0$, then

$$R = g^{ab} R_{ab} = k\, g^{ab} T_{ab} = k T$$

$$\nabla_b R = 0 \implies \nabla_b T = 0 \implies T = \text{constant } \underline{\text{everywhere}}$$

but that is not possible because, for example, the spacetime of a star has:



opt 2:

$$\boxed{G_{ab} \equiv R_{ab} - \frac{1}{2} R\, g_{ab} = k\, G\, T_{ab}}$$

Looks good!

Bianchi identities $\implies$ local energy conservation.

<u>Determine k from Newton limit</u>

- Take the trace:

$$\underbrace{g^{ab}}\left(R_{ab} - \frac{1}{2} R\, g_{ab}\right) = k G\, g^{ab} T_{ab}$$

$$R - \frac{1}{2} R \underbrace{g^{ab} g_{ab}}_{Tr(g^{-1}g) = Tr(\mathbb{1}) = 4 \quad (n=4)} = k G T$$

$$- R = k G T$$

Reinsert in the equations:

$$R_{ab} - \frac{1}{2} R\, g_{ab} = R_{ab} + \frac{1}{2} k G T g_{ab} = k G T_{ab}$$

$$\boxed{R_{ab} = k G \left(T_{ab} - \frac{1}{2} T g_{ab}\right)}$$ Trace-reverse equations

Observation

In vacuum: $G_{ab} = 0 \iff R_{ab} = 0$.

- Take the Newtonian, static weak-field limit of Trace-reverse equations.

$$g_{00} = -1 + h_{00} \quad ; \quad g^{00} = -1 - h_{00}$$

$$R_{00} = R^{\mu}{}_{0\mu0} \underset{\uparrow}{=} R^{i}{}_{0i0}$$

$$R^{0}{}_{000} = 0$$

$$R^{i}{}_{0j0} = \partial_j \Gamma^{i}_{00} - \underbrace{\partial_0 \Gamma^{i}_{j0}}_{\substack{=0 \\ \text{time derivatives}}} + \Gamma^{i}_{j\lambda}\Gamma^{\lambda}_{00} - \Gamma^{i}_{0\lambda}\Gamma^{\lambda}_{j0} \simeq \partial_j \Gamma^{i}_{00}$$

$\Gamma^2 \sim (\partial g)^2 \sim (\partial h)^2$

can be neglected in linearization

$$R_{00} = R^{j}{}_{0j0} = \partial_j \Gamma^{j}_{00} = \partial_j \left[\frac{1}{2} \underbrace{g^{i\lambda}}_{=\eta^{i\lambda}} \left(\underbrace{\partial_0 g_{\lambda 0}}_{=0} + \underbrace{\partial_0 g_{0\lambda}}_{=0} - \underbrace{\partial_\lambda g_{00}}_{\partial_i h_{00}} \right) \right] =$$

$$= -\frac{1}{2} \delta^{ij} \partial_i \partial_j h_{00} = \underbrace{-\frac{1}{2} \Delta h_{00}}_{\text{LHS.}}$$

$$T_{00} = \mathcal{E}$$

$$g_{00} T = (\eta_{00} + h_{00}) T = \eta_{00} T = \eta_{00} (g^{\mu\nu} T_{\mu\nu}) =$$

$$= \underbrace{\eta_{00}}_{=-1} \left(\eta^{00} T_{00} + \underbrace{\eta^{ij}}_{\delta^{ij}} T_{ij} \right) = - \left(\underbrace{\eta^{00}}_{=-1} T_{00} + \delta^{ij} T_{ij} \right) \simeq + \mathcal{E}$$

$|T_{00}| \gg |T_{ij}|$ "$\rho \gg \frac{P}{c^2}$" Always true in geometric units and weak-field

$$T_{00} - \frac{1}{2} g_{00} T \cong \mathcal{E} - \frac{1}{2}\mathcal{E} = \underbrace{\frac{1}{2}\mathcal{E}}_{\text{RHS.}}$$

Put together:

$$-\frac{1}{\cancel{2}} \Delta h_{00} = \frac{1}{\cancel{2}} K G \mathcal{E}$$

Compare with Newton law using $h_{00} = -2\phi$ and $\mathcal{E}$ ρ

$$\Delta \phi = \frac{1}{2} K G \mathcal{E}$$

$$= 4\pi G \rho$$

$$\boxed{K = 8\pi}$$

Final result:

$$\boxed{\boxed{R_{ab} - \frac{1}{2} R g_{ab} = 8\pi G T_{ab}}}$$

Einstein field equations (EFE)

Observations

$$G_{ab}[\partial^2 g, \partial g, g] = 8\pi G \, T_{ab}[g]$$

- 10 PDEs coupled and nonlinear and involving the metric 2nd, 1st derivatives in some coord. system.

