These semi-private notes are constructed from the following books:

- R.Wald, "General Relativity" University of Chicago Press, 1984
- S.M.Carrol, "Spacetime and Geometry, An Introduction to General Relativity", Addison-Wesley, 2003.
- B.F.Schutz, "A First Course in General Relativity", Cambridge University Press, 1985.
- E.Gourgoulhon GR notes
- M.Maggiore, "Gravitational Waves, Vol.1: Theory and Experiments", Oxford University Press, 2008.
- R.A.Isaacson Gravitational radiation in the high-frequency limit (See App.C)
- T.Damour, J.H.Taylor On the orbital period change of the binary pulsar PSR1913+16

If you decide to use them to study or teach, please
(0) be careful and refer to the original books
(1) cite/refer to my website
(2) let me know and send feedbacks.

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WEAK FIELD, LINEAR GR, AND GRAVITATIONAL WAVES

In the regime of weak gravity one assumes that there exist a global inertial coordinate system in which the metric can be written as:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \quad \text{with} \quad |h_{\mu\nu}| \ll 1 .$$

Since the components of $h_{\mu\nu}$ are "small" in the sense above, one linearizes GR equations at linear order in $h_{\mu\nu}$. Linearized equations apply, for example, to the Solar system where:

$$|h_{\mu\nu}| \sim \frac{\phi}{c^2} \lesssim \frac{G M_\odot}{c^2 R_\odot} \sim 10^{-6} ,$$

and describe

- Newtonian gravity (weak field metric)
- Gravitoelectric and gravitomagnetic phenomena
- Propagation of gravitational waves.

Formally, the linearized theory can be regarded as a field theory in which:

- $\eta_{\mu\nu}$ is a background metric;
- The gravitational field generated by matter distribution $T_{\mu\nu}$ does not back-react on the source;
- $h_{\mu\nu}$ is the main field, Lorentz covariant in flat space.

Consider in fact a Lorentz transformation of coordinates:

$$x^\mu = \Lambda^\mu{}_\nu \, x'^\nu \qquad \text{with } \Lambda^\mu{}_\nu \text{ Lorentz "matrix":}$$

$$\frac{\partial x^\mu}{\partial x'^\nu} = \Lambda^\mu{}_{\nu} \qquad \Lambda^T \eta \Lambda = \eta$$

then: $$g_{\mu\nu} \longrightarrow g_{\mu'\nu'}$$

$$g_{\mu'\nu'} = \frac{\partial x^{\mu}}{\partial x^{\mu'}} \frac{\partial x^{\nu}}{\partial x^{\nu'}} g_{\mu\nu} =$$

$$= \Lambda^{\mu}{}_{\mu'} \Lambda^{\nu}{}_{\nu'} (\eta_{\mu\nu} + h_{\mu\nu}) =$$

$$= \Lambda^{\mu}{}_{\mu'} \Lambda^{\nu}{}_{\nu'} \eta_{\mu\nu} + \Lambda^{\mu}{}_{\mu'} \Lambda^{\nu}{}_{\nu'} h_{\mu\nu} =$$

$$= \underbrace{\eta_{\mu'\nu'}} + \Lambda^{\mu}{}_{\mu'} \Lambda^{\nu}{}_{\nu'} h_{\mu\nu}$$

$= \eta_{\mu\nu}$, because Λ is Lorentz $\Rightarrow$ compare: $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \Rightarrow$

$h_{\mu\nu}$ transform like a tensor under Lorentz group

$$h_{\mu\nu} \longrightarrow h_{\mu'\nu'} = \Lambda^{\mu}{}_{\mu'} \Lambda^{\nu}{}_{\nu'} h_{\mu\nu} .$$

LINEARIZED EINSTEIN TENSOR

Work at linear order in $h_{\mu\nu}$, raise indexes with background metric $\eta_{\mu\nu}, \eta^{\mu\nu}$.

We obtain:

$$g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu} + O(2)$$

$$\Gamma^{\mu}{}_{\alpha\beta} = \frac{1}{2} \eta^{\mu\lambda} (\partial_{\alpha} h_{\lambda\beta} + \partial_{\beta} h_{\lambda\alpha} - \partial_{\lambda} h_{\alpha\beta}) + O(2)$$

$$R_{\mu\nu} = \text{“} \partial\Gamma - \partial\Gamma + \Gamma\Gamma - \Gamma\Gamma \text{”} = \partial_{\alpha} \Gamma^{\alpha}_{\mu\nu} - \partial_{\mu} \Gamma^{\alpha}{}_{\alpha\nu} + O(2)$$

$$= \partial^{\alpha} \partial_{(\mu} h_{\nu)\alpha} - \frac{1}{2} \partial^{\lambda} \partial_{\lambda} h_{\mu\nu} - \frac{1}{2} \partial_{\mu} \partial_{\nu} h \qquad \text{with } h \equiv h^{\alpha}{}_{\alpha} = \eta^{\alpha\beta} h_{\alpha\beta} .$$

$$R = \eta^{\mu\nu} R_{\mu\nu} = \frac{1}{2} \left(\eta^{\mu\nu} \partial^{\alpha} \partial_{\mu} h_{\nu\alpha} + \eta^{\mu\nu} \partial^{\alpha} \partial_{\nu} h_{\mu\alpha} \right) - \frac{1}{2} \partial^{\lambda} \partial_{\lambda} \underbrace{\eta^{\mu\nu} h_{\mu\nu}}_{h} - \frac{1}{2} \underbrace{\eta^{\mu\nu} \partial_{\mu} \partial_{\nu}}_{\partial^{\lambda}\partial_{\lambda}} h =$$

$$= \frac{1}{\cancel{2}} \cancel{2}\, \partial^{\alpha} \partial^{\nu} h_{\nu\alpha} - \partial^{\lambda} \partial_{\lambda} h$$

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R \eta_{\mu\nu} = -\frac{1}{2}\partial^\lambda\partial_\lambda h_{\mu\nu} + \partial^\alpha \partial_{(\mu} h_{\nu)\alpha} - \frac{1}{2}\partial_\mu\partial_\nu h - \frac{1}{2}\eta_{\mu\nu}\left(\partial^\alpha\partial^\beta h_{\alpha\beta} - \partial^\lambda\partial_\lambda h\right)$$

INFINITESIMAL DIFFEOMORPHISM INVARIANCE

Consider the coordinate transformation:

$$x^\alpha \longmapsto x'^\alpha = x^\alpha + \xi^\alpha(x^\beta)$$

where $|\partial_\beta \xi^\alpha| \sim |h_{\mu\nu}| << 1$ and thus:

$$\frac{\partial x'^\alpha}{\partial x^\beta} = \delta^\alpha{}_\beta + \partial_\beta \xi^\alpha$$

$$\frac{\partial x^\alpha}{\partial x'^\beta} = \delta^\alpha{}_\beta - \partial_\beta \xi^\alpha + O(|\partial\xi|^2)$$

invert the above matrix:

$$\left[(1+\delta A)^{-1} \simeq 1 - \delta A + (\delta A^2) \right]$$

Abusing the index notation we write the above equations as

NOTE ON THE MEANING [Schutz] (*) :

$$\frac{\partial x^{\alpha'}}{\partial x^\beta} = \delta^\alpha{}_\beta + \partial_\beta \xi^\alpha$$

$$\frac{\partial x^\alpha}{\partial x^{\beta'}} = \delta^\alpha{}_\beta - \partial_\beta \xi^\alpha$$

α' refers to tensor component in x' coords
α refers to tensor component in x coords
BUT indexes on RHS/LHS take the same values!

this works as far as we use different letters for prime/unprime coordinates and it is useful to apply usual formulas in presence of the δ's:

$$g_{\alpha'\beta'} = \frac{\partial x^\mu}{\partial x^{\alpha'}} \frac{\partial x^\nu}{\partial x^{\beta'}} g_{\mu\nu}$$

$$= \left(\delta^\mu{}_\alpha - \partial_\alpha \xi^\mu\right)\left(\delta^\nu{}_\beta - \partial_\beta \xi^\nu\right)\left(\eta_{\mu\nu} + h_{\mu\nu}\right) =$$

$$= " \left(\delta\delta - \partial\xi\delta - \partial\xi\delta + \partial\xi\partial\xi\right)(\eta + h) " =$$

$$= " \delta\delta\eta - \delta\eta\partial\xi - \delta\eta\partial\xi + \delta\delta h - \underbrace{\delta h \partial\xi - \delta h \partial\xi + \partial\xi\partial\xi h}_{O(2)} "$$

$$= \delta^\mu_\alpha \delta^\nu_\beta \eta_{\mu\nu} + \delta^\mu_\alpha \delta^\nu_\beta h_{\mu\nu} - \delta^\nu_\beta \eta_{\mu\nu} \partial_\alpha \xi^\mu - \delta^\mu_\alpha \eta_{\mu\nu} \partial_\beta \xi^\nu + O(2) =$$

$$g_{\alpha'\beta'} = \eta_{\alpha\beta} + h_{\alpha\beta} - \delta^{\nu}{}_{\beta}\,\eta_{\mu\nu}\,\partial_\alpha \xi^\mu - \delta^{\mu}{}_{\alpha}\,\eta_{\mu\nu}\,\partial_\beta \xi^\nu + O(2)$$

$$= \eta_{\alpha\beta} + h_{\alpha\beta} - \partial_\alpha \xi_\beta - \partial_\beta \xi_\alpha \qquad (*)$$

→ An infinitesimal coordinate change has the effect of re-defining

$$h_{\alpha\beta} \longrightarrow h_{\alpha\beta} - \partial_\alpha \xi_\beta - \partial_\beta \xi_\alpha$$

as far as the perturbation remains <u>small</u> the transformation is allowed!

Moreover, one can verify that the linearized Einstein tensor is invariant such transf.

<u>Observations</u>

- If one considers ξ^α as the components of a vector, then from the definition of <u>Lie derivative</u>:

$$\mathcal{L}_\xi \eta_{\mu\nu} = 2\,\partial_{(\mu}\xi_{\nu)}$$

The infinitesimal coordinate transformation is then interpreted as an infinitesimal diffeomorphism generated by ξ^α:

$$h_{\mu\nu} \longrightarrow h_{\mu\nu} + \mathcal{L}_\xi \eta_{\mu\nu}$$

- The above transformation is the analogue of the <u>gauge transformation</u> in Maxwell theory:

$$A_\alpha \longrightarrow A_\alpha + \partial_\alpha X$$

- (*) Again, the meaning of the equation is: $g_{\alpha\beta}(x') = \eta_{\alpha\beta} + h_{\alpha\beta}(x) - 2\,\partial_{(\beta}\xi_{\alpha)}$, i.e. the α,β component of g in coordinate x' is equal the α,β component of η in coordinate x, plus the α,β component of h coordinate x, plus the $\partial\xi$ in coordinate x.

WEAK FIELD EQUATIONS

The linearized Einstein tensor

$$G_{\mu\nu} = \underbrace{\partial^\alpha \partial_{(\mu} h_{\nu)\alpha}}_{\mathrm{I.}} \underbrace{- \frac{1}{2} \eta_{\alpha\beta} \partial^\alpha \partial^\beta h_{\mu\nu}}_{\mathrm{II.}} \underbrace{- \frac{1}{2} \partial_\mu \partial_\nu h}_{\mathrm{III.}} \underbrace{- \frac{1}{2} \eta_{\mu\nu} \partial^\alpha \partial^\beta h_{\alpha\beta}}_{\mathrm{IV.}} \underbrace{+ \frac{1}{2} \eta_{\mu\nu} \eta_{\alpha\beta} \partial^\alpha \partial^\beta h}_{\mathrm{V.}}$$

can be written in a simpler form considering the <u>trace-reverse variable</u>:

$$\bar{h}_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} h \quad ,$$

Note that : $\bar{h} \equiv \eta^{\mu\nu} \bar{h}_{\mu\nu} = \underbrace{\eta^{\mu\nu} h_{\mu\nu}}_{h} - \frac{1}{2} \underbrace{\eta^{\mu\nu} \eta_{\mu\nu}}_{4} h = -h$.

