These semi-private notes are constructed from the following books:

- R.Wald, "General Relativity" University of Chicago Press, 1984
- S.M.Carrol, "Spacetime and Geometry, An Introduction to General Relativity", Addison-Wesley, 2003.
- B.F.Schutz, "A First Course in General Relativity", Cambridge University Press, 1985.
- B.F.Schutz, "Geometrical Methods of Mathematical Physics", Cambridge University Press, 1980.

If you decide to use them to study or teach, please
(0) be careful and refer to the original books
(1) cite/refer to my website
(2) let me know and send feedbacks.

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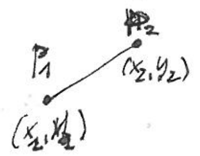
DIFFERENTIAL GEOMETRY

An event in spacetime is characterized by 4 numbers, $x^\mu = (t, x, y, z)$
In pre-relativity and SR, spacetime is <u>globally</u> in a one-to-one correspondence with $\mathbb{R}^4$.
In GR we do not want "to fix" the spacetime, rather Einstein equations determine its global structure.
Thus, we need to introduce and use some more general matematical formalism (differential geometry) that allow us to describe non-Euclidean geometries – in fact <u>arbitrary geometries</u>, and write differential epuations in those geometries.

$\boxed{\text{Examples}}$

- Measure distances on a sphere, S^2
- Describe waves on a sphere (sea waves)

<u>Locally</u> a sphere behaves like $\mathbb{R}^2$, i.e. one can compute "small distances"

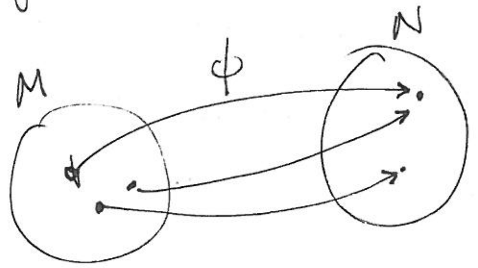


$\overline{P_1P_2} \simeq \sqrt{(x_1-x_2)^2-(y_1-y_2)^2}$; but globally there is no one-to-one correspondence between S^2 and $\mathbb{R}^2$ in a continuous manner (as we shall also see later).
A way to approach the problem is to <u>embed</u> S^2 in $\mathbb{R}^3$ and consider the sphere immerse in a Euclidean space of higher dimensions (3). This approach allows one to calculate meridians and parallels as paths of minimal distance between 2 points on the sphere; and it allows to write wave epuations on S^2 using the standard transformation from Cartesian to spherical coordinates.
However that is not the approach we can take in GR for the spacetime: we experience only a 4D spacetime and do not how to embed the latter in a higher dimensional Euclidean space.

MANIFOLD

The concept of <u>manifold</u> is introduced in order to map a generic set into $\mathbb{R}^N$. The map is not always possible for the global set, but it is possible <u>locally</u>. Similarly to an Atlas, that "covers" Earth with many charts, a manifold is a structure that maps a generic geometry into $\mathbb{R}^N$, eventually "piece-by-piece".

<u>Def</u>: <u>MAP between two sets</u> $\phi: M \longmapsto N$ assign to each element of M one element of N.



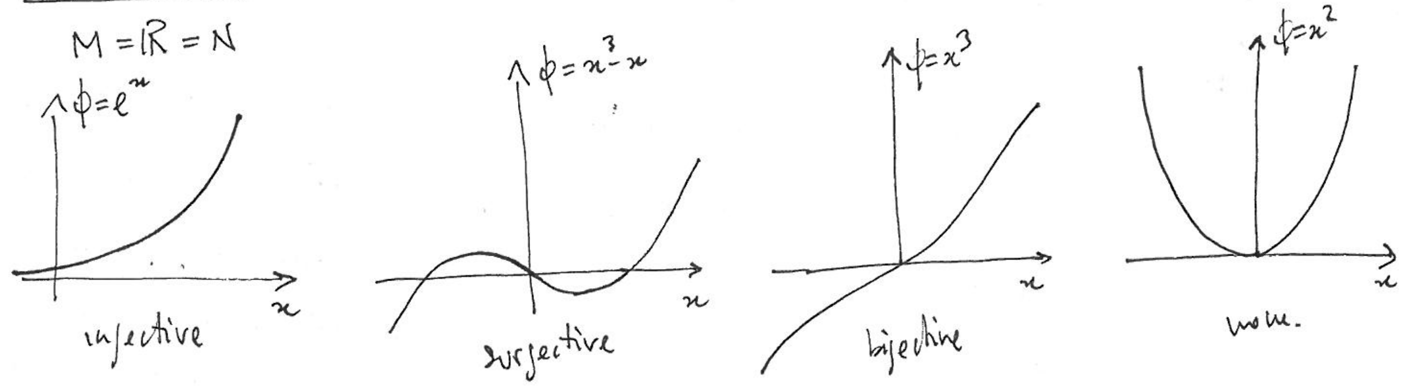
A map can be:

- <u>injective</u> : $\forall$ element of N $\exists$ <u>at most</u> 1 element of M
- <u>surjective</u> : $\forall$ element of N $\exists$ <u>at least</u> 1 element of M
- <u>bijective (invertible)</u>: injective $\wedge$ surjective $\Rightarrow \exists\, \phi^{-1}$: $\underbrace{\phi^{-1}\circ\phi}(x) = 0$

  – map composition –
- None of the above.

$\boxed{\text{Examples}}$

$M = \mathbb{R} = N$



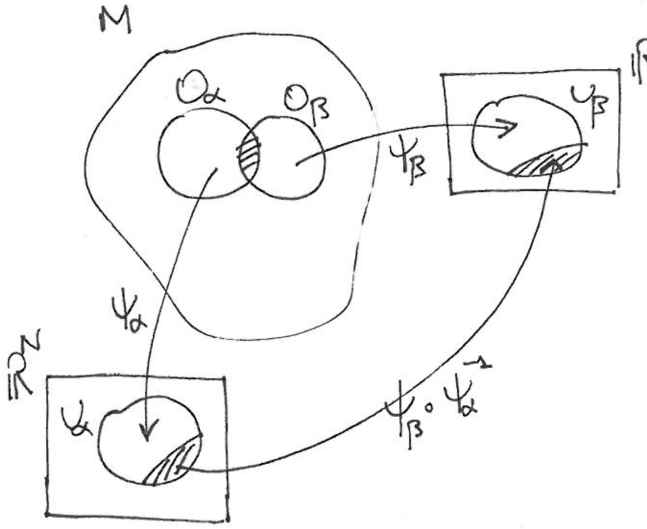
<u>Def</u>: OPEN BALL in $\mathbb{R}^N$ of radius r around $y \in \mathbb{R}^N$ is the set of points

$$B_{r,y} = \left\{ x \in \mathbb{R}^N : \; |x-y| = \left[\sum_{i=1}^{N} (x_i - y_i)^2 \right]^{1/2} < r \right\}$$

<u>Def</u>: OPEN SET in $\mathbb{R}^N$ = any set of $\mathbb{R}^N$ that can be expressed as <u>union of balls</u>.

Def: MANIFOLD is a set M together with a set of subset $\{O_\alpha\}$ such that:

1. $\{O_\alpha\}$ covers M; i.e. each $p \in M$ is contained in at least one O_α.
2. $\forall \alpha$ $\exists$ a bijective map to an open subset of $\mathbb{R}^N$, $\psi_\alpha : O_\alpha \mapsto U_\alpha \subset \mathbb{R}^N$
3. if any two sets overlap $O_\alpha \cap O_\beta \neq \phi$, then $\psi_\beta \circ \psi_\alpha^{-1} : \mathbb{R}^N \mapsto \mathbb{R}^N$ is C^∞.



ψ_α is called a chart or a coordinate system.

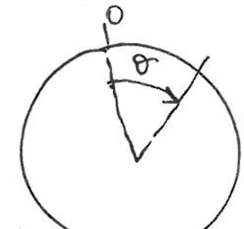
$\{\psi_\alpha\}$ is called an atlas.

$(M, \{O_\alpha\}, \{\psi_\alpha\})$ is usually shortly indicated a "manifold M"

Examples

- $M = \mathbb{R}^N$: $O = \mathbb{R}^N$, ψ = identity (1 chart)
- $M = S^1$: the natural coordinate system for the unit circle is $\theta : S^1 \mapsto \mathbb{R}$.

Does that define a manifold?