- $T_{ab}[g]$ depends on g_{ab}. One cannot specify a matter distribution and then calculate g_{ab} via EFE (different from electromagnetism).

$\rightarrow$ the dynamics of the matter need to be solved <u>together</u> with the metric.

Actually, EFE contain already information about matter dynamics:

$$\nabla^a T_{ab} = 0 .$$

For example, for a perfect fluid these are <u>all</u> the equations one needs!

Moreover, EFE contain the <u>geodesic hypotesis</u>. For a perfect fluid with $P=0$ ("dust") the local conservation of energy-momentum <u>implies</u> the geodesic equation:

$$u^a \nabla_a u^b = 0$$

That has been shown to hold true for <u>any</u> body with sufficiently <u>weak self-gravity</u>.

The common approach is, as a matter of fact, to <u>postulate</u> an expression for T_{ab} and use the "conservation law" from EFE and the equation of motion for particles and fields.

<u>Examples</u>

- Perfect fluid : $T_{ab} = (\rho + P) u_a u_b + P g_{ab}$
- Scalar field : $T_{ab} = \nabla_a \phi \nabla_b \phi - \frac{1}{2} g_{ab} \left(\nabla_c \phi \nabla^c \phi + m^2 \phi^2 \right)$
- EM field : $T_{ab} = \frac{1}{4\pi} \left(F_{ac} F^c{}_b - \frac{1}{4} g_{ab} F_{de} F^{de} \right)$

  Note: this gives <u>vacuum</u> EM equations. Equations of motion [and thus T_{ab}] for the charges/currents need to be added.

In general we will see that "matter" fields can be added to the <u>action</u>, from which one can derive T_{ab} with all the contributions from the different fields (and eventual interactions among them).

STRUCTURE OF EFE

Consider EFE in $n=4$ and in vacuum:

$$G_{ab}[g] = 0 = R_{ab}[g] \ ,$$

10 equations $\longrightarrow$ 10 metric components g_{ab}.

In order to obtain the metric components, one would need to

- fix a coordinate system $\rightarrow$ obtain PDEs
- solve (somehow, see below) the resulting PDE system defined by $R_{\mu\nu}[g_{\mu\nu}] = 0$.

Problem: a coordinate change can in principle fix 4 metric components. The number of metric components with physical meaning is

$$10 - 4 = 6$$

Are there too many equations? No, Bianchi identities are precisely 4 equations:

$$\nabla^{\mu} G_{\mu\nu} = 0$$

that can be interpreted as constraints for the 4 functions:

$$R_{\mu\nu}[g_{\mu\nu}(x)] \ .$$

Let us look more in detail what type of equations are EFE.

In a coord. sys. the Ricci tensor reads:

$$R_{\mu\nu} = \underbrace{-\frac{1}{2} g^{\alpha\beta} \partial_\alpha \partial_\beta g_{\mu\nu} + g^{\alpha\beta} \partial_\alpha \partial_{(\mu} g_{\nu)\beta} - \frac{1}{2} g^{\alpha\beta} \partial_\mu \partial_\nu g_{\alpha\beta}}_{\substack{\text{principal part} \\ \text{(2nd derivatives)}}} + \underbrace{Q_{\mu\nu}(\partial g, g)}_{\substack{\text{non-principal part} \\ \text{(lower derivatives)}}}$$

Cauchy problem for Einstein equations

Looking at the Ricci tensor

$$0 = R_{\mu\nu} = -\frac{1}{2} \underbrace{g^{\alpha\beta} \partial_\alpha \partial_\beta g_{\mu\nu}}_{\sim \Box g_{\mu\nu}} + \dots$$

we are tempted to consider it a hyperbolic PDE and define an initial value problem (IVP) or Cauchy problem for the spacetime metric.