Consider the different terms and write them in terms of $\bar{h}_{\mu\nu}$:

$$\mathrm{I.}\quad \partial^\alpha \partial_{(\mu} h_{\nu)\alpha} = \partial^\alpha \partial_{(\mu} \bar{h}_{\nu)\alpha} + \frac{1}{4} \partial^\alpha \partial_\mu (\eta_{\nu\alpha} h) + \frac{1}{4} \partial^\alpha \partial_\nu (\eta_{\mu\alpha} h) = \underbrace{\partial^\alpha \partial_{(\mu} \bar{h}_{\nu)\alpha}}_{\mathrm{Ia.}} + \underbrace{\frac{1}{2} \partial_{(\mu} \partial_{\nu)} h}_{\mathrm{Ib.}}$$

$$\mathrm{II.}\quad -\frac{1}{2} \eta_{\alpha\beta} \partial^\alpha \partial^\beta h_{\mu\nu} = \underbrace{-\frac{1}{2} \eta_{\alpha\beta} \partial^\alpha \partial^\beta \bar{h}_{\mu\nu}}_{\mathrm{IIa.}} \underbrace{- \frac{1}{4} \eta_{\alpha\beta} \partial^\alpha \partial^\beta h \, \eta_{\mu\nu}}_{\mathrm{IIb.}}$$

$$\mathrm{IV.}\quad -\frac{1}{2} \eta_{\mu\nu} \partial^\alpha \partial^\beta h_{\alpha\beta} = -\frac{1}{2} \eta_{\mu\nu} \partial^\alpha \partial^\beta \left(\bar{h}_{\alpha\beta} + \frac{1}{2} \eta_{\alpha\beta} h \right) = \underbrace{-\frac{1}{2} \eta_{\mu\nu} \partial^\alpha \partial^\beta \bar{h}_{\alpha\beta}}_{\mathrm{IVa.}} \underbrace{- \frac{1}{4} \eta_{\mu\nu} \eta_{\alpha\beta} \partial^\alpha \partial^\beta h}_{\mathrm{IVb.}}$$

Note:

$$\mathrm{V} + \mathrm{IIb.} + \mathrm{IVb} = 0$$

$$\mathrm{III} + \mathrm{Ib.} = 0$$

$$G_{\mu\nu} = \mathrm{Ia.} + \mathrm{IIa} + \mathrm{IVa}$$

$$\boxed{G_{\mu\nu} = -\frac{1}{2} \eta_{\alpha\beta} \partial^\alpha \partial^\beta \bar{h}_{\mu\nu} + \partial^\alpha \partial_{(\mu} \bar{h}_{\nu)\alpha} - \frac{1}{2} \eta_{\mu\nu} \partial^\alpha \partial^\beta \bar{h}_{\alpha\beta}}$$

From the expression above one can further note that the last 2 terms contain $\sim \partial^\alpha \bar{h}_{\mu\alpha}$, this suggest to fix the gauge as :

$$\partial^\alpha \bar{h}_{\mu\alpha} = 0 \qquad \underline{\text{Hilbert gauge}}$$

How to fix Hilbert gauge?

$$h_{\mu\nu} \to h_{\mu\nu} + 2\partial_{(\mu}\xi_{\nu)}$$

$$\bar{h}_{\mu\nu} \to \bar{h}_{\mu\nu} + 2\partial_{(\mu}\xi_{\nu)} - \eta_{\mu\nu}\partial_\lambda \xi^\lambda$$

$$\partial^\mu \bar{h}_{\mu\nu} \to \partial^\mu \bar{h}_{\mu\nu} + \Box \xi_\nu + \cancel{\partial^\mu \partial_\nu \xi_\mu} - \cancel{\partial_\nu \partial_\lambda \xi^\lambda} = \partial^\mu \bar{h}_{\mu\nu} + \Box \xi_\nu$$

if $\partial^\mu \bar{h}_{\mu\nu} \neq 0$, then one solves $\Box \xi_\nu = -\partial^\mu \bar{h}_{\mu\nu}$ for ξ_ν and make a transformation

Linearized GR in Hilbert gauge:

$$\boxed{\Box \bar{h}_{\mu\nu} = -16\pi T_{\mu\nu}}$$

Observations:

- linear wave equation for the components of $\bar{h}_{\mu\nu}$
- $T_{\mu\nu} = T_{\mu\nu}[\eta]$, so one can specify the source in this case and solve for the gravitational field using Green functions.
- The Bianchi identity in weak field reads

$$\partial_\nu G^{\mu\nu} = 0$$

since the derivative associated to $\eta_{\mu\nu}$ is simply ∂_μ. Hence, the weak field equations imply that

$$\partial_\nu T^{\mu\nu} = 0 .$$

SOLUTION FOR STATIC SOURCE : WEAK FIELD METRIC

Consider a vector along the "time direction" of the global inertial coordinates:

$$t^a \equiv \left(\frac{\partial}{\partial x^0}\right)^a$$

Assume the stress-energy tensor is of the form:

$$T_{\mu\nu} = \rho\, t_\mu t_\nu \qquad \text{i.e.} \quad T_{00} = \rho \;,\; T_{0i} = 0 = T_{ij}$$

where ρ is a scalar describing the energy-density of the matter, and it does <u>not</u> depend on time. Then one can also assume that the gravitational field is time-independent:

$$\partial_t \bar{h}_{\mu\nu} = 0 \;.$$

The linearized EFE reads:

$$\begin{cases} \Delta \bar{h}_{\mu\nu} = -16\pi\rho & \mu=\nu=0 \\ \Delta \bar{h}_{\mu\nu} = 0 & \text{otherwise} \end{cases}$$

Comparing to Newton equations for the grav. potential ϕ one obtains:

$$\begin{cases} \bar{h}_{\mu\nu} = -4\phi & \mu=\nu=0 \\ \bar{h}_{\mu\nu} = 0 & \text{otherwise} \end{cases}$$

with: $\Delta\phi = 4\pi\rho$. In compact notation one can write:

$$\bar{h}_{\mu\nu} = -4\phi\, t_\mu t_\nu$$

$$\Rightarrow \qquad \bar{h} = \eta^{\mu\nu}\bar{h}_{\mu\nu} = +4\phi$$

$$h_{\mu\nu} = \bar{h}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\bar{h} = -4\phi\, t_\mu t_\nu - \frac{1}{2}\eta_{\mu\nu}(4\phi)$$

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} = \eta_{\mu\nu} + \bar{h}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\bar{h} = \eta_{\mu\nu}\left(1 - \frac{\bar{h}}{2}\right) + \bar{h}_{\mu\nu}$$

$$g_{\mu\nu} = \eta_{\mu\nu}(1-2\phi) + (-4\phi\, t_\mu t_\nu)$$

thus:

$$\boxed{g = -(1+2\phi)\,dt^2 + (1-2\phi)\,\delta_{ij}\,dx^i dx^j}$$

Note that far from the source one has $\phi \sim -\frac{M}{r} + \mathcal{O}(r^{-2})$.

SOLUTION FOR NON STATIC SOURCE WITH NO STRESSES

Consider a source with mass-energy current vector J^μ; the stress-energy tensor is assumed to be:

$$T_{\mu\nu} = 2\, J_{(\mu} t_{\nu)} - 2\rho\, t_\mu t_\nu$$

and the spatial terms are now neglected $T_{ij} \equiv 0$.

Linearized EFE are:

$$\begin{cases} \Box \bar{h}_{\mu 0} = -16\pi\, T_{\mu 0} & \mu = 0 \ldots 4 \\ \Box \bar{h}_{ij} = 0 & i,j = 1,2,3 \end{cases} \quad (\times)$$

Assuming $\partial_t \bar{h}_{ij} = 0 \quad \Rightarrow \quad \Delta \bar{h}_{ij} = 0 \quad \Rightarrow \quad \bar{h}_{ij} = 0$

The field is fully determined by $\bar{h}_{\mu 0}$.

However $(\times)$ is "Maxwell equation" for the field:

$$A_\mu \equiv -\frac{1}{4}\bar{h}_{\mu 0} = -\frac{1}{4}\bar{h}_{\mu\nu} t^\nu$$

in Lorentz gauge!

The metric is given by [Exercise]:

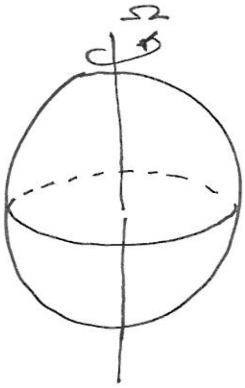
$$g_{00} = -1 + 2A_0 \;; \quad g_{0i} = 4A_i \;; \quad g_{ij} = (1+2A_0)\,\delta_{ij}$$

$$= -\left(1 + \frac{2\phi}{c^2}\right) \qquad = \frac{4}{c}A_i \qquad = \left(1 - \frac{2\phi}{c^2}\right)\delta_{ij}$$

If one assumes $\bar{h}_{\mu 0}$ time-independent, then a formal solution for the potential is:

$$\partial_t \bar{h}_{\mu 0} = 0 \quad \Rightarrow \quad \begin{cases} A_0 = -\phi & : \quad \Delta\phi = 4\pi\rho \\ A_i = \displaystyle\int \frac{T_{0i}(\vec{x}')}{|\vec{x} - \vec{x}'|} \, d^3x' \end{cases} .$$

An example of source generating the above stationary field is a (weakly gravitating) planet or star in slow rotation. We might write in this case:



$$J^\mu = \rho u^\mu = \rho \, (1, 0, 0, v_\varphi)$$

and study the motion of a test-body in the field of the rotating object. The Lagrangian is

$$L = \sqrt{-g_{\mu\nu} \frac{dx^\mu}{dt} \frac{dx^\nu}{dt}} \overset{\bullet}{=} \sqrt{-\left(-c^2 + 2c^2 A_0 + 2 \cdot 4 c A_i \frac{v^i}{c} + \frac{v^i v_i}{c^2} + \underbrace{+2 A_0 \frac{v^i v_i}{c^2}}_{O(c^{-4})}\right)}$$

$$\approx \frac{1}{2} \vec{v}^2 + A_0 + 4 \vec{A} \cdot \vec{v}$$

Thus, the equations of motion at lowest order are similar to those of a charged particle subjected to Lorentz force:

$$\ddot{\vec{x}} = \vec{E} + 4 \dot{\vec{x}} \times \vec{B}$$

with the differences:

- there is no charge here!
- $\vec{E}$ and $\vec{B}$ are calculated from $A_\mu = -\frac{1}{4} \bar{h}_{\mu 0}$ using the same formula as if A_μ was the EM potential
- there is a factor "4" in front of $\vec{B}$.

The fields $\vec{E}$ and $\vec{B}$ are the gravitoelectric and gravitomagnetic fields.

• Remember: $x^0 = ct$, $A_0 = O(c^{-2})$, $A_i = O(c^{-1})$

$$\left.\begin{aligned} & x^0 = ct \\ & A_0 = O(c^{-2}) \\ & A_i = O(c^{-1}) \end{aligned}\right\} \Rightarrow \underbrace{-c^2 + 2c^2 A_0}_{O(1)} + \underbrace{8 c A_i \frac{v^i}{c}}_{O(\frac{1}{c})} + \underbrace{\frac{v^i v_i}{c^2}}_{O(\frac{1}{c^2})} + \underbrace{2 A_0}_{O(\frac{1}{c^2})} \underbrace{\frac{v^i v_i}{c^2}}_{O(\frac{1}{c^2})}$$

Example : Lense-Thirring effect

The precession motion of a gyroscope due to the gravitomagnetic field of a rotating object can be obtained from the spin-precession formula in electromagnetism :

$$\frac{d\vec{S}}{dt} = \underbrace{\vec{\mu}}_{\substack{\text{magnetic moment}\\ \text{of the electron}}} \times \vec{B} = \frac{1}{2}\frac{e}{m}\vec{S}\times\vec{B}$$

where :

$$\frac{d\vec{S}}{dt} = \vec{\Omega}\times\vec{B} \quad \Rightarrow \quad \Omega = -\frac{e}{m}B \qquad \text{(EM)}$$

with the substitution $e \to m$, $\vec{B} \to 4\vec{B}$ one obtains the gravitomagnetic precession frequency :

$$\Omega_{LS} = -2\vec{B} = -2\,\nabla\times\vec{A} \qquad \text{(GR)} .$$

In the field of the Earth one obtains that a free-falling body acquires an angular velocity :

$$\Omega = \frac{d\varphi}{dt} \sim 0.22''\ \text{year}^{-1}\left(\frac{R_\oplus}{r}\right)^3$$

this "frame dragging" effect has been measured by the satellite mission Gravity-Probe B in 2004 with confidence ~ 20%..