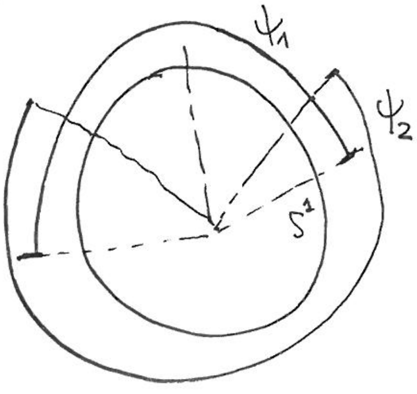
1. θ covers S^1 ✓

$[0, 2\pi)$ or $(0, 2\pi]$

2. θ maps to an open set of $\mathbb{R}$ ✗

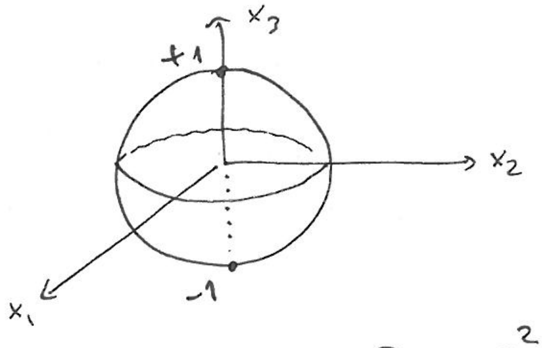
_ including either $\theta = 0$ or $\theta = 2\pi$ gives a close interval
_ excluding both $\theta = 0$ and $\theta = 2\pi$ does not cover the circle.

$\Rightarrow$ the circle cannot be covered with one chart!



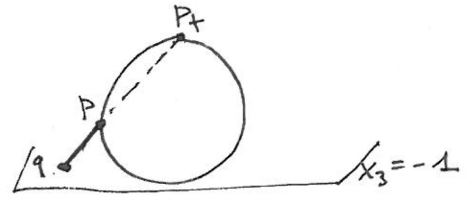
At least two charts are needed.

- $M = S^2 = \{ p = (x_1, x_2, x_3) \in \mathbb{R}^3 : \sum_{i=1}^{3} (x_i)^2 = 1 \}$



lets consider the two subsets : $O_{\pm} = S^2 - \{(0,0,\pm 1)\}$,
these are two open sets.

Each of the two open sets can be mapped to $\mathbb{R}^2$ by using stereographic projections, e.g.



- take O_+ (S^2 – north pole)
- take the plane "below" (thought the south pole) $x_3 = -1$
- map each point on the sphere to the plane by "drawing" the line $\overline{PP_+}$ that hits the plane

$$\psi_{\pm}(p) = \psi_{\pm}(x_1, x_2, x_3) = q = (y_1, y_2) = \left(\frac{2\,x_1}{1 \mp x_3}, \frac{2\,x_2}{1 \mp x_3} \right)$$

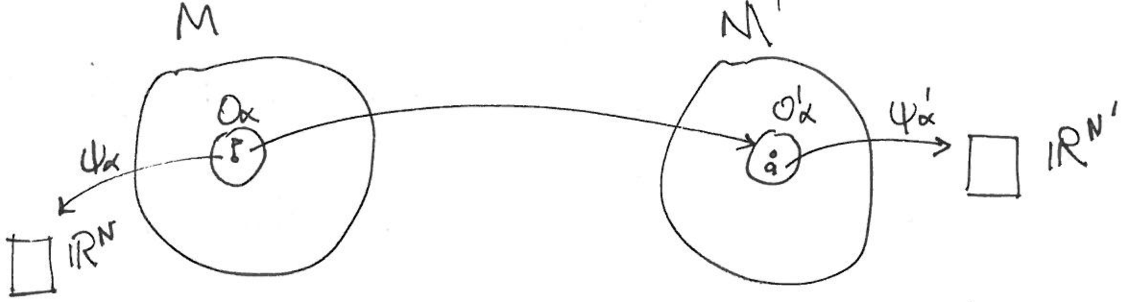
check :

1. $O_{\pm}$ cover the sphere ✓
2. open sets
3. Overlap in region $-1 < x_3 < 1$ and there

$$\psi_- \circ \psi_+^{-1}(y_i) = \frac{4\,y_i}{y_1^2 + y_2^2} \quad \text{is } C^\infty$$

Key point: MANIFOLD $\rightarrow$ set (space) locally $\sim \mathbb{R}^N$, but globally different/generic.

This concept allow us to import all the analysis tools in generic spaces (not only $\mathbb{R}^N$).



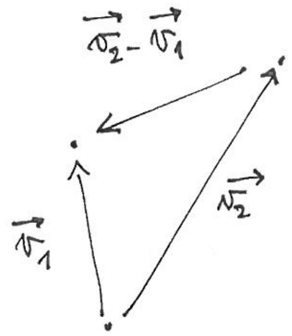
$f: M \rightarrow M'$ can be understood as a "regular" one:

$$F = \psi_\alpha \circ f \circ \psi'_\alpha : \mathbb{R}^N \longrightarrow \mathbb{R}^{N'}$$

In particular if: $F \in C^\infty$ and invertible $\Longleftrightarrow$ DIFFEOMORFISM.