That is indeed possible (including proving well-posedness) although one has to clarify a couple of non-trivial points:

1 – meaning of "time" and time-coordinate

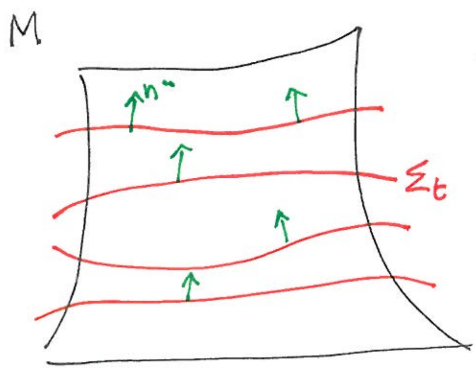
2 – structure of the equations and role of coordinates (gauge).

1: Globally hyperbolic spacetimes

To give a meaning to "time coordinate" we restrict to a class of manifolds in which there exist a smooth function t whose gradient is timelike vector:

$$\exists\ t : M \to \mathbb{R} \ : \quad \mathrm{grad}(t) = dt \propto n_a \quad \text{with} \quad n_a n^a < 0$$

This function foliate the spacetime in spacelike hypersurfaces Σ_t:



$(\mathrm{grad}(t) \neq 0)$

Def: Globally hyperbolic spacetime.

Alternative definition:

A spacetime in which exist a Cauchy surface, i.e. in which causal curves intersec Σ_t only once.

Observations

- In a globally hyperbolic spacetime the wave equation $\Box_g \phi = 0$ admits a well-posed IVP.

- If we restrict to this class one can always define <u>adapted coordinates</u> to the structure and label $x^0 = t$, "time".

- Although we are restricting to a particular class of spacetimes, globally hyperbolic spacetimes comprise (or well approximate) many (most) of the astrophysical and cosmological spacetimes of interest.
  For example one can argue that they comprise the spacetimes of <u>isolated objects</u>: from black holes, stars to a galaxy.

- Globally hyperbolic spacetimes are the starting point for <u>Hamiltonian GR formalism</u> (or ADM formalism) which, in turn, is necessary to develop:
  - quantum theory of gravity
  - concepts of energy and mass in GR
  - Hamiltonian post-Newtonian formalism to approximate GR solutions.

  Strictly related, globally hyperbolic spacetimes are assumed for <u>3+1 GR</u>: the formalism currently employed for numerical solution (<u>numerical relativity</u>).

To gain insight on GR, we first consider Maxwell equations.
The initial value problem in electromagnetism has strong analogies to GR if formulated on the vector potential. The key aspects are:

- the equations are composed of <u>evolution</u> (hyperbolic-type) and <u>constraints</u> (elliptic-type) equations;
- a well-posed IVP can be set up by working on a specific <u>gauge</u>.

Resolution:

→ However, physically, we have "gauge freedom": two solutions

$$A_\alpha \quad , \quad A_\alpha + \partial_\alpha \chi$$

correspond to the same $\vec{E}$ and $\vec{B}$ and are to be considered equivalent.

One can exploit this fact to define a well-posed system.

For example, chose the Lorentz gauge:

$$\partial_\alpha A^\alpha = 0 \, ,$$

in this gauge Maxwell eqs for A_α read

$$0 = \partial^\alpha \partial_\alpha A_\beta - \partial^\alpha \partial_\beta A_\alpha = \partial^\alpha \partial_\alpha A_\beta - \partial_\beta (\underbrace{\partial^\alpha A_\alpha}_{=0}) \overset{L.G.}{=} \partial^\alpha \partial_\alpha A_\beta$$

i.e.

$$\Box_\eta A_\beta = 0 \qquad \text{(L.G.)}$$

the $\beta = 0$ equation is a dynamical eq. now (∂^2_{tt} do not cancel anymore) and we have 4 eqs for 4 unknowns.

Note also those are linear wave equations for each of the components of A_β; hence given initial data at $t=0$ one can obtain a unique solution for all $t>0$.