An extreme version of this phenomenon happens around rotating black-holes (not weak-field!). Particles close to the black-hole horizon are "dragged around" at a speed comparable to the rotation of the black-hole

$$\Omega \sim \Omega_H .$$

The Lense-Thirring effect in strong-field plays an important role to understand high-energy particle emission from matter accreting black-holes (quasar, AGN, ...)

GRAVITATIONAL WAVE PROPAGATION

The linearized EFE <u>in vacuum</u> are homogeneous wave equations

$$\Box_\eta \bar{h}_{\alpha\beta} = \eta^{\mu\nu}\partial_\mu\partial_\nu \bar{h}_{\alpha\beta} = 0$$

for each component of the metric perturbation.

Solutions can be constructed as superposition of <u>plane waves</u>: $\bar{h}_{\alpha\beta} = A_{\alpha\beta}\, e^{ik_\mu x^\mu}$

where $k^\mu = \text{const}$ is the wave vector. Using the fact that

$$\partial_\mu \bar{h}_{\alpha\beta} = A_{\alpha\beta}\,\partial_\mu e^{ik_\rho x^\rho} = \bar{h}_{\alpha\beta}\,\partial_\mu(ik_\rho x^\rho) = i\,\bar{h}_{\alpha\beta}\,k_\rho\,\delta^\rho_{\ \mu} = ik_\mu \bar{h}_{\alpha\beta},$$

and plugging the plane wave ansatz into the wave equation one obtains:

$$\left.\begin{array}{l}\Box \bar{h}_{\alpha\beta} = 0 \\ \bar{h}_{\alpha\beta} = A_{\alpha\beta}\, e^{ik_\mu x^\mu}\end{array}\right\} \rightarrow -\eta^{\mu\nu} k_\mu k_\nu \bar{h}_{\alpha\beta} = 0 \implies \eta^{\mu\nu}k_\mu k_\nu = k_\mu k^\mu = 0;$$

$\implies$ the wave vector is NULL (in Minkowski / flat background).

A photon moving along k^μ has worldline:

$$x^\mu(\lambda) = k^\mu\lambda + x_0^{\ \mu}, \qquad x_0^{\ \mu} \equiv x^\mu(0) \ \text{initial value}$$

and because

$$k_\mu x^\mu(\lambda) = \underbrace{k_\mu k^\mu}_{=0}\lambda + k_\mu x_0^{\ \mu} = k_\mu x_0^{\ \mu} = \text{const},$$

it moves along with the wave front, staying "in phase" with:

$$\phi = k_\mu x^\mu = -k_0 x^0 + \vec{k}\cdot\vec{x} = -\omega t + \vec{k}\cdot\vec{x}$$

where we have defined $k^\mu = (\omega, \vec{k})$ and $k_\mu k^\mu = 0 \implies$ the <u>dispersion relation</u>:

$$\omega^2 = |\vec{k}|^2$$

$\rightarrow$ GWs propagate at the speed of light.

Note that in general the frequency observed by an observer of 4-velocity u^μ is $\omega = -k_\mu u^\mu$.

Transverse – Traceless (TT) gauge and physical degrees of freedom (d.o.f.)

Linearized EFE in vacuum are $\underline{10}$ equations.

Hilbert gauge : $0 = \partial^\mu \bar{h}_{\mu\nu} = i k^\mu A_{\mu\nu} e^{i k_\rho x^\rho} \rightarrow k^\mu A_{\mu\nu} = 0$ 4 equations

$\hookrightarrow$ $10 - 4 = 6$ remaining d.o.f.

Consider an infinitesimal coord. transformation :

$$x^\alpha \longrightarrow x'^\alpha = x^\alpha + \xi^\alpha$$

$$h_{\alpha\beta} \longrightarrow h'_{\alpha\beta} = h_{\alpha\beta} - 2\,\partial_{(\alpha}\xi_{\beta)}$$

$$h = \eta^{\alpha\beta} h_{\alpha\beta} \rightarrow h' = h - \eta^{\alpha\beta}\partial_\alpha \xi_\beta - \eta^{\alpha\beta}\partial_\beta \xi_\alpha = h - 2\partial_\alpha \xi^\alpha$$

$$\bar{h}_{\alpha\beta} = h_{\alpha\beta} - \frac{1}{2}\eta_{\alpha\beta} h \longrightarrow \bar{h}'_{\alpha\beta} = h_{\alpha\beta} - 2\,\partial_{(\alpha}\xi_{\beta)} + \frac{1}{\cancel{2}}\cancel{2}\,\eta_{\alpha\beta}\partial_\mu \xi^\mu - \frac{1}{2}\eta_{\alpha\beta} h =$$

$$= \bar{h}_{\alpha\beta} - 2\,\partial_{(\alpha}\xi_{\beta)} + \eta_{\alpha\beta}\partial_\alpha \xi^\mu$$

Define : $V_\alpha \equiv \partial^\mu \bar{h}_{\mu\alpha}$

$$V_\alpha \rightarrow V'_\alpha = V_\alpha - \underline{\eta^{\mu\beta}\partial_\mu\partial_\alpha \xi_\beta} - \eta^{\mu\beta}\partial_\mu\partial_\beta \xi_\alpha + \underbrace{\eta^{\mu\beta}\eta_{\alpha\beta}}_{\delta^\mu{}_\alpha}\partial_\mu\partial_\rho \xi^\rho$$

$$= \partial_\alpha \partial_\rho \xi^\rho$$

$$= \underline{\eta^{\rho\beta}\partial_\rho\partial_\alpha \xi_\beta}$$

$$= V_\alpha - \eta^{\mu\beta}\partial_\mu\partial_\beta \xi_\alpha =$$

$$= V_\alpha - \Box \xi_\alpha$$

Hence :

$$V'_\alpha = 0 \iff \Box \xi_\alpha = V_\alpha$$

$\rightarrow$ the Hilbert gauge is defined up to 4 harmonic functions !

There is a remaining freedom to chose these 4 harmonic functions. Let

$$\xi^\alpha := B^\alpha e^{i k_\mu x^\mu}$$

with this assignment ξ^α are solutions of the wave equation in vacuum; they depend on 4 constants B^α. Fix these 4 constants by requiring that:

$$\begin{cases} A^\mu{}_\mu = 0 & \leftrightarrow \quad \bar{h} = 0 \quad \leftrightarrow \quad \bar{h}_{\mu\nu} = h_{\mu\nu} \\ A_{0\mu} = 0 & \leftrightarrow \quad \bar{h}_{0\mu} = 0 \end{cases}$$

4 Equations:

$6 - 4 = 2$ remaining d.o.f. $\Rightarrow$ <u>physical degrees of freedom of GWs</u> (see later).

In order to impose the 4 conditions one simply observes that:

$$\bar{h}_{\mu\nu} \to \bar{h}_{\mu\nu} - 2\partial_{(\mu}\xi_{\nu)} - \eta_{\mu\nu}\partial_\lambda \xi^\lambda \quad \Rightarrow$$

$$A_{\mu\nu} \to A_{\mu\nu} - i\,2 K_{(\mu}B_{\nu)} + i\,\eta_{\mu\nu} K_\lambda B^\lambda$$

and the 4 equations become a <u>linear system</u> in B^μ which is invertible...

The conditions:

$$\begin{cases} \partial^\mu \bar{h}_{\mu\nu} = 0 & \\ \bar{h} = 0 & : \text{Traceless} \\ \bar{h}_{0\mu} = 0 & : \text{Transverse to direction } \partial_t \end{cases}$$

fix the gauge called <u>"Transverse-Traceless"</u> or <u>"TT" gauge</u>.

Choose a wave propagating along the $\hat{z}$ axis:

$$K^\mu = (\omega, 0, 0, K_3)$$

then:

- $K_\mu K^\mu = 0 \quad \to \quad -K_3 = \omega$
- Hilbert gauge $\quad K^\mu A_{\mu\nu} = 0 \quad \to \quad K^0 A_{0\nu} + K^3 A_{3\nu} = \omega A_{0\nu} - \omega A_{3\nu} = 0$
- Transverse condition $\quad A_{0\mu} = 0 \quad \longrightarrow \quad A_{3\nu} = 0$

we have then that

$$A_{\mu\nu} = \begin{pmatrix} \boxed{0 \quad 0 \quad 0 \quad 0} \\ 0 \quad A_{11} \quad A_{12} \quad 0 \\ 0 \quad A_{12} \quad A_{22} \quad 0 \\ \boxed{0 \quad 0 \quad 0 \quad 0} \end{pmatrix}$$

→ transverse gauge condition

→ Hilbert + transverse gauge condition

by symmetry

- Traceless condition $\rightarrow$ $A_{22} = -A_{11}$

Hence:

$$h^{TT}_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & A_+ & A_+ & 0 \\ 0 & A_+ & -A_+ & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} e^{i k_\sigma x^\sigma}$$

with $k_\sigma x^\sigma = \omega t - \omega z = \omega\left(t - \frac{z}{c}\right)$.

Define:

$$h_+\left(t - \frac{z}{c}\right) \equiv a_+ \, e^{i\omega\left(t - \frac{z}{c}\right)}$$

$$h_\times\left(t - \frac{z}{c}\right) \equiv a_\times \, e^{i\omega\left(t - \frac{z}{c}\right)}$$

Polarizations of the GWs

Observations

- the TT gauge can be defined <u>only in vacuum</u>.

  In case matter is present: $\Box \bar{h}_{\mu\nu} \neq 0$ (Hilbert gauge)

  we still have the freedom to rescale $\bar{h}_{\mu\nu}$ with an infinitesimal coord. transformation "generated by harmonic functions" $\Box \xi^\alpha = 0$, BUT we <u>cannot</u> set to zero components of $\bar{h}_{\mu\nu}$ inside a source.

- the <u>metric</u> in TT gauge reads:

$$g = -dt^2 + dz^2 + (1 + h_+)dx^2 + (1 - h_+)dy^2 + 2h_\times \, dx\,dy \qquad \text{(propagation along } \hat{z}\text{)}$$

$$= -dt^2 + \left(\delta_{ij} + h^{TT}_{ij}\right)dx^i dx^j \qquad \text{(in general)}$$

• Given a plane wave solution $\bar{h}_{\mu\nu}$ in Lorentz gauge propagating in direction $\hat{n}$ and outside the source, one obtains the solution in TT gauge by the following projection:

$$h_{ij}^{TT} = \Lambda_{ij,kl}\, \bar{h}_{kl} \qquad \text{(sum over } k,l\text{)}$$

where:

$$\Lambda_{ij,kl}(\hat{n}) \equiv P_{ik} P_{jl} - \frac{1}{2} P_{ij} P_{kl}$$

$$P_{ij}(\hat{n}) \equiv \delta_{ij} - n_i n_j \; .$$

One can show [exercise]:

- P_{ij} is symmetric
- P_{ij} is transverse : $n^i P_{ij} = 0$
- P_{ij} is a projector : $P_{ik} P_{kj} = P_{ij}$
- P_{ij} has trace $P_{ii} = 2$
- Λ is a projector : $\Lambda_{ij,kl} \Lambda_{kl,mn} = \Lambda_{ij,mn}$
- Λ is transverse in all indexes
- Λ is traceless in ij and kl : $\Lambda_{ii,kl} = \Lambda_{ij,kk} = 0$
- Λ is symmetric in ij kl : $\Lambda_{ij,kl} = \Lambda_{kl,ij}$

and obviously $\Box \bar{h}_{ij} = 0 \Rightarrow \Box h_{ij}^{TT} = 0$.

