Geometry of spherically symmetric spacetime

Sebastiano Bernuzzi*
* *Theoretisch-Physikalisches Institut, Friedrich-Schiller-Universität Jena, 07743, Jena, Germany*
(Dated: January 6, 2019)

Collection of general formulas for spherically symmetric spacetime in $n = 4$ dimensions. These calculations are preparatory to prove Birkhoff theorem.
GR class @ Jena FSU, WS 2018/19 `https://bernuzzi.gitlab.io/gr/`

Follow notation of [1].
Assume the metric:

$$g = -\mathrm{e}^{2\alpha(\mathrm{t,r})}\mathrm{dt}^2 + \mathrm{e}^{2\beta(\mathrm{t,r})}\mathrm{dr}^2 + \mathrm{r}^2\mathrm{d}\Omega^2 \tag{1a}$$

with the 2-sphere metric

$$\mathrm{d}\Omega^2 = \mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2 \tag{1b}$$

Christoffel symbols (only nonvanishing components):

$$\Gamma^t_{tt} = \partial_t\alpha \qquad \Gamma^r_{tt} = \mathrm{e}^{2(\alpha-\beta)}\partial_\mathrm{r}\alpha \qquad \Gamma^r_{\theta\theta} = -r\mathrm{e}^{-2\beta} \qquad \Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta \tag{2a}$$

$$\Gamma^t_{tr} = \partial_r\alpha \qquad \Gamma^r_{tr} = \partial_t\beta \qquad \Gamma^r_{\phi\phi} = -r\mathrm{e}^{-2\beta}\sin^2\theta \qquad \Gamma^\phi_{r\theta} = r^{-1} \tag{2b}$$

$$\Gamma^t_{rr} = \mathrm{e}^{2(\beta-\alpha)}\partial_\mathrm{t}\beta \qquad \Gamma^r_{rr} = \partial_r\beta \qquad \Gamma^\theta_{r\theta} = r^{-1} \qquad \Gamma^\phi_{\theta\phi} = \frac{\cos\theta}{\sin\theta} \tag{2c}$$

Riemann tensor:

$$R^t_{rtr} = \mathrm{e}^{2(\beta-\alpha)}(\partial^2_\mathrm{t}\beta + (\partial_\mathrm{t}\beta)^2 - \partial_\mathrm{t}\alpha\partial_\mathrm{t}\beta) + (\partial_\mathrm{r}\alpha\partial_\mathrm{r}\beta - \partial^2_\mathrm{r}\alpha - (\partial_\mathrm{r}\alpha)^2) \tag{3a}$$

$$R^t_{\theta t\theta} = -r\mathrm{e}^{-2\beta}\partial_\mathrm{r}\alpha \qquad R^t_{\phi t\phi} = -r\mathrm{e}^{-2\beta}\sin^2\theta\partial_\mathrm{r}\alpha \qquad R^t_{\theta r\theta} = -r\mathrm{e}^{-2\alpha}\partial_\mathrm{t}\beta \tag{3b}$$

$$R^t_{\phi r\phi} = -r\mathrm{e}^{-2\alpha}\sin^2\theta\partial_\mathrm{t}\beta \qquad R^r_{\theta r\theta} = r\mathrm{e}^{-2\beta}\partial_\mathrm{r}\beta \qquad R^r_{\phi r\phi} = r\mathrm{e}^{-2\beta}\sin^2\theta\partial_\mathrm{r}\beta \tag{3c}$$

$$R^\theta_{\phi\theta\phi} = (1 - \mathrm{e}^{-2\beta})\sin^2\theta \tag{3d}$$

Ricci tensor:

$$R_{tt} = (\partial^2_t\beta + (\partial_t\beta)^2 - \partial_t\alpha\partial_t\beta) + \mathrm{e}^{2(\alpha-\beta)}(\partial^2_\mathrm{r}\alpha + (\partial_\mathrm{r}\alpha)^2 - \partial_\mathrm{r}\alpha\partial_\mathrm{r}\beta + 2\mathrm{r}^{-1}\partial_\mathrm{r}\alpha) \tag{4a}$$

$$R_{rr} = -(\partial^2_r\alpha + (\partial_r\alpha)^2 - \partial_r\alpha\partial_r\beta - 2r^{-1}\partial_r\beta) + \mathrm{e}^{2(\beta-\alpha)}(\partial^2_\mathrm{t}\beta + (\partial_\mathrm{t}\beta)^2 - \partial_\mathrm{t}\alpha\partial_\mathrm{t}\beta) \tag{4b}$$

$$R_{tr} = 2r^{-1}\partial_t\beta \qquad R_{\theta\theta} = \mathrm{e}^{-2\beta}(\mathrm{r}\partial_\mathrm{r}\beta - \mathrm{r}\partial_\mathrm{r}\alpha - 1) + 1 \qquad R_{\phi\phi} = R_{\theta\theta}\sin^2\theta \tag{4c}$$

---

[1] S. M. Carroll, "Spacetime and geometry: An introduction to general relativity," San Francisco, USA: Addison-Wesley (2004) 513 p