Even more importantly, if one chooses

$$A_\beta \, , \partial_t A_\beta \quad \text{on } \Sigma_0 \ (t=0) \ : \ \begin{cases} \partial^\beta A_\beta = 0 \\ \dfrac{\partial}{\partial t}(\partial^\beta A_\beta) = 0 \end{cases} \qquad (*)$$

then the gauge choice is satisfied $\forall t$:

$$0 = \Box A_\beta \Rightarrow \partial_\beta \Box A^\beta = \Box \partial_\beta A^\beta = 0 \qquad \forall t \, .$$

And finally $(*)$ are equivalent to the constraint C ($\beta = 0$ equation <u>before</u> gauge fixing) because:

$$C = \Box A_0 - \partial^\alpha \partial_0 A_\alpha = \Box A_0 - \partial_0 (\partial^\alpha A_\alpha) \underset{\Box A_\beta = 0}{\overset{L.G.}{=}} - \partial_0 (\partial^\alpha A_\alpha) \underset{(*)}{\overset{L.G.}{=}} 0$$

$\Rightarrow$ constraints are satisfied on Σ_0
and remain such during the dynamics.

Summary: Maxwell eqs in A_α define a well-posed IVP if we work in an appropriate gauge and if initial data satisfy the constraint equation.

2: IVP for GR

The equation structure in GR is entirely analog to what discussed above. In particular, in a globally hyperbolic spacetime, the 4 equations

$$G_{ab} n^b = 0 \qquad\qquad (\text{vacuum})$$

are constraints in the sense that

$$C_\mu := G_{\mu\nu} n^\nu$$

do not depend on 2^{nd} time derivatives, and

$$C_\mu [\partial_t g, \partial_i^2 g, \partial_i g, g] = 0$$

is required to hold on the initial hypersurface Σ_0.

The constraint equations are a system of 4 equations, nonlinear with mixed components in the system. They are "elliptic-type" of the equations but of no known type in general. Solving these equation constitute the INITIAL DATA PROBLEM in GR and requires:

i) Specific formalism to obtain mathematically sound equations;

ii) Specific choice of which data to solve for and which data to specify.

In this context the <u>Bianchi identities</u> $\nabla^\mu G_{\mu\nu} = 0$, play the same role as the identity in Maxwell theory: they guarantee that the constraints are propagated along the dynamics.

For example, assume that in a specific coord. sys. (gauge) one has

$$0 = C_\mu = G_{0\mu} ,$$

the Bianchi identities implies that the constraints contain at most first time derivatives of g:

$$\nabla^\mu G_{\mu\nu} = 0 \implies \partial_t G^{0\beta} = \underbrace{- \partial_k G^{k\beta} \underbrace{- \Gamma^\alpha_{\alpha\rho} G^{\beta\rho} - \Gamma^\beta_{\lambda\gamma} G^{\lambda\gamma}}_{\text{these terms contain \underline{at most} } \partial_t^2 g}}$$

$\implies$ the RHS contain at most $\partial_t^2 g$

$\implies$ the LHS —"—

$\implies G^{0\beta} = C^\beta = C^\beta[\partial_t g, \partial_k^2 g, \partial_k g, g]$,

must contain at most $\partial_t g$.

Moreover:
$$\begin{cases} C_\mu = 0 \quad \text{at} \quad t = 0 \\ G_{\mu\nu} = 0 \quad \text{EFE in vacuum} \end{cases} \implies \partial_t G^{0\beta} = \partial_t C^\beta = 0 \quad \forall t$$

(from the same eq. above).

How to write the <u>evolution equations</u> in order to have a well-posed IVP?

Let us take another look to the Ricci tensor in terms of partial derivatives, it can be written as:

$$0 \doteq R_{\mu\nu} = -\frac{1}{2} g^{\alpha\beta} \partial_\alpha \partial_\beta g_{\mu\nu} - g_{\alpha(\mu} \partial_{\nu)} H^\alpha + \tilde{Q}_{\mu\nu}(\partial g, g)$$

where:

$$H^{\alpha} \equiv \partial_\mu g^{\alpha\mu} + \frac{1}{2} g^{\mu\beta} g^{\rho\sigma} \partial_\beta g_{\rho\sigma} .$$

Hence, if one asks:

$$H^{\alpha} \equiv 0 \qquad \underline{\text{Harmonic gauge / coordinates}}$$

Then one has quasi-linear wave equations for the metric components:

$$0 = -\frac{1}{2} g^{\alpha\beta} \partial_\alpha \partial_\beta g_{\mu\nu} + \tilde{Q}_{\mu\nu}(\partial g, g) \qquad \underline{\text{reduced EFE}} .$$

→ In this gauge EFE have been proven to admit a well posed IVP by Choquet-Bruhat (1962).