TANGENT VECTOR SPACE

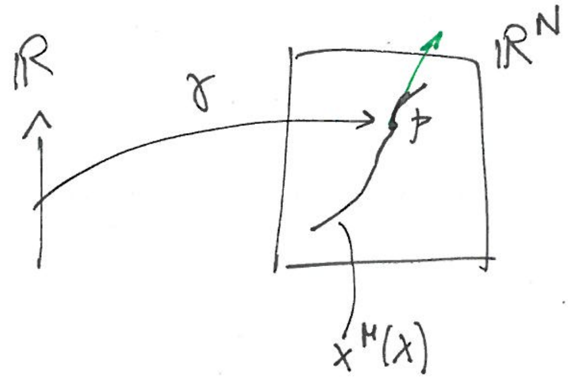
In Euclidean geometry one uses vectors as the basic way to describe "displacements".



- Vectors form a vector space;
- Vectors are defined globally in $\mathbb{R}^N$; intuitive notion how to "rigidly trasport" them around, sum and subtract them;
- Each vector introduces a direction, and can be naturally associated to the derivative in that direction:

$$\underset{\text{vector/direction}}{\vec{v} = (v_1, \dots, v_N)} \quad \longleftrightarrow \quad \underset{\text{derivate of functions in direction } \vec{v}}{\sum_\mu v^\mu \partial_\mu}$$

In particular, given a curve:



- the tangent vector to γ at point p is: $v_\gamma^\mu(p) = \frac{dx^\mu}{d\lambda}$

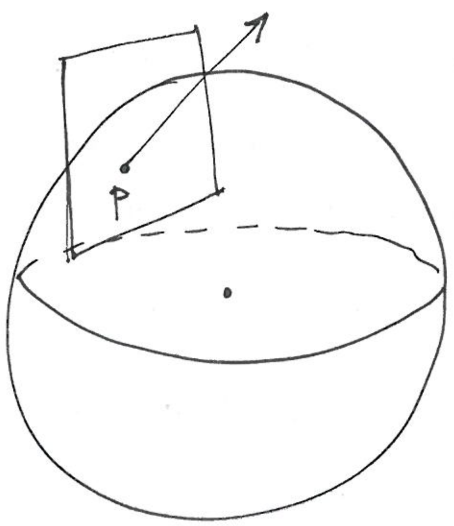
- any function derivative can be written:

$$\frac{df}{d\lambda} = \sum_\mu \frac{dx^\mu}{d\lambda} \frac{\partial f}{\partial x^\mu} = \sum_\mu v^\mu \partial_\mu f \qquad \forall f .$$

<u>Remark</u>: Properties of derivatives

1. Linear
2. Leibnitz

In <u>arbitrary geometry</u> the concept of tangent vector is globally lost.



Tangent vectors can be defined only at a point.

The idea can be viz. by <u>embedding</u> the manifold in a larger $\mathbb{R}^N$ space, e.g. S_2 in $\mathbb{R}^3$.

But in general, one needs an <u>intrinsic way</u> of characterizing vectors.

$\longrightarrow$ generalize the concept of vector using its identification with derivatives

Consider: $\mathcal{F} = \{ f : M \to \mathbb{R} \text{ , smooth} \}$ $\qquad \dim M = N$

<u>Def</u>: TANGENT VECTOR SPACE AT p, $T_pM = \{ v_p : \mathcal{F} \longrightarrow \mathbb{R} \}$ such that

$v \in T_pM$:

1. linear $\quad v(af+bg) = a\,v(f) + b\,v(g)$, $\quad a,b \in \mathbb{R} \quad f,g \in \mathcal{F}$
2. leibnitz $\quad v(fg) = f\,v(g) + g\,v(f)$
3. $(v_1 + v_2)(f) = v_1(f) + v_2(f)$
4. $(av)(f) = a\,v(f)$

<u>Observations</u>:

— $f(p) = \text{const} \equiv k \implies v(f) = 0$

<u>PROOF.</u>

$$\left. \begin{aligned} v(f^2) &\overset{2.}{=} f(p)\,v(f) + f(p)\,v(f) = 2k\,v(f) \\ &\overset{1.}{=} v(kf) = k\,v(f) \end{aligned} \right\} \implies v(f) = 0$$

$\square$

— T_pM is a vector space $\quad v = \sum_\mu v^\mu \underbrace{e_\mu}_{\text{basis}} \Leftarrow \qquad$ 3.,4. (trivial)

— $\dim(T_pM) = N \quad$ and $\quad v = \sum_\mu v^\mu \partial_\mu$.