More in general, for any symmetric tensor S_{ij} one obtains a symmetric, transverse and traceless (STF) tensor using the Λ projector:

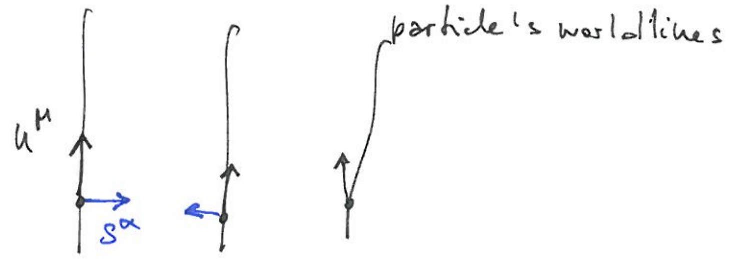
$$S_{ij}^{STF} = \Lambda_{ij,kl}\, S_{kl} \; .$$

EFFECT OF GWs ON TEST-MASSES

To study the effect of GW on test masses one considers the geodesic <u>deviation equation</u>

$$u^\mu \nabla_\mu u^\nu \nabla_\nu S^\alpha = R^\alpha{}_{\nu\rho\sigma} u^\nu u^\rho S^\sigma$$

for the displacement vector S^α



and thus study, in a coordinate independent way, the "relative motion" of nearby particles (geodesics) at the passage of the wave.

In linearized GR, Riemann $\approx \mathcal{O}(h)$ and particle that are <u>initially at rest</u>* acquire a velocity due to the perturbation:

$$\frac{dx^i}{d\tau} \approx \mathcal{O}(h)$$

and similarly

$$\frac{dx^0}{d\tau} \approx 1 + \mathcal{O}(h)$$

Hence:

$$u^\mu = c\,\frac{dx^\mu}{d\tau} \approx (1,0,0,0) + \mathcal{O}(h)$$

because the Riemann is $\mathcal{O}(h)$. Using coordinate time t in place of τ, one finds:

$$\frac{d^2 S^\alpha}{dt^2} = R^\alpha{}_{00\sigma} S^\sigma$$

and the Riemann tensor can be expressed in TT gauge as:

---

* We work in the global inertial reference system "of the background metric"...

$$R_{\mu 00 \nu} = \frac{1}{2} \frac{\partial^2}{\partial t^2} h_{\mu\nu}^{TT} \qquad (\text{TT gauge}).$$

This formula shows that the 2 d.o.f. of GW are physical: they cannot be further gauged away.

Restricting to the spatial indexes, the geodesic deviation gives:

$$\frac{d^2}{dt^2} s^i = \frac{1}{2} \ddot{h}_{ij}^{TT} s^j$$

Let: $s^i(t) = s_0^i + \delta s^i(t)$ where $s_0^i = s^i(t=0)$ is the initial position of the masses before the wave arrives. At first order in $h_{\alpha\beta}$ the above equation is:

$$\frac{d^2}{dt^2} \delta s^i = \frac{1}{2} \ddot{h}_{ij}^{TT} s_0^j$$

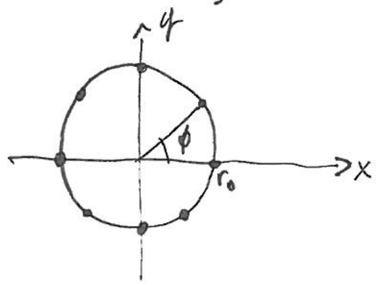
If $\dot{s}^i(0) = 0$, then the solution is

$$\delta s^i(t) = \left(\delta_{ij} + \frac{1}{2} h_{ij}^{TT}(t) \right) s_0^j .$$

For a wave propagating along $\hat{z}$ one has:

$$\begin{cases} \delta x^i(t) = x_0 + \frac{1}{2}\left(h_+(t)\, x_0 + h_\times(t)\, y_0\right) \\ \delta y^i(t) = y_0 + \frac{1}{2}\left(-h_+(t)\, y_0 + h_\times(t)\, x_0\right) \\ \delta z^i(t) = z_0 \end{cases}$$

Consider a set of test masses positioned initially on a circle in the xy



and let

$$\begin{cases} x_0 = r_0 \cos\phi \\ y_0 = r_0 \sin\phi \end{cases}$$

The effect of the "+" polarization ($h_x=0$) is:

$$\begin{cases} \delta x^i(t) = x_0 + \frac{1}{2} h_+(t)\, x_0 = r_0 \cos\phi \left(1 + \frac{1}{2} h_+(t)\right) \\ \delta y^i(t) = y_0 - \frac{1}{2} h_+(t)\, y_0 = r_0 \sin\phi \left(1 - \frac{1}{2} h_+(t)\right) \end{cases}$$

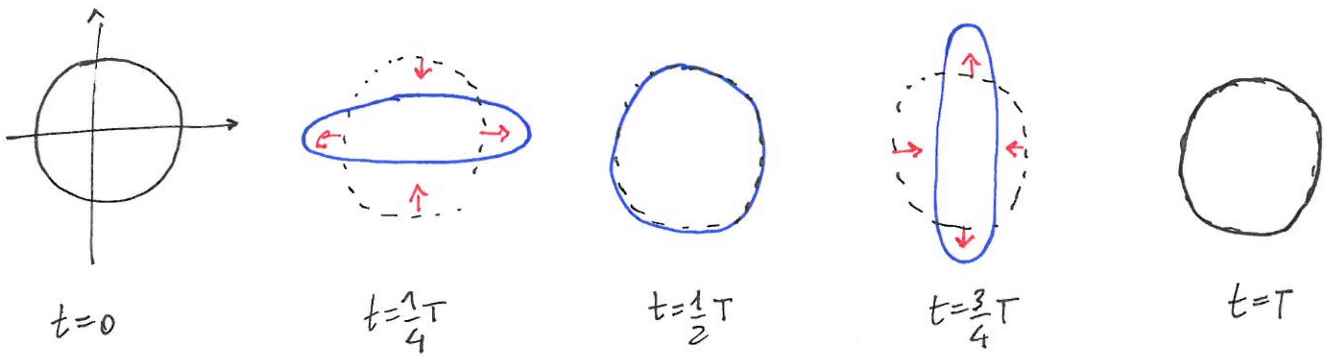
take the square and sum up the equations to eliminate "ϕ":

$$\frac{\delta x^{i\,2}}{r_0^2 \left(1+\frac{1}{2} h_+\right)^2} + \frac{\delta y^{i\,2}}{r^0 \left(1-\frac{1}{2} h_+\right)^2} = \cos^2\phi + \sin^2\phi = 1$$

the above equation is an ellipsis of semi-major axes along x and y given by:

$$A \equiv r_0 \left(1 + \frac{1}{2} h_+\right)$$
$$B \equiv r_0 \left(1 - \frac{1}{2} h_+\right)$$

Because $h_+(t)$ is an oscillating function, one axis gets shorter while the other gets longer, periodically. Hence (T is the period of h_+):



Similarly, the effect of the "x" polarization (h_+) is:

$$\begin{cases} \delta x^i = x_0 + \frac{1}{2} h_x(t)\, y_0 \\ \delta y^i = y_0 + \frac{1}{2} h_x(t)\, x_0 \end{cases}$$

the RHS matrix

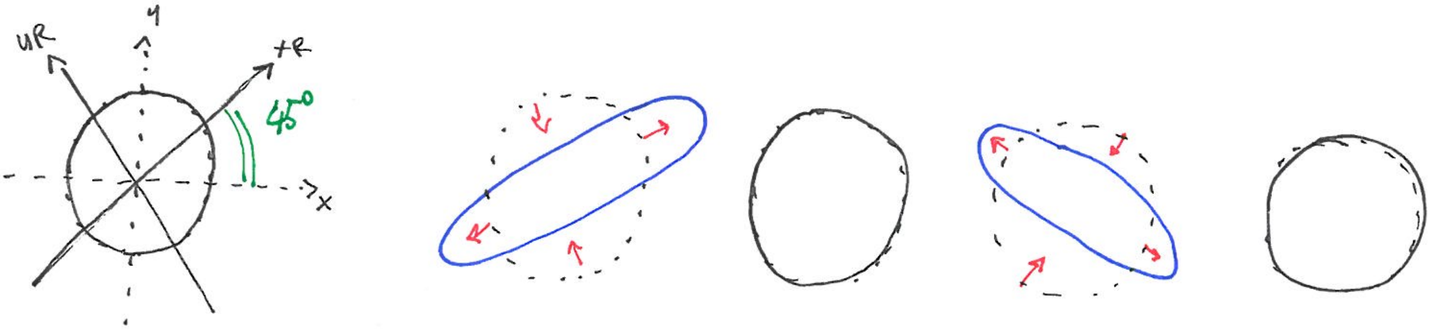
$$M = \begin{pmatrix} 1 & +\frac{1}{2} h_x \\ \frac{1}{2} h_x & 1 \end{pmatrix}$$

can be diagonalized with a 45° rotation:

Eigenvalues: $\lambda = \pm \frac{1}{2} h_x$ Eigenvectors: $R = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \Big|_{\alpha=\frac{\pi}{4}}$

If one performs the rotation, one finds in the rotated frame the same equation for the "+" polarization:

$$\begin{cases} \delta x_R^i = x_{0R} + \frac{1}{2} h_x \, x_{0R} \\ \delta y_R^i = y_{0R} - \frac{1}{2} h_x \, y_{0R} \end{cases} \Rightarrow$$



$t=0$ $t=\frac{1}{4}T$ $t=\frac{1}{2}T$ $t=\frac{3}{4}T$ $t=T$

SOURCES OF GWs

Consider the full linearized EFE :

$$\Box \bar{h}_{\alpha\beta} = -16\pi T_{\alpha\beta}$$

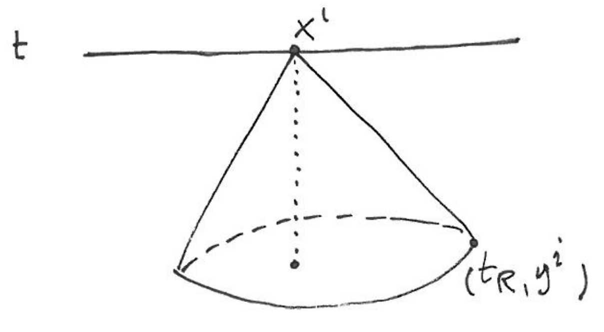
a formal solution to this equation can be given in terms of Green function with retarded time :

$$t_R \equiv t - \frac{1}{c}|\vec{x} - \vec{x}'|$$

$$\bar{h}_{\alpha\beta}(t, \vec{x}) = 4 \int \frac{T_{\alpha\beta}(t_R, \vec{x}')}{|\vec{x} - \vec{x}'|} d^3x'$$

with $\vec{x} = (x, y, z)$, $|\vec{x}| = \sqrt{\delta_{ij} x^i x^j} \equiv r$ and $\vec{n} \equiv \frac{\vec{x}}{r}$.

The physical picture is that the solution of time t must be calculated in terms of events in the past light cone :



The solution of the inhomogeneous equation is obtained as :

$$\bar{h}_{\mu\nu} = -16\pi \int G(x - x') T_{\mu\nu}(x') d^4x'$$

where "G" is the solution of :

$$\Box_{(x)} G(x - x') = \delta^4(x - x')$$

given by the retarded component only :

$$G_R(x - x') = -\frac{1}{4\pi} \frac{1}{|\vec{x} - \vec{x}'|} \delta(t_R - t) .$$

We evaluate the solution under the hypothesis:

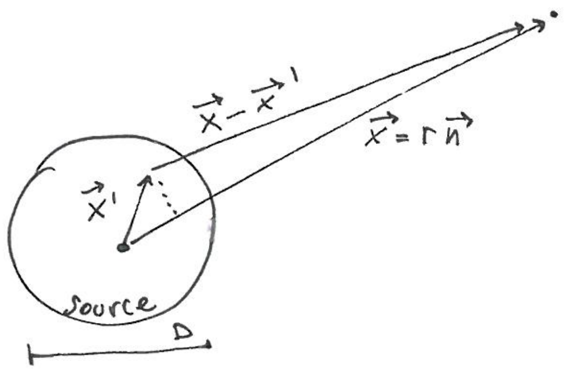
i) large distance from the source.

ii) sourc motion is slow compare to c.