<u>Observations</u>

- Meaning of harmonic coordinates:

$$\begin{aligned}
\Box_g x^\mu &= \nabla_a \nabla^a x^\mu \\
&= \nabla_\alpha \partial^\alpha x^\mu \\
&= \frac{1}{\sqrt{-g}} \partial_\alpha \Big(\sqrt{-g}\, g^{\alpha\beta} \underbrace{\partial_\beta x^\mu}_{\delta^\mu{}_\beta} \Big) = \\
&= \frac{1}{\sqrt{-g}} \partial_\alpha \left(\sqrt{-g}\, g^{\alpha\mu}\right) = \partial_\alpha g^{\alpha\mu} + \frac{1}{2} g^{\alpha\mu} g^{\rho\sigma} \partial_\alpha g_{\rho\sigma} = H^\mu
\end{aligned}$$

Note: x^μ is <u>not</u> a vector; $\nabla^a x^\mu$ is a vector

and: $g^{\alpha\beta} \nabla_\beta (\nabla_\alpha x^\mu) = g^{\alpha\beta} \nabla_\beta (\delta^\mu_\alpha) = g^{\alpha\beta} \left[\partial_\beta (\delta^\mu_\alpha) - \delta^\mu_\sigma \Gamma^\sigma_{\alpha\beta}\right] = -g^{\alpha\beta} \Gamma^\mu_{\alpha\beta} = H^\mu$

- There exists other gauges/coordinates and formulations of EFE leading to a well-posed IVP, but the idea behind the different "reductions" is in many cases analogous to what done in for harmonic coords.

- Solving the IVP constitutes the <u>DYNAMICAL or EVOLUTION PROBLEM</u>.

HILBERT ACTION AND LAGRANGIAN FORMULATION

Einstein eqs can be derived from an action. In fact, they were derived first by Lorentz and Hilbert using that method. The action/Lagrangian approach has some advantages:

- Easier to postulate (or guess) since it is a scalar
- Automatically implement the symmetries (diffeomorfism invariant) and the tensor form of the equations of motion
- Permit a general definition of the stress-energy tensor.

However the Hilbert action approach has some differences with respect to the action of other fields and there are subtle points.

Let us start with the general scheme for an action and the related EOM for field ψ:

$$S = \int \mathcal{L}[\psi, \nabla\psi]$$

integral on M — Lagrangian density depending on the field ψ and its derivatives

variation of the fields and derivatives (variations are $=0$ at boundary):

$$\psi \longrightarrow \psi + \delta\psi \quad ; \quad \nabla\psi \longrightarrow \nabla\psi + \delta\nabla\psi$$

leads to the variation of the action δS and the EOM:

$$\delta S = 0 \quad \longrightarrow$$ EOM in the 2^{nd} derivatives of the field

$$F[\nabla^2\psi, \nabla\psi, \psi] = 0$$

<u>In what the GR action is "special"?</u>

(i) $\displaystyle \int = \int f\,\epsilon = \int f\sqrt{-g}\,dx^1\wedge\ldots\wedge dx^4$, with f a scalar

$\Rightarrow$ the measure ϵ contains the metric, i.e. it contains the field to vary!

(ii) $\nabla g = 0$ for metric compatibility $\Rightarrow$

- we can vary only g_{ab}
- $\mathcal{L}$ must contain 2^{nd} derivatives of g to obtain EOM with 2^{nd} derivatives
- there is a boundary term that does not vanish if we do not ask explicitly that ∇g are fixed at the boundary

---

(i) $\Rightarrow$ $\mathcal{L}$ cannot be a scalar. Ways to proceed:

- Redefine $\mathcal{L}$ to be a 4-form: $\mathcal{L} \longrightarrow \mathcal{L}\epsilon = \mathcal{L}\sqrt{-g}\, dx^1 \wedge \dots \wedge dx^4$
but that complicates the functional derivatives for the variation...