To show this one must introduce a <u>basis</u> of T_pM $\{e_\mu\}$ such that

$$v = \sum_\mu v^\mu e_\mu$$

and show that the choice $e_\mu = \partial_\mu$ actually gives

- N independent vectors that
- span T_pM.

(i) Introduce <u>local coordinates</u> $\psi(p) = x^\mu \in \mathbb{R}^N$

$\{ \partial_\mu = \frac{\partial}{\partial x^\mu} \}$ N tangent vectors, linearly independent

$$e_\mu = \partial_\mu \rightarrow e_\mu(f) = \partial_\mu(f) \equiv \frac{\partial}{\partial x^\mu} \underbrace{(f \circ \psi^{-1})}_{\text{Coordinate representation of } f} \Big|_{\psi(p)}$$

(ii) ∂_μ span T_pM.

– Consider $F : O \subset \mathbb{R}^N \to \mathbb{R}$

$$F(x) = F(0) + \sum_\mu x^\mu \frac{\partial F(x)}{\partial x^\mu} \longleftrightarrow F(x) - F(0) = \int_0^1 \frac{d}{dt} F(tx^1, \dots, tx^N)\, dt =$$

$$= \sum_\mu x^\mu \underbrace{\int_0^1 \partial_\mu F(\ )\, dt}_{H_\mu(x)}$$

Particular case $H_\mu(0) = \partial_\mu F \big|_{x=0}$.

– $v(f) = v(f - f(p))$ $[\, v(f(p)) = 0 \,]$

$= v\left(\sum_\mu x^\mu \partial_\mu F \circ \psi \right)$ $[\, F = f \circ \psi^{-1} \,]$

$\underset{1.\ 2.}{=} \sum_\mu v(x^\mu)\, \partial_\mu f$ $[\, f = F \circ \psi \,]$

- $\{\partial_\mu\} \rightarrow$ "natural basis", "coordinate basis"
- THE VALUE OF $v_p(f)$ DOES NOT DEPEND ON THE BASIS

Change coordinate : $\varphi_\alpha \longrightarrow \varphi_\alpha'$ i.e. $x^\mu \longrightarrow x^{\mu'}$

The basis changes ! $\partial_\mu \longrightarrow \partial_{\mu'}$

and thus the components : $v^\mu \longrightarrow v^{\mu'}$.

Relation :

- $$\partial_{\mu'} = \frac{\partial}{\partial x^{\mu'}} \underset{\uparrow}{=} \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial}{\partial x^\mu} = \partial_\mu$$ with $\frac{\partial x^\mu}{\partial x^{\mu'}} \rightarrow \mu^{TH}$ component of the map : $\varphi' \circ \varphi^{-1}$

chain rule!

- $\rightarrow$ $$v^{\mu'} = v^\mu \frac{\partial x^{\mu'}}{\partial x^\mu}$$ "vector transformation law" (component)

TANGENT VECTOR FIELD

Assignment of T_pM $\forall p \in M$.

Remarks

- $p \neq q \rightarrow T_pM \neq T_qM$, tangent spaces at different pts are different!
- $\forall p$ $v_p(f) \in T_pM$ is a function : $M \rightarrow \mathbb{R}$

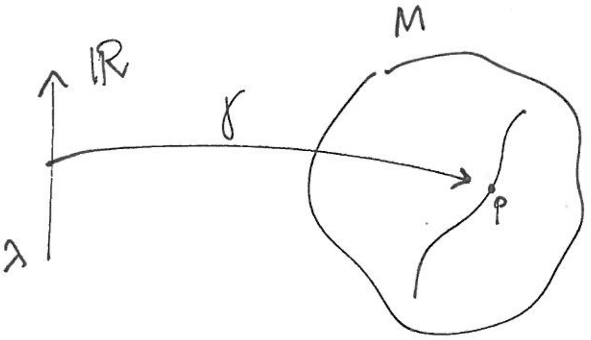
  Vector field is smooth $\leftrightarrow$ $\forall f \in C^\infty \Rightarrow v(f) \in C^\infty$.