Note in addition that the source should not be self-gravitating since we are using weak field equations.

i) $\Rightarrow$ $|\vec{x}-\vec{x}'| = r\left|\vec{n}-\frac{\vec{x}'}{r}\right| = r\sqrt{\left(\vec{n}-\frac{\vec{x}'}{r}\right)\cdot\left(\vec{n}-\frac{\vec{x}'}{r}\right)} = r\sqrt{1-\vec{n}\cdot\frac{\vec{x}'}{r}+\underbrace{\left(\frac{\vec{x}'}{r}\right)^2}_{\substack{O(r^{-2})\\ \text{"small"}\\ \text{for } r\to\infty}}}$

$$\simeq r - \vec{x}'\cdot\vec{n}$$



$$\Rightarrow \quad \bar{h}_{\alpha\beta} \cong \frac{4}{r}\int T_{\alpha\beta}\left(t-\frac{r}{c}+\frac{\vec{n}\cdot\vec{x}'}{c}, \vec{x}'\right)d^3x'$$

retain only the leading order term in $|\vec{x}-\vec{x}'|$

ii) $\Rightarrow$ if ω is a characteristic frequency related to motion of the source, then $\omega\frac{|\vec{x}'|}{c} \ll 1$, where $|\vec{x}'|\sim D$ the typical size of the source.

$\Rightarrow$ one can neglect the term $\frac{\vec{n}\cdot\vec{x}'}{c}$ in the $T_{\alpha\beta}$ argument:

$$\bar{h}_{\alpha\beta} \cong \frac{4}{r}\int T_{\alpha\beta}(t-r, \vec{x}')\,d^3x'$$

Save this result for later and turn attention to the source.

the conservation law for $T_{\alpha\beta}$ gives:

$$0 = \partial^{\mu} T_{\alpha\mu} = \eta^{\mu\nu} \partial_\nu T_{\alpha\mu} \quad \begin{cases} \alpha = 0 : & -\partial_t T_{00} + \partial T_{0} = 0 \\ \alpha = k : & -\partial_t T_{k0} + \partial_\ell T_{k\ell} = 0 \end{cases}$$

Derive ∂_t the equation for $\alpha = 0$:

$$-\partial_{tt} T_{00} + \partial_t \partial_k T_{0k} = 0$$

substitute $\quad \partial T_0 = \partial_k \partial_\ell T_{kl}$ using the $\alpha = k$ equation and obtain:

$$-\partial_{tt} T_{00} + \partial_k \partial_\ell T_{kl} = 0$$

Multiply now for $x^i x^j$ and integrate:

$$\int \partial_{tt} T_{00} \, x^i x^j \, d^3x = \int \partial_k \partial_\ell T_{k\ell} \, x^i x^j \, d^3x =$$

$$= \int \Big[\underbrace{\partial_k \left(\partial_\ell T_{k\ell} \, x^i x^j \right)}_{\text{divergence term,}} - \partial_\ell T_{k\ell} \underbrace{\partial_k (x^i x^j)}_{= (\delta^i_k x^j + \delta^j_k x^i)} \Big] d^3x =$$

Gauss theorem $\int_V \to \int_S$

but because the source is compact the surface integral $\int_S = 0$

$$= -\int \left(\partial_\ell T_{i\ell} \, x^j + \partial_\ell T_{j\ell} \, x^i \right) d^3x =$$

$$= -\int \Big[\underbrace{\partial_\ell \left(T_{i\ell} x^j \right)}_{\text{divergence}} - T_{i\ell} \underbrace{\frac{\partial x^j}{\partial x^\ell}}_{\delta^j_\ell} + \underbrace{\partial_\ell \left(T_{j\ell} x^i \right)}_{\text{divergence}} + $$

$$- T_{j\ell} \underbrace{\frac{\partial x^i}{\partial x^\ell}}_{\delta^i_\ell} \Big] d^3x =$$

$$= +2 \int T_{ij} \, d^3x$$

Hence:

$$\frac{d^2}{dt^2} \int T_{00} \, x^i x^j \, d^3x = 2 \int T_{ij} \, d^3x$$

Note now that for slowly moving sources:

$$T_{00} = \mathcal{E} \simeq \rho c^2 \qquad (\gamma \sim 1)$$

$$\rightarrow \quad 2\int T_{ij}\, d^3x = \frac{1}{c^2}\frac{d}{dt^2}\int T_{00}\, x^i x^j\, d^3x = \frac{1}{c^2}\frac{d}{dt^2}\underbrace{\int c^2 \rho\, x^i x^j\, d^3x}_{\equiv I_{ij}} = \frac{d}{dt^2} I_{ij}$$

Moment of inertia tensor

Putting things together:

$$\boxed{\bar{h}_{ij}(t,\vec{x}) = \frac{2G}{c^4 r}\ddot{I}_{ij}\left(t - \frac{r}{c}\right)}$$

Gravitational waves are sourced by variations of the mass quadrupole ("first multipole with 2 indexes"...)

The solution outside the source ($|\vec{x}| \gg D$) can be projected to TT gauge form using:

$$h^{TT}_{ij} = \Lambda_{ij\,kl}\, \bar{h}_{kl} = \frac{2}{r}\Lambda_{ij\,kl}\, \ddot{I}_{kl}\left(t - \frac{r}{c}\right)$$

this formula is more often written as

$$\boxed{h^{TT}_{ij} = \frac{2G}{c^4 r}\Lambda_{ij\,kl}\, \ddot{Q}^{kl}\left(t - \frac{r}{c}\right)} \quad \text{Quadrupole formula}$$

where one defines the quadrupole moment of mass as:

$$Q_{ij} \equiv I_{ij} - \frac{1}{3}(\underbrace{\delta^{kl} I_{kl}}_{I})\,\delta_{ij} =$$

$$= \int \rho(t,\vec{x}')\left(x^i x^j - \frac{1}{3}\vec{x}\cdot\vec{x}\,\delta_{ij}\right) d^3x'$$

The quadrupole moment is the tensor that appear naturally in the multipolar expansion of the Newtonian potential :

$$\phi(t,\vec{x}) = -\frac{GM}{r} + \frac{3G\, Q_{ij}(t)\, n^i n^j}{2r^3} + \dots$$

[Note that the dipolar term is just the center of mass vector that can be "gauged away" using relative positions. The situation is different in electromagnetism in which the dipole term approximate the charge distribution.]

The quadrupole formula for GW represents only the first contribution of the matter source to the GW. In fact all the multipoles of the source should in principle appear in the expansion of the $T_{\alpha\beta}$; the expansion just start at the ($l=2$) quadrupolar order because of mass and momentum conservation : $\dot{M}=0$, $\dot{P}^i=0$.

Using dimension analysis one immediately finds that GWs are given by :

$$h \sim \left(\frac{R}{D}\right)\left(\frac{GM}{c^2 R}\right)\left(\frac{v}{c}\right)^2$$

where :

R : size of the source

D : distance from the source

M : mass of the source

v : velocity of the source

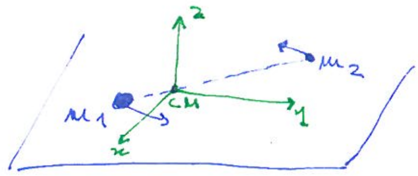
The formula indicate that GW are produced by physical systems that are :

- very compact
- strongly gravitating
- rapidly moving .

Exercise: Quadrupole moment of a Newtonian binary system

Consider a two-body system in Newtonian gravity, with the two objects (point-masses) at coordinates $\vec{x}_1$ and $\vec{x}_2$. Indicating m_1 and m_2 the two masses, the center of mass is

$$\vec{x}_{CM} = \frac{m_1 \vec{x}_1 + m_2 \vec{x}_2}{m_1 + m_2}$$



The <u>moment of inertia</u> is

$$I^{ij} = \int \rho x^i x^j d^3x = \int \left[m_1 \delta^3(\vec{x} - \vec{x}_1) + m_2 \delta^3(\vec{x} - \vec{x}_2) \right] x^i x^j d^3x =$$

$$= m_1 x_1^i x_1^j + m_2 x_2^i x_2^j = m\, x_{CM}^i x_{CM}^j + \mu r^i r^j$$

where:

$$m \equiv m_1 + m_2 \qquad \mu = \frac{m_1 m_2}{m} \quad \text{reduced mass}$$

$$\vec{r} \equiv \vec{x}_1 - \vec{x}_2 \quad \text{relative distance.}$$

In the CM-frame ($x_{CM}^i = 0$) one has simply:

$$\rho(t, \vec{x}) = \mu \delta^3(\vec{x} - \vec{r}) \quad \text{and} \quad I^{ij}(t) = \mu r^i(t) r^j(t)$$

Trace:

$$I = \delta_{ij} I^{ij} = \mu \delta_{ij} r^i r^j = \mu \vec{r} \cdot \vec{r} = \mu r^2$$

<u>Quadrupole moment</u>:

$$Q^{ij} = I^{ij} - \frac{1}{3} I \delta^{ij} = \mu \left(r^i r^j - \frac{1}{3} r^2 \delta^{ij} \right).$$

Specify now for a <u>circular orbit</u> in the plane (x, y):

$$\vec{r}(t) = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} \quad \text{with} \quad \begin{cases} x(t) = R \cos(\Omega t + \frac{\pi}{2}) \\ y(t) = R \sin(\Omega t + \frac{\pi}{2}) \\ z(t) = 0 \end{cases}$$

where $\frac{\pi}{2}$ is an initial phase (chosen arbitrarily to simplify later calculations) and

$$\Omega = \frac{2\pi}{T} = \left(\frac{G m}{R^3} \right)^{1/2} \quad \text{is the } \underline{\text{orbital frequency}}$$

as given by the <u>kepler law</u>. The latter can be derived by equating the gravitational force to the centripetal force:

$$\frac{G m_1 m_2}{r^2} = \mu \frac{v^2}{r}$$

and using the fact that the orbital period is the time to complete a revolution

$$T = \frac{2\pi R}{v}$$

$$\rightarrow \quad v^2 = \frac{Gm}{r} \quad \rightarrow \quad T = 2\pi R \left(G m R^{-1}\right)^{-1/2} = 2\pi \left(G m R^3\right)^{-1/2}$$

Note kepler law can be easily derived by dimensional analysis just remembering that it involves the quantities Ω, G, m and R [Exercise].

The inertia moment for circular orbit is then:

$$I^{11} = \mu R^2 \cos^2\left(\Omega t + \frac{\pi}{2}\right) = \mu R^2 \frac{1 + \cos(2\Omega t + \pi)}{2} = \mu R^2 \frac{1 - \cos(2\Omega t)}{2}$$

$$I^{22} = \mu R^2 \sin^2\left(\Omega t + \frac{\pi}{2}\right) = \mu R^2 \frac{1 - \cos(2\Omega t + \pi)}{2} = \mu R^2 \frac{1 + \cos(2\Omega t)}{2}$$

$$I^{12} = \mu R^2 \cos\left(\Omega t + \frac{\pi}{2}\right)\sin\left(\Omega t + \frac{\pi}{2}\right) = \mu R^2 \frac{1}{2}\left[\sin\left(\Omega t + \frac{\pi}{2} + \Omega t + \frac{\pi}{2}\right) + \sin(0)\right] =$$

$$= -\mu R^2 \frac{1}{2} \sin(2\Omega t)$$

$$I^{i3} = 0$$

From the formula: $\quad \bar{h}_{ij} \propto \frac{1}{r} \ddot{I}_{ij} \quad$ one concludes that:

- the GW emitted from a source "moving" at monocromatic frequency Ω, has frequency

$$\Omega_{GW} = 2\,\Omega$$

this is always true (any source) and just a consequence of the fact the radiation is generated by a "quadrupole".