- Work with a scalar lagrangian $\hat{\mathcal{L}}$ and a tensor density $\mathcal{L} = \hat{\mathcal{L}}\sqrt{g}$, vary $\mathcal{L}$ but write the Lagrange-Euler equations in terms of $\hat{\mathcal{L}}$.
This works for fields other than g_{ab}

- For g_{ab} we pragmatically proceed by varying $\frac{\delta S}{\delta g}$ in coordinates.

(ii) $\Rightarrow$ $S \quad \int \sqrt{-g}\, \tilde{\mathcal{L}}[g_{\mu\nu}]$ with $\tilde{\mathcal{L}}$ scalar built out of $\partial^2 g_{\mu\nu}$.

- Riemann in $N=4$ has 20 components: $\sim \partial^2 g_{\mu\nu}$.
In a local inertial frame we can fix coordinate by performing Lorentz transformation and eliminating 6 of the Riemann components.
One is left with $20-6=14$ curvature invariants; turns out that only one of those is linear in $\partial^2 g$: R.

The Hilbert action is : $\boxed{S = \int \sqrt{-g}\, R}$

Does it lead to EFE in vacuum?

The variation is more easily performed in the inverse metric $g^{\mu\nu}$.

$$g^{\mu\alpha} g_{\nu\alpha} = \delta^{\mu}{}_{\nu}$$

vary:

$$\delta\left(g^{\mu\alpha} g_{\nu\alpha}\right) = 0 \qquad (\delta^{\mu}{}_{\nu} \text{ does not vary})$$

$$\delta g^{\mu\alpha} g_{\nu\alpha} + g^{\mu\alpha}\, \delta g_{\nu\alpha} = 0$$

$$g_{\nu\alpha}\, \delta g^{\mu\alpha} + g^{\mu\alpha}\, \delta g_{\nu\alpha} = 0$$

multiply by $g_{\mu\rho}$ and contract :

$$g_{\mu\rho} g_{\nu\alpha}\, \delta g^{\mu\alpha} + \underbrace{g_{\mu\rho} g^{\mu\alpha}}_{\delta^{\alpha}_{\rho}}\, \delta g_{\nu\alpha} = 0$$

$$\boxed{\delta g_{\nu\rho} = - g_{\mu\rho} g_{\nu\alpha}\, \delta g^{\mu\alpha}}$$

$\Rightarrow$ stationary points of S w.r.t. $g_{\mu\nu}$ are stationary points of S w.r.t. $g^{\mu\nu}$.

Perform variation :

$$\delta S = \int \delta\left(\sqrt{-g} R\right) = \int \delta\left(\sqrt{-g}\, g^{\mu\nu} R_{\mu\nu}\right) =$$

$$= \underbrace{\int \sqrt{-g}\, R_{\mu\nu}\, \delta g^{\mu\nu}}_{\mathrm{I}} + \underbrace{\int \delta\left(\sqrt{-g}\right) R}_{\mathrm{II}} + \underbrace{\int \sqrt{-g}\, g^{\mu\nu}\, \delta R_{\mu\nu}}_{\mathrm{III}}$$

Term by term :

I : ok!

II : Make use of the algebraic identity : $\ln(\det A) = \mathrm{Tr}(\ln A)$ (for any sparse matrix A), and its variation :

$$\frac{1}{\det A}\,\delta(\det A) = \mathrm{Tr}\,(A^{-1}\,\delta A)$$

$$\delta g = g \cdot g^{\mu\nu}\,\delta g_{\mu\nu} = g\,\underbrace{\underbrace{g^{\mu\nu}(-g_{\mu\alpha}}_{\delta^\nu_\alpha}\,g_{\nu\beta}}_{g_{\alpha\beta}}\,\delta g^{\alpha\beta}) =$$

$$= -g\, g_{\alpha\beta}\,\delta g^{\alpha\beta}$$

$$\delta(\sqrt{-g}\,) = \frac{1}{2}\frac{1}{\sqrt{g}}(-\delta g) = -\frac{1}{2}\frac{1}{\sqrt{-g}}(-g)\,g_{\alpha\beta}\,\delta g^{\alpha\beta}$$

$$= -\frac{1}{2}\sqrt{-g}\;g_{\alpha\beta}\,\delta g^{\alpha\beta}$$

$$\mathrm{II} = \int \sqrt{-g}\left(-\tfrac{1}{2}\right) g_{\alpha\beta}\,R\,\delta g^{\alpha\beta} \quad .$$