  Note that:

  $\left.\begin{array}{l}\partial_\mu \text{ is smooth} \\ v_p \text{ smooth}\end{array}\right\} \rightarrow v^\mu$ are smooth functions.

- From now on we will not distinguish between tangent vector and tangent vector field
  - Extension of point-objects to fields is done in a similar way as for vectors
  - Context should define of what object one is referring to.

SMOOTH CURVE ON M

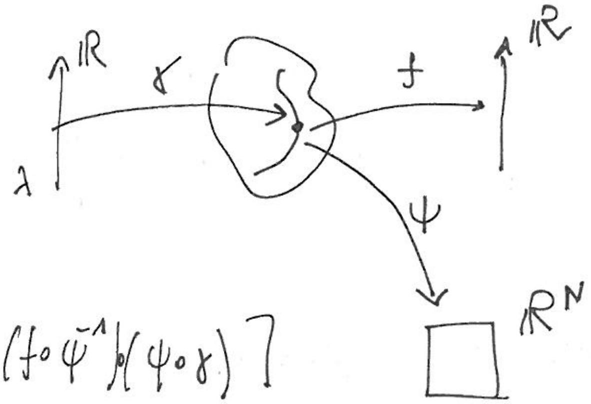


$\gamma : \mathbb{R} \to M$, C^∞

$\gamma(\lambda) = p$

$\forall p \in M$ one associates a tangent vector

$v_\gamma : F \to \mathbb{R}$



$$v_\gamma(f) \equiv \frac{d}{d\lambda}\left(f \circ \gamma\right)$$

$$= \frac{d}{d\lambda}\left(f \circ \psi^{-1} \circ \psi \circ \gamma\right) = \frac{d}{d\lambda}\Big[\underbrace{(f \circ \psi^{-1})}_{\mathbb{R}^N \to \mathbb{R}} \underbrace{(\psi \circ \gamma)}_{x^\mu(\lambda)} \Big]$$

$$= \sum_\mu \underbrace{\frac{dx^\mu}{d\lambda}}_{\substack{v_\gamma^\mu \\ \text{comp.}}} \underbrace{\frac{\partial}{\partial x^\mu}}_{\substack{e_\mu \\ \text{basis.}}} (f \circ \psi^{-1}) = \frac{dx^\mu}{d\lambda}\, \partial_\mu (f \circ \psi^{-1})$$

→ the components of the tangent vector to the curve are derivatives of the coordinates.

<u>Remark</u>:

- Given a curve γ, one can find the $v_\gamma(p)$ from the comp. $v^\mu \equiv \frac{dx^\mu}{d\lambda}$

- Given a vector at one point p, one can construct the curve γ by solving in $\mathbb{R}^N$ for the components:

$$\frac{dx^\mu}{d\lambda} = v^\mu(\bar{x})$$

1^{st} order ODE system →
Local existence and uniqueness of solution!

- Interpretation of tangent vector as infinitesimal displacement.

$\phi_t : \mathbb{R} \times M \longrightarrow M$, 1-parameter family of diffeomorfism.

$t, s \in \mathbb{R}$: $\phi_t \circ \phi_s = \phi_{t+s}$ $\approx$ smooth transformation on M.

Given $p \in M$ and ϕ_t one can <u>define</u> the curve :

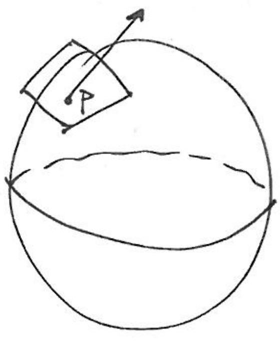
$$\gamma_p \equiv \phi_t(p) : \mathbb{R} \longrightarrow M$$

this curves has a corresponding tangent vector...

$\longrightarrow$ tangent vectors as generator of diffeomorfism.

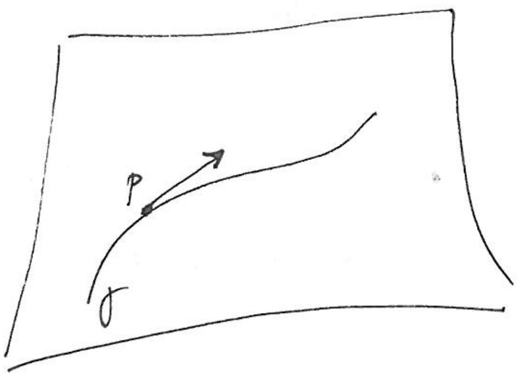
<u>Recap</u> :