[Note the "2Ω" comes from: $x^i x^j \rightsquigarrow \cos^2(\Omega t) \rightsquigarrow \cos(2\Omega t)$]

Taking the derivatives:

$$\ddot{I}_{11} = 2\mu R^2 \Omega^2 \cos(2\Omega t) \quad , \quad \ddot{I}_{22} = -\ddot{I}_{11} \quad ,$$
$$\ddot{I}_{12} = 2\mu R^2 \Omega^2 \sin(2\Omega t)$$

Away from the source, one can obtain the wave form in TT gauge by projecting using the STF projector:

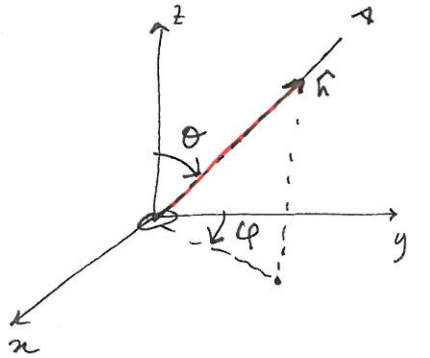
$$h_{ij}^{TT} = \frac{1}{r}\frac{2G}{c^4}\Lambda_{ij,kl}\ddot{I}^{kl}\left(t-\frac{r}{c}\right)$$

and obtain [calculations are left as exercise]:

$$h_+ = \frac{1}{r}\left(\ddot{I}_{11} - \ddot{I}_{22}\right) \quad , \quad h_\times = \frac{1}{r}\, 2\ddot{I}_{12}$$

that give, for a wave detected at distance r and propagating along $\hat{n}$:

$$h_+(t;\theta,\varphi) = \frac{1}{r}\frac{G}{c^4}\, 4\mu R^2\Omega^2 \left(\frac{1+\cos^2\theta}{2}\right)\cos(2\Omega t_R + \varphi)$$
$$h_\times(t;\theta,\varphi) = \frac{1}{r}\frac{G}{c^2}\, 4\mu R^2\Omega^2 \quad \cos\theta \quad \sin(2\Omega t_R + \varphi)$$



The generic angular dependence is obtained by rotating I_{ij} to I'_{ij} $(I' = R^T I R)$ and using the latter in the h^{TT} formula [see e.g. Maggiore book, Sec. 3.3].
The rotation gives the factors:

$$\sim \frac{1+\cos^2\theta}{2} \quad , \quad \cos\theta$$

for $h_{+,\times}$ that are very general for a source such that $I_{i3}=0$ $\ddot{I}_{22} = \ddot{I}_{11}$

The factor φ enter the phase simply because the rotation around z "picks" another point in the trajectory of the source (the latter is not rotational invariant in general). In case of circular orbit one simply pick a factor $\Delta\varphi = \Omega\Delta t$ with a rotation of $\Delta\varphi$.

ENERGY CARRIED BY GWs

GWs carry energy because they can do work (move) on test masses. However the definition of energy of "GW energy" is nontrivial and requires careful discussion.

In general:

NO DEFINITION OF LOCAL ENERGY DENSITY EXIST IN GR

Main difficulty: separate "background" from "dynamical" part of g_{ab}.

For isolated system one can define the notion of total energy (and total energy flux) at large distances from the system and assuming a flat background.

The logic goes as follows:

- Energy must be quadratic in the perturbation $h_{\alpha\beta}$, so one must consider 2nd order perturbations;
- Any form of energy must generate curvature through a stress-energy tensor;
- Far away from the source (in vacuum), one can find an effective stress energy tensor, quadratic in $h_{\alpha\beta}$, from which one can define a gauge invariant energy and energy flux. The key hypothesis is that the spacetime is asymptotically flat.

We first discuss this case and then outline a more general procedure to separate background from dynamics.

GW energy for asymptotically flat spacetimes

Take linearized GR:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} + \mathcal{O}(2)$$

where

$\eta_{\mu\nu}$: flat (Minkowski) spacetime

$|h_{\mu\nu}| \simeq \epsilon |\eta_{\mu\nu}| \approx \epsilon \ll 1$: perturbation

and "push it" to second-order in ϵ :

$$g_{\mu\nu} = \eta_{\mu\nu} + \underset{\mathcal{O}(\epsilon)}{\overset{(1)}{h}_{\mu\nu}} + \underset{\mathcal{O}(\epsilon^2)}{\overset{(2)}{h}_{\mu\nu}} + \mathcal{O}(3)$$

The Ricci tensor is:

$$R_{\alpha\beta} = \overset{(0)}{R}_{\alpha\beta} + \overset{(1)}{R}_{\alpha\beta} + \overset{(2)}{R}_{\alpha\beta} + \mathcal{O}(3) \quad \text{where:}$$

$$\overset{(0)}{R}_{\alpha\beta} = R_{\alpha\beta}[\eta_{\mu\nu}] \sim \eta\, \partial^2 \eta$$

$$\overset{(1)}{R}_{\alpha\beta} = R'_{\alpha\beta}[\overset{(1)}{h}_{\mu\nu}] \sim \eta\, \partial^2 h$$

Expression quadratic in ϵ, applied to $\overset{(1)}{h}_{\mu\nu}$

$$\overset{(2)}{R}_{\alpha\beta} = R'_{\alpha\beta}[\overset{(2)}{h}_{\mu\nu}] + R''[\overset{(1)}{h}_{\mu\nu}] \sim \eta\, \partial^2 \overset{(2)}{h} + \overset{(1)}{h}\, \partial^2 \overset{(1)}{h}$$

Expression linear in ϵ, applied to $\overset{(2)}{h}_{\mu\nu}$

EFE in vacuum:

$$R_{\alpha\beta} = \overset{(0)}{R}_{\alpha\beta} + \overset{(1)}{R}_{\alpha\beta} + \overset{(2)}{R}_{\alpha\beta} = 0$$

where: $\overset{(0)}{R}_{\alpha\beta}[\eta_{\mu\nu}] = 0$: flat background

$$\overset{(1)}{R}_{\alpha\beta}[\overset{(1)}{h}_{\mu\nu}] = 0 \quad : \quad \text{determine } \overset{(1)}{h}_{\mu\nu} \text{ up to gauge}$$

Plug $\eta_{\mu\nu}$ and $\overset{(1)}{h}_{\mu\nu}$ into $R_{\alpha\beta}$ and obtain:

$$\overset{(2)}{R}_{\alpha\beta} = R'_{\alpha\beta}[\overset{(2)}{h}_{\mu\nu}] + R''_{\alpha\beta}[\overset{(1)}{h}_{\mu\nu}] = 0 \quad \rightarrow \text{"solve" for } \overset{(2)}{h}_{\mu\nu}.$$

the equation above can be written as:

$$G'_{\alpha\beta}[\overset{(2)}{h}_{\mu\nu}] := R'_{\alpha\beta}[\overset{(2)}{h}_{\mu\nu}] - \frac{1}{2} R'[\overset{(2)}{h}_{\mu\nu}]\eta_{\alpha\beta} = 8\pi t_{\alpha\beta}$$

with:

$$t_{\alpha\beta} \equiv -\frac{1}{8\pi} G''_{\alpha\beta}[\overset{(1)}{h}_{\mu\nu}] = R''_{\alpha\beta}[\overset{(1)}{h}_{\mu\nu}] - \frac{1}{2} R''[\overset{(1)}{h}_{\mu\nu}]\eta_{\alpha\beta}$$

Attempt to interpret this quantity as stress-energy tensor of GW:

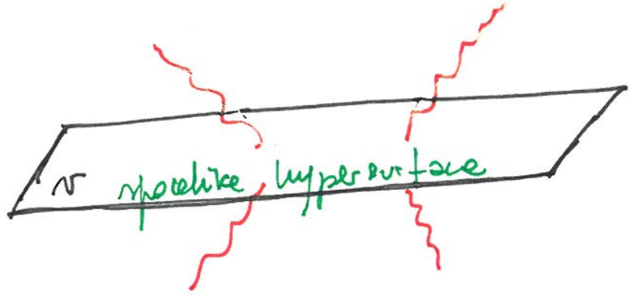
+ symmetric
+ conserved on flat background $\partial^\alpha t_{\alpha\beta} = 0$
+ quadratic in $h_{\alpha\beta}$, i.e. $\mathcal{O}(\epsilon^2)$
- NOT gauge invariant
- NOT unique: one can add any tensor $\partial^\alpha \partial^\beta U_{\mu\alpha\nu\beta}$:
    - quadratic in $h_{\alpha\beta}$
    - $U_{\mu\alpha\nu\beta} = U_{[\mu\alpha]\nu\beta} = U_{\mu\alpha[\nu\beta]} = U_{\nu\beta\mu\alpha}$

  and obtain the same $t_{\alpha\beta}$.

  Landau-Lifshitz construct a similar quantity called LL pseudotensor that differ of $\partial^\alpha \partial^\beta U_{\mu\alpha\nu\beta}$.

However, the quantity:

$$E \equiv \int_V d^3x \, t_{00}$$



is gauge invariant $E[h_{\alpha\beta}] = E[h_{\alpha\beta} + 2\partial_{(\alpha}\xi_{\beta)}]$

under infinitesimal coordinate transformation

and unique

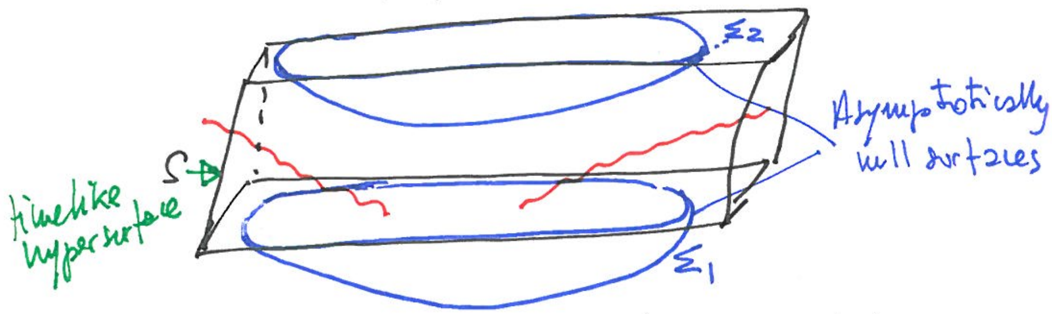
IF at large distances from the isolated source one has:

$$\left.\begin{aligned} \overset{(1)}{h}_{\alpha\beta} &\sim \mathcal{O}(r^{-1}) \\ \partial_\mu \overset{(1)}{h}_{\alpha\beta} &\sim \mathcal{O}(r^{-2}) \\ \partial_\mu\partial_\nu \overset{(1)}{h}_{\alpha\beta} &\sim \mathcal{O}(r^{-3}) \end{aligned}\right\} \quad \text{Asymptotically flat conditions.}$$

$\Rightarrow$ E can be interpreted as total energy associated with $h_{\alpha\beta}$.

Similarly, while the energy flux $-t^a{}_0$ is not gauge invariant, one can define:

$$\Delta E = -\int_S t_{a0} \, ds^a$$



that can be meaningfully interpreted as the radiated energy between two stationary regimes "bracketed" by the two null surfaces Σ_1 and Σ_2.

[See Wald Chap. 11 for details on how the limit $r \to +\infty$ is taken]

General procedure to define GWs on any background

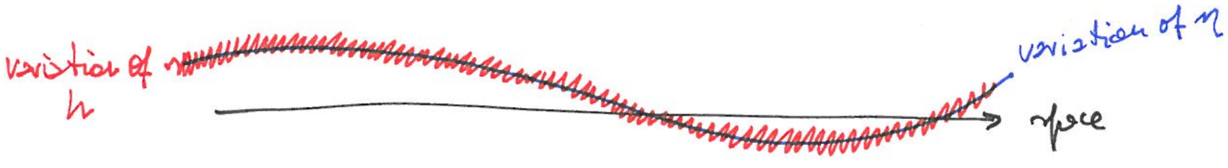
Consider the metric as given by a generic background plus a perturbation:

$$g = \eta + h$$

one can talk about "perturbation" and "background" only if the typical scales of variation of η and h are very different, for example

$$\lambda\!\!\!^{-} \equiv \frac{\lambda}{2\pi} \qquad \text{SCALE OF SPATIAL VARIATION OF } h$$

$$L \qquad \text{SCALE OF SPATIAL VARIATION OF } \eta$$



then, one can divide short wavelength perturbations from long wavelength background. Similarly, one can have high frequency perturbations and low frequency background if considers the scales of temporal variation.

The combination of "small amplitude" $|h| \ll |\eta|$ and "short-wavelength" $\lambda\!\!\!^{-} \ll L$ perturbation allows one to define a stress-energy tensor for GWs on generic background.

<u>Remark</u>: the spatial and temporal scales of variation of the background are typically <u>not</u> related.

One needs only one of the two, but also needs to identify which one!

Example: GW measurement on Earth.