III : One can show that

$$g^{\mu\nu}\,\delta R_{\mu\nu} = \nabla^\alpha \underbrace{\left[\nabla^\beta(\delta g_{\alpha\beta}) - g^{\sigma\rho}\,\nabla_\alpha(\delta g_{\sigma\rho})\right]}_{\equiv V_\alpha} ,$$

→ term III is a "total derivative" (divergence) generating ~

$$\int_M \nabla_\alpha V^\alpha = \int_{\partial M} V^\alpha n_\alpha$$

a <u>boundary term</u>. The latter is nontrivial and it can be shown to be related to the variation of the <u>trace of the extrinsic curvature of the boundary</u> δK.

With no extra conditions on the derivatives of δg_{ab} the term is $\delta K \neq 0$

If we require term III to vanish and put together terms I and II:

$$\delta S = \int \sqrt{-g}\, \underbrace{\left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R\right)}_{= G_{\mu\nu}} \delta g^{\mu\nu}$$

and

$$\frac{1}{\sqrt{-g}} \frac{\delta S}{\delta g^{\mu\nu}} = 0 \quad \Rightarrow \quad G_{\mu\nu} = 0 \quad , \text{ EFE in vacuum.}$$

Alternatively, one needs to define a more general action with an extra term:

$$S = \int_M \sqrt{-g}\, R + \int_{\partial M} 2K$$

in order to obtain EFE.

Observations

- The proper definition of boundary terms ply a relevant role in the Hamiltonian formulation of GR, asymptotically flat spacetime, and definition of mass-energy in those cases.

- An alternative variational approach to GR is the so-called Palatini approach in which one varies the connection together with the metric:

$$S[g_{ab}, \nabla_a]$$

This is possible because R_{ab} can be viewed as dependent only on the connection (Christoffel symbols) and not (explicitely) on the metric.

The variation of that action lead to:

— EFE

— metric compatibility condition $\nabla_a g_{bc} = 0$

without need of discussing boundary terms!

How to include matter fields?

One can immediately notice that the action

$$S = \frac{1}{16\pi G} S_H + S_M$$

where S_H is the Hilbert action leads to the complete (non-vacuum) EFE under the variation:

$$\frac{1}{\sqrt{-g}} \frac{\delta S}{\delta g^{\mu\nu}} = 0$$

if one defines:

$$\boxed{T_{\mu\nu} \equiv -\frac{2}{\sqrt{-g}} \frac{\delta S_M}{\delta g^{\mu\nu}}}$$

In this way one can define a stress-energy (0,2) symmetric tensor given an action term for the matter.

One can immediately verify that the action terms with Lagrangian densities:

$$\mathcal{L}_{KG} \equiv -\frac{1}{2}\sqrt{-g}\left(\nabla_a \phi \nabla^a \phi + m^2 \phi^2\right)$$

$$\mathcal{L}_{EM} \equiv -\frac{1}{4}\sqrt{-g}\, F_{ab} F^{ab}$$

yield

- the EOM for KG and EM, if variation is w.r.t. the field
- the T_{ab} given above, if variation is w.r.t. the g^{ab}

Coupled Einstein-KG or Einstein-EM equations can be thus obtained.

Structure of EM and GR

Field	A_α	$g_{\alpha\beta}$
Constraint $C \equiv 0$	$\partial^\alpha F_{\alpha\beta}[A_\mu]\, n^\beta = 0$	$G_{\mu\nu}[g_{\alpha\beta}]\, n^\nu = 0$
Identity $\partial_t C \equiv 0$	$\partial^\alpha \partial^\beta F_{\alpha\beta}[A_\mu] = 0$	$\nabla^\mu G_{\mu\nu}[g_{\alpha\beta}] = 0$
gauge choice (not unique)	$\partial^\alpha A_\alpha = 0$ Lorentz	$H^\alpha \equiv \Box x^\alpha = 0$ Harmonic
Dynamical Eqs. for well-posed IVP	$\Box_\eta A_\alpha = 0$	$-\frac{1}{2} g^{\mu\nu} \partial_\mu \partial_\nu g_{\alpha\beta} + Q_{\alpha\beta}[\partial g_{\alpha\beta}, g_{\alpha\beta}] = 0$