- The definition of tangent vector is compatible and extend the definition of 4-vectors given in SR.
  The definition given here is "intrinsic" and apply to abirary geometries, arbitrary dimensions and arbitrary coordinate systems. Steps :



- intuitive def. of tangent vector as infinitesimal displacement

- $v \longleftrightarrow$ directional derivative



- compatible with tangent to the curve γ at point p

COTANGENT VECTOR SPACE

$$T_p^* M \equiv \{ \omega : T_pM \to \mathbb{R},\ \text{linear} \}$$

- ω : <u>dual vector</u> or <u>1-form</u>

- T_p^*M is a vector space with the rule: $(a\omega_1 + b\omega_2)(v) = a\,\omega_1(v) + b\,\omega_2(v)$

  $a, b \in \mathbb{R}$

  $v_{1,2} \in T_pM$

- Given a basis $\{e_\mu\}$ of T_pM, one <u>defines</u>:

$$e^{*\mu} : \quad e^{*\mu}(e_\nu) = \delta^\mu_\nu$$

- $\{e^{*\mu}\}$ is a basis of T_p^*M, $\quad \omega = \sum_\mu \omega_\mu e^{*\mu}$
- $\dim T_p^*M = \dim T_pM$
- The action of ω is simply given by the action on the basis:

$$\omega(v) = \sum_\mu \omega_\mu e^{*\mu}(v) = \sum_\mu \omega_\mu\, e^{*\mu}\Big(\sum_\nu v^\nu e_\nu\Big) =$$

$$\underset{\text{linearity}}{=} \sum_\mu \sum_\nu \omega_\mu v^\nu e^{*\mu}(e_\nu) = \sum_{\mu,\nu} \omega_\mu v^\nu \delta^\mu_\nu = \underbrace{\sum_\mu \omega_\mu v^\mu}_{\text{a function!}}$$

- For a <u>given basis</u> there is a correspondence between vectors and covectors: one can think of vectors as linear maps on duals:

$$v(\omega) \equiv \omega(v) = \omega_\mu v^\mu .$$

In this sense one has $T_p^{**}M = T_pM$.

Example of dual vector : gradient of a scalar field

Worldline $x^\mu(\lambda)$ in Minkowski spacetime (curve, γ)

Scalar field $\phi(x^\mu)$

the tangent vector to γ has components : $v^\mu = \frac{dx^\mu}{d\lambda}$

the value of the scalar field is parametrized by λ : $\phi(x^\mu)\big|_\gamma = \phi(x^\mu(\lambda))$

$\rightarrow$ $\frac{d\phi}{d\lambda}$ is "the rate of change" of the scalar field along γ.

$$\boxed{\frac{d\phi}{d\lambda} = \frac{\partial\phi}{\partial x^\mu}\frac{dx^\mu}{d\lambda} = \frac{\partial\phi}{\partial x^\mu} v^\mu}$$

this equation can be viewed as a map

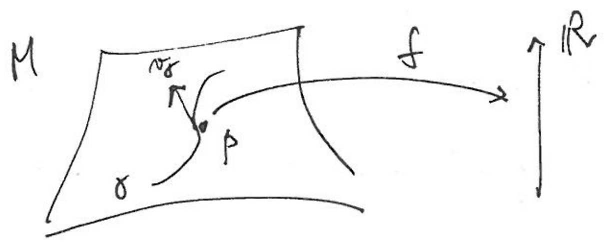
$$v \longrightarrow \frac{d\phi}{d\lambda} \in \mathbb{R}$$

$\Rightarrow$ is a 1-form!

$$\left.\begin{aligned} \omega(v) &= \frac{d\phi}{d\lambda} = \frac{\partial\phi}{\partial x^\mu} v^\mu \\ &= \omega_\mu v^\mu \end{aligned}\right\} \rightarrow \underbrace{\omega_\mu = \frac{\partial\phi}{\partial x^\mu} = \left(\frac{\partial\phi}{\partial t}, \frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}\right)}_{\text{components of the grad}(\phi).}$$

Example

In general, the simplest 1-form is the grad of a function on M.



$$v_\gamma(f) = \frac{df}{d\lambda} \quad \forall f$$

$$\rightarrow \quad v_\gamma = \frac{d}{d\lambda}$$

$$\omega(v_\gamma) \equiv df(v_\gamma) = df\left(\frac{d}{d\lambda}\right) = \frac{df}{d\lambda}$$

Q: Why the grad is the simplest choice?