Take a GW with

$$f_{GW} \sim 10^2 - 10^3 \text{ Hz} \quad , \quad \lambdabar_{GW} \sim 500 - 50 \text{ km}$$
$$h_{GW} \sim 10^{-21}$$

Take a GW LAB on Earth, where $\phi_\oplus \sim \frac{GM_\oplus}{c^2 R_\oplus} \sim 10^{-6}$

- $\phi_\oplus$ is <u>not</u> smooth on <u>length scales</u> $L \sim \lambdabar_{GW}$: there're variations of $h \sim 10^{-9} \gg h_{GW}$ due to e.g. mountains ...
  And also $L_{LAB} \ll \lambdabar_{GW}$!
  $\Rightarrow$ one cannot separate the length scales!

- $\phi_\oplus$ is <u>static</u> with $f \ll f_{GW}$
  $\Rightarrow$ one can separate timescales!

$\rightarrow$ GW experiments on Earth measure temporal variations, not length variations.

Small amplitude and short wavelength perturbations

Consider the metric expansion in the "amplitude":

$$g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta} \quad \text{with} \quad |h_{\alpha\beta}| \ll |\eta_{\alpha\beta}|$$

To make this a bit more formal we write:

$$|h_{\alpha\beta}| \approx \epsilon |\eta_{\alpha\beta}| \quad \text{with } \epsilon \ll 1 \quad \text{a small parameter.}$$

We use the symbol "$\approx$" and the absolute value "$|\cdot|$" to indicate that:

$|h_{\alpha\beta}|$ "has typical values" of $\epsilon |\eta_{\alpha\beta}|$

this is just notation but useful to make it explicit.

The metric at order $\mathcal{O}(\epsilon^2) = \mathcal{O}(2)$ is:

$$g_{\alpha\beta} = \eta_{\alpha\beta} + \overset{(1)}{h}_{\alpha\beta} + \overset{(2)}{h}_{\alpha\beta} + \mathcal{O}(3)$$

where the terms $\mathcal{O}(1)$ and $\mathcal{O}(2)$ are indicated with superscripts.

The expansion of the Ricci is thus:

$$R_{\alpha\beta} = \overset{(0)}{R}_{\alpha\beta} + \overset{(1)}{R}_{\alpha\beta} + \overset{(2)}{R}_{\alpha\beta} + \mathcal{O}(3)$$

where:

$$\overset{(0)}{R}_{\alpha\beta} = R_{\alpha\beta}[\eta_{\mu\nu}] \sim \eta\, \partial^2 \eta \quad , \quad \text{here: "$\sim$" = "goes like"}$$

$$\overset{(1)}{R}_{\alpha\beta} = R'_{\alpha\beta}[\overset{(1)}{h}_{\mu\nu}] \sim \eta\, \partial^2 \overset{(1)}{h}$$

$$\overset{(2)}{R}_{\alpha\beta} = \underbrace{R'_{\alpha\beta}[\overset{(2)}{h}_{\mu\nu}]}_{\substack{R_{\alpha\beta}\text{ expression linear-in-}\epsilon \\ \text{applied to } \overset{(2)}{h}}} + \underbrace{R''_{\alpha\beta}[\overset{(1)}{h}_{\mu\nu}]}_{\substack{R_{\alpha\beta}\text{ expression quadratic-in-}\epsilon \\ \text{applied to } \overset{(1)}{h}}} \sim \eta\, \partial^2 \overset{(2)}{h} + \overset{(1)}{h}\, \partial^2 \overset{(1)}{h}$$

The EFE at order $\mathcal{O}(2)$ are then (Trace reverse):

$$\overset{(0)}{R}_{\alpha\beta} + \overset{(1)}{R}_{\alpha\beta} + \overset{(2)}{R}_{\alpha\beta} = 8\pi\left(T_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} T\right) \equiv 8\pi \bar{T}_{\alpha\beta}[g_{\alpha\beta}]$$

where $g_{\alpha\beta}$ is the metric up to $\mathcal{O}(2)$.

Consider the vacuum case $(\bar{T}_{\alpha\beta}=0)$. One has the equation for the background:

$$\overset{(0)}{R}_{\alpha\beta} = 0$$

and the one for the linear perturbation:

$$R'_{\alpha\beta}[\overset{(1)}{h}_{\mu\nu}] = 0$$

that determines $\overset{(1)}{h}_{\mu\nu}$ up to gauge.

Thus, at $\mathcal{O}(2)$ one is left with:

$$\overset{(2)}{R}_{\alpha\beta} = R'_{\alpha\beta}[\overset{(2)}{h}_{\mu\nu}] + R''[\overset{(1)}{h}_{\mu\nu}] = 0$$

which we might write:

$$\underbrace{R'_{\alpha\beta}[\overset{(2)}{h}_{\mu\nu}]}_{\text{same equation LHS as for linearised EFE but applied to } \overset{(2)}{h}_{\mu\nu}} = -R''_{\alpha\beta}[\overset{(1)}{h}_{\mu\nu}] \equiv 8\pi \underbrace{t_{\alpha\beta}[\overset{(1)}{h}_{\mu\nu}]}_{\text{a "stress-energy"-like term that is produced by the curvature of the linear pert. } \overset{(1)}{h}_{\mu\nu}}$$

<u>Observations</u>

+ $t_{\alpha\beta}$ is symmetric
+ $t_{\alpha\beta}$ is quadratic in $h_{\alpha\beta}$ $\mathcal{O}(2)$

+ $E_{\mu\nu}$ is conserved on flat background, $\partial^\mu E_{\mu\nu} = 0$

- $E_{\mu\nu}$ is not invariant under infinitesimal coordinate transformations
- $E_{\mu\nu}$ is not a tensor in the full theory
- Since we can always choose normal coordinates at a point, we can always make the metric flat at a point and make $E_{\mu\nu} = 0$ at the point even in the full theory!

  What we did is correct but IT IS NOT POSSIBLE TO GIVE A LOCAL MEASURE OF ENERGY OF THE GRAVITATIONAL FIELD.

We still want to obtain an expression for the energy transported "far away" by the GWs.
To do that one needs to consider an appropriate procedure to average on the perturbation leng scale.
Let us introduce:

$\lambda\!\!\!^-$: short wavelength scale of variation of $h_{\alpha\beta}$

L : long length scale of the background variation

$$|\partial_\alpha \eta_{\mu\nu}| \approx \frac{1}{L} |\eta_{\mu\nu}| \approx \frac{1}{L}$$

$$|\partial_\alpha h_{\mu\nu}| \approx \frac{1}{\lambda\!\!\!^-} |h_{\mu\nu}| \qquad \lambda\!\!\!^- \ll L$$

Note the above estimates imply:

$$|\partial^2 \eta| \approx \frac{1}{L^2} |\eta_{\mu\nu}| \approx \frac{1}{L^2}$$

$$|\eta \partial^2 h| \approx \frac{1}{\lambda\!\!\!^{-2}} |h_{\mu\nu}|$$

$$|h \partial^2 h| \approx \frac{1}{\lambda\!\!\!^-} |h_{\alpha\beta}| |h_{\mu\nu}|$$

Consider again EFE at $\mathcal{O}(\epsilon^2)$:

$$\overset{(0)}{R}_{\mu\nu} + \overset{(1)}{R}_{\mu\nu} + \overset{(2)}{R}_{\mu\nu} = 8\pi \, T_{\mu\nu}$$

and look at the characteristic variation scales :

$$\overset{(0)}{R}_{\mu\nu} \sim \eta \, \partial^2 \eta \approx \frac{1}{L^2} \qquad \rightarrow \text{ long wavelength or Low frequency}$$

$$\overset{(1)}{R}_{\mu\nu} \sim \eta \, \partial^2 \overset{(1)}{h} \approx \frac{1}{\lambda\!\!\!^{-2}} \epsilon \qquad \rightarrow \text{ High frequency or short wavelength}$$

$$\overset{(2)}{R}_{\mu\nu} \sim \eta \, \partial^2 \overset{(2)}{h} + \overset{(1)}{h} \, \partial^2 \overset{(1)}{h} \approx \underbrace{\frac{1}{\lambda\!\!\!^{-2}} |h_{\alpha\beta}| |h_{\mu\nu}|} \approx \frac{\epsilon^2}{\lambda\!\!\!^{-2}} \quad \ldots \text{ but}$$

this term can contain <u>both</u> low and high frequencies since two high frequency mode could combine to give a low-frequency one.

Indicate this contribution as

$$\approx \frac{h^2}{\lambda\!\!\!^{-2}} .$$

Separate the $\mathcal{O}(2)$ EFE according to the two scales (*) :

$$\text{High freq :} \qquad \overset{(1)}{R}_{\mu\nu} = - \overset{(2)}{R}{}^{\text{High}}_{\mu\nu} + 8\pi \, T^{\text{High}}_{\mu\nu}$$

$$\text{Low freq :} \qquad \overset{(0)}{R}_{\mu\nu} = - \overset{(2)}{R}{}^{\text{low}}_{\mu\nu} + 8\pi \, T^{\text{low}}_{\mu\nu}$$

also these equations indicate that curvature becomes a source of gravity.
Consider in particular the low frequencies equation :

$$\text{if } T_{\mu\nu} = 0 \text{, then} \qquad \frac{1}{L^2} \approx \frac{h^2}{\lambda\!\!\!^{-2}} \Rightarrow h \approx \frac{\lambda\!\!\!^{-}}{L}$$

is the typical curvature determined by GWs

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* Note we have now <u>two</u> parameters, so we can equate terms that before had different orders (powers) of ϵ ...

if $\begin{cases} T_{\mu\nu} \neq 0 \\ |T^{low}_{\mu\nu}| \gg |R^{(2)\,low}_{\mu\nu}| \end{cases}$, then $\frac{1}{L^2} \approx \frac{h^2}{\lambda\!\!\!^{-2}} + (\text{matter}) \gg \frac{h^2}{\lambda\!\!\!^{-2}} \Rightarrow h \ll \frac{\lambda\!\!\!^{-}}{L}$

the typical curvature determined by the matter in the background dominates the one from GWs.

This analysis leads to important:

Observations

- GWs are defined only for small amplitudes.
  it is otherwise impossibile to define perturbation/waves from the separation of two scales:

$$|h_{\alpha\beta}| \sim 1 \Rightarrow \frac{\lambda\!\!\!^{-}}{L} \sim 1 \quad (!)$$

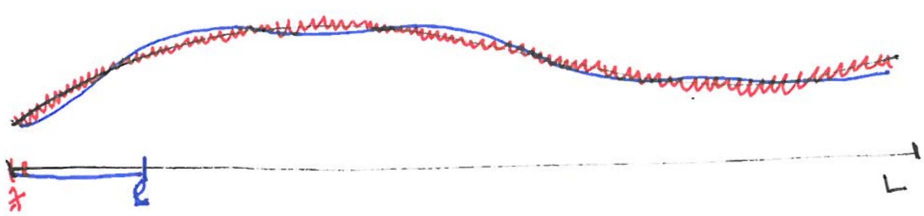
- if $\eta_{\alpha\beta}$ is Minkowski, then $L = 0$.

$\Rightarrow$ it is impossible to satisfy $h \lesssim \frac{\lambda\!\!\!^{-}}{L}$

$\Rightarrow$ one cannot extend the expansion in powers of ϵ beyond linear order, the expansion would have no domain of validity.
If one allows the background to curve, then one can think that GWs "back-react" on it and "generate" curvature.

Stress energy tensor for GWs

The stress-energy "tensor" for GWs can be defined by extracting $\overset{(2)}{R}{}^{low}_{\mu\nu}$ using an averaging procedure on wavelengths $\lambda \ll l \ll L$:



$$\overset{(2)}{R}{}^{low}_{\mu\nu} \leftarrow \langle \overset{(2)}{R}_{\mu\nu} \rangle \quad , \text{ and}$$

$$t_{\mu\nu} \equiv -\frac{1}{8\pi} \langle \overset{(2)}{R}_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} \overset{(2)}{R} \rangle$$ <u>effective stress-energy tensor for GWs</u>

Physically, the average procedure eliminate (integrate away) all the high-frequency modes

Matematically, the Ricci tensor:

$$\overset{(2)}{R}_{\mu\nu} = \frac{1}{2}\Big[\frac{1}{2} \partial_\mu h_{\alpha\beta} \partial_\nu h^{\alpha\beta} + h^{\alpha\beta} \partial_\mu \partial_\nu h_{\alpha\beta} + \dots$$
$$+ 11 \text{ terms containing } \partial_\mu h_{\nu\mu} \dots \Big]$$

reduces to:

$$\langle \overset{(2)}{R}_{\mu\nu} \rangle = -\frac{1}{4} \langle \partial_\mu h_{\alpha\beta} \partial_\nu h^{\alpha\beta} \rangle \quad \text{in TT gauge}$$

since all the 11 terms can be integrated by parts and eliminated using the gauge and the equations of motion.

The specific average operator has been introduced by Brill & Hartle and Isaacson (1968):

$$\langle T_{\alpha\beta} \rangle \equiv \int \eta_\alpha{}^{\mu'}(x,x') \, \eta_\beta{}^{\nu'}(x,x') \, T_{\mu'\nu'}(x') \, f(x,x') \sqrt{-g(x')} \, d^4x'$$

and similarly for generic tensors.

In the above expression:

- $\eta^{\mu'}_{\ \alpha}(x, x')$ is an operator that transport $T_{\mu'\nu'}(x')$ in a neighbourg of x' and transform as a tensor at x' in the index μ' and as a tensor at x in the index α. it is called the <u>bivector of geodetic parallel displacement</u> (see Brill & Hartle, Isaacson, and MTW 35.14)

- $f(x, x')$ is a weighting function that goes rapidly to 0 if x and x' are separated by many wavelengths

Operatively, one has the following rules:

- $\langle T_{\mu\nu}{}^{\rho}{}_{;\rho} \rangle = 0$
- one can integrate by parts
- Covariant derivatives inside the average commute.

The final result is that:

$$\boxed{t_{\alpha\beta} \equiv \frac{c^4}{32\pi G} \langle \partial_\alpha h_{ij}^{TT} \partial_\beta h^{ij\,TT} \rangle}$$

is the effective stress-energy tensor for GWs.

The expression is in TT gauge and it is valid <u>far from the source</u>.

$t_{\alpha\beta}$ is conserved on flat background:

$$\partial^\alpha t_{\alpha\beta} = 0$$

and the energy of GWs contained in a spatial volume V is:

$$E_{GW} = \int_V d^3x \, t_{00} \, .$$

From the $\beta = 0$ component of the conservation equation one obtains:

$$0 = \int_V \left(\partial_0 t^{00} + \partial_i t^{0i} \right) = -\dot{E}_{GW} + \int_V \partial_i t^{0i} = -\dot{E}_{GW} + \int_\Sigma t^{0i} n_i =$$

take $\Sigma = S_r$ a sphere of radius $r \gg 1$:

$$= -\dot{E}_{GW} + \int_{S_r} t^{0r} n_r \underset{\uparrow}{=} -\dot{E}_{GW} + \int_{S_r} r^2 \langle \partial^0 h_{ij} \, \partial_r h_{ij} \rangle =$$

use def. of $t_{\alpha\beta}$
(do not write the constants ... will get them back later...)

observe that :

$$h_{ij} \sim \frac{1}{r} f\left(t - \frac{r}{c}\right) \quad \text{and} \quad \partial_r f = -\frac{1}{r}\frac{1}{c}\partial_t f + O(r^{-2})$$

$$\Rightarrow \quad \partial_r h_{ij} = -\partial_t h_{ij}$$
$$= -\partial_0 h_{ij} = +\partial^0 h_{ij}$$

hence:

$$= -\dot{E}_{GW} + \int_{S_r} r^2 \langle \partial_t h_{ij} \, \partial_t h_{ij} \rangle$$

$\rightarrow$

$$\frac{dE_{GW}}{dt} = \frac{c^3}{32\pi G} \int_{\Sigma} \langle \partial_t h_{ij}^{TT} \, \partial_t h_{ij}^{TT} \rangle \, da$$

$$= \frac{c^3}{32\pi G} r^2 \int_{S_r} \langle \partial_t h_{ij}^{TT} \, \partial_t h_{ij}^{TT} \rangle \, d\Omega$$

Observation: the integral exists because $h_{ij} \sim \frac{1}{r}$.

In terms of the $\times, +$ amplitudes one can also obtain:

$$\frac{dE_{GW}}{dt} = \frac{c^3}{16\pi G} \langle \dot{h}_+^2 + \dot{h}_\times^2 \rangle$$

And finally using the quadrupole formula:

$$\frac{dE_{GW}}{dt} = \frac{1}{5} \frac{G}{c^5} \langle \dddot{Q}_{ij} \dddot{Q}^{ij} \rangle$$

Dimension analysis of the GW luminosity formula

$[Q] = a\, M L^2$ with "a" factor depending on angles and $a=0$ for spherically symmetric bodies

$[\dddot{Q}] = a\, T^{-3} M L^2 \sim a\, \Omega^3 M L^2$

$[\dot{E}_{GW}] = \frac{G}{c^5} [\dddot{Q}]^2 = \frac{G}{c^5} a^2 \Omega^6 M^2 L^4$

Observe that:

- $\left[\frac{c^5}{G}\right] = E T^{-1}$ not $\frac{G}{c^5}$!
- $\frac{c^5}{G} \sim 10^{52}$ W $\Rightarrow$ if $\dddot{Q} \sim$ laboratory experiments, then the power in GW is very small
- However, let us re-express the formula in terms of $\frac{c^5}{G}$:

R = characteristic size of the source

$\sigma = \frac{GM}{c^2 R}$ "compactness" or "surface gravity" or "self-gravity parameter" $\rightarrow M = \frac{c^2 R \sigma}{G}$

$\Omega = \frac{v}{R} = \frac{v}{c} \frac{c}{R}$

$$[\dot{E}_{GW}] \sim \frac{G}{c^5} \left(\frac{v}{c}\right)^6 \left(\frac{c}{R}\right)^6 \frac{c^4 R^2 \sigma^2}{G^2} R^4 = \frac{c^5}{G} \left(\frac{v}{c}\right)^6 \sigma^2 = \frac{c^5}{G} \left(\frac{v}{c}\right)^6 \left(\frac{GM}{c^2 R}\right)^2$$

this formula shows that for a source with

- strong self-gravity
- high velocity

$\Longrightarrow$ GW are the most luminous radiation in the Universe!

The formula is due to Weber. For $\sigma \sim 1$, $v \sim c$:

$$\dot{E}_{GW} \sim 10^{52}\, W \sim 10^{26} \times \dot{E}_{\odot} \text{ (in EM)}.$$

Exercise : Binary pulsar

The formula applied to the binary system is almost exact:

$$\dot{E}_{GW} = a \frac{G}{c^5} \mu^2 r^4 \Omega^6$$

where :

$a = \frac{32}{5}$

μ : reduced mass

r : relative distance

Ω : orbital frequency

Note that from Kepler law :

$$\Omega^2 = m r^{-3} \rightarrow r^{-1} = m^{-1/3} \Omega^{2/3}$$

substituting in $\dot{E}_{GW}$:

$$\dot{E}_{GW} = a \frac{G}{c^5} \mu^2 r^4 \Omega^6 = a \frac{G}{c^5} \mu^2 m^{+4/3} \Omega^{-8/3} \Omega^6 =$$

$$= a \frac{G}{c^5} \mu^2 m^{4/3} \Omega^{10/3}$$

$=$ $m_1 = m_2 = \frac{m}{2}$, $\mu = \frac{m_1 m_2}{m} = \frac{m}{4}$

$$\sim (m\Omega)^{10/3}$$

Compare with the orbital energy :

$$E_0 = \frac{1}{2} \mu v^2 - \frac{m\mu}{r} = \frac{1}{2} \mu \Omega^2 r^2 - \mu \frac{m}{r} =$$ Ω^2 from Kepler law

$$= \frac{1}{2} \mu r^2 m r^{-3} - \mu \frac{m}{r} =$$

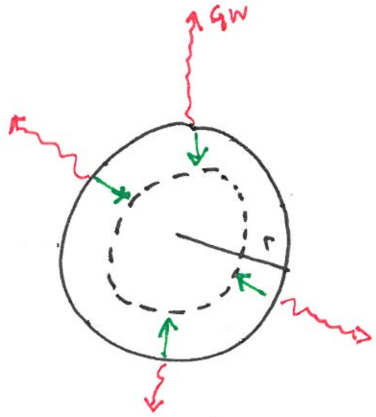
$$= \mu \frac{m}{r} \left(\frac{1}{2} - 1\right) = -\frac{1}{2} \frac{\mu m}{2r}$$

$$m_1 = m_2 = \frac{m}{2} \longrightarrow \quad E_0 = -\frac{m^2}{8r} \sim -m^{5/3}\,\Omega^{2/3} = \underbrace{-\,m\,\underbrace{(m\Omega)^{2/3}}_{>0}}_{<0}$$

(bound orbits)

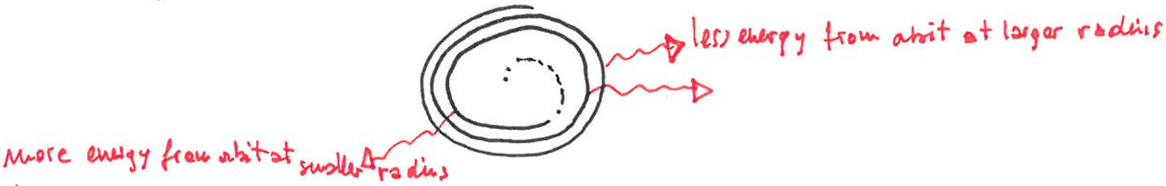
Observation:

If we remove energy via GWs, then by the formula above:



- E_0 becomes more negative
- r becomes <u>smaller</u>
- Ω <u>increases</u> (T <u>decreases</u>)

Indeed the evolution of circular orbits under gravitational radiation is a slow <u>inspiral</u>



during which the binary accelerate the orbit and more radiation is emitted. The motion can only end with the <u>collision</u> of the two bodies!

Let: $\quad E_0 = \alpha\,\Omega^{2/3}$

$$\frac{d}{dt}\ln E_0 = \frac{d}{dt}\ln\left(\alpha\Omega^{2/3}\right) \Rightarrow \frac{1}{E_0}\dot{E}_0 = \frac{2}{3}\frac{\alpha\,\Omega^{-1/3}}{\alpha\,\Omega^{2/3}}\dot{\Omega}$$

$$= \frac{2}{3}\frac{1}{\Omega}\dot{\Omega} = -\frac{2}{3}\frac{1}{T}\dot{T}$$

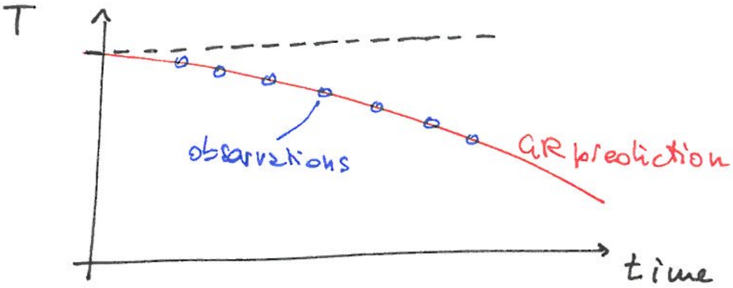
where $T = \frac{2\pi}{\Omega}$ is the orbital period.

Then: $\quad \dot{T} = -\frac{3}{2}\frac{T}{E_0}\dot{E}_0 = +\frac{3}{2}\frac{T}{E_0}\dot{E}_{GW} =$

$$\approx -T\,m^{-1}(m\Omega)^{-2/3}(m\Omega)^{10/3} = -T\,m^{-1}(m\Omega)^{8/3}$$

$$\approx -2\cdot 10^{-13} \sim -6\cdot 10^{-6}\ \text{s/yr}$$

Hulse & Taylor observed this effect in the binary pulsar.



The estimate above is an order of magnitude wrong ... one needs to:

- include eccentricity
- include "some" self-gravity, see e.g. Damour & Taylor 1991 ApJ paper.

and obtain: $\dot{T} \sim -2.3 \cdot 10^{-12}$