A: The value of the grad depends only on the <u>point</u> p.
If I had taken the function itself, at p, I would not know how to take the derivative!

Natural basis for T_p^*M

$\{\partial_\mu\}$ natural basis for T_pM

The grad of the coordinates is a natural basis for T_p^*M since:

$$dx^\mu(\partial_\nu) = \frac{\partial x^\mu}{\partial x^\nu} = \delta^\mu{}_\nu$$

$\rightarrow \omega \in T_p^*M \,, \quad \omega = \omega_\mu \, dx^\mu .$

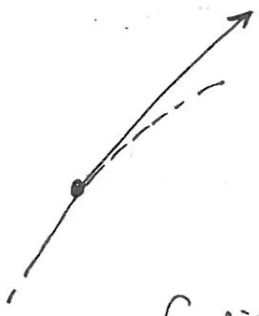
How do basis and component transform under coord. changes?

$x^\mu \longrightarrow x^{\mu'}$

$$dx^{\mu'} = \frac{\partial x^{\mu'}}{\partial x^\mu} dx^\mu$$

$$\omega_{\mu'} = \frac{\partial x^\mu}{\partial x^{\mu'}} \omega_\mu$$

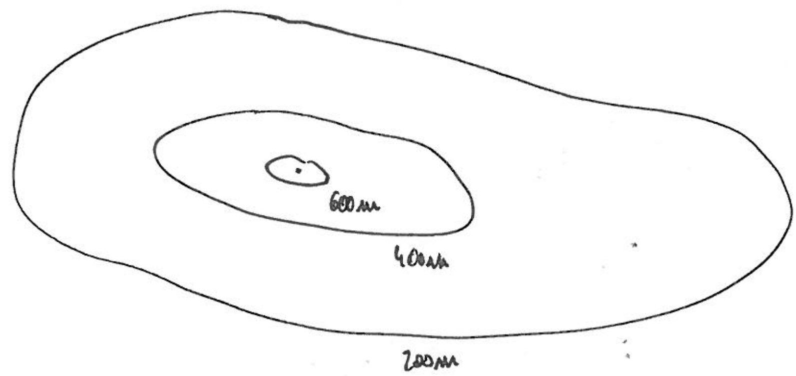
Abstract visualisation of 1-forms



"vector" $\rightarrow$ direction, tangent to curve, derivative.

How about a 1-form?

Consider a topographical map = contours of constant elevation
equisurfaces of a scalar field

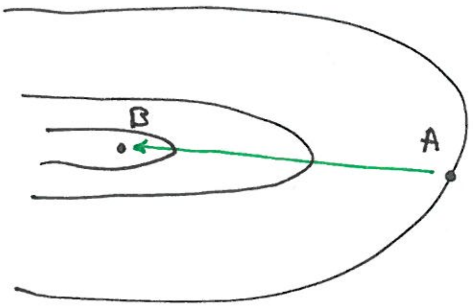


h = elevation

dh = grad(h)

Closer lines = dh is largest

In order to know the elevation from point A to point B,

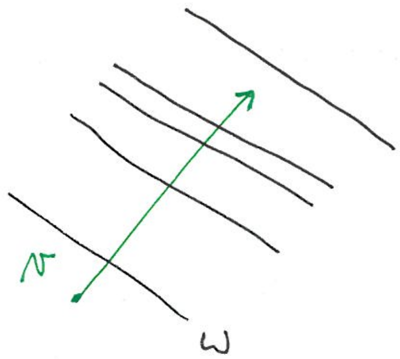


one draws the vector $\overrightarrow{AB}$ and count the lines that the vector crosses

$\longrightarrow$ The value of grad(h) = # surfaces crossed by $\overrightarrow{AB}$

In gen:

$\Rightarrow$ 1-forms can be represented by a series of surfaces



– Surfaces are // because we are at one point in the manifold

– Number of crossings with $\vec{v}$ gives the value $\omega(v)$

$\omega(v) \simeq 4.5$

TENSORS

Tangent vectors and co-vectors generalize to tensors.

A tensor of type (<u>rank</u>) (k,ℓ) is a multilinear map:

$$T : \underbrace{T_p^*M \times \dots \times T_p^*M}_{k} \times \underbrace{T_pM \times \dots \times T_pM}_{\ell} \longrightarrow \mathbb{R}$$

$\omega_{1,2} \in T_p^*M$ — $T(\dots, a\omega_1 + b\omega_2, \dots) = aT(\dots, \omega_1, \dots) + bT(\dots, \omega_2, \dots)$

$v_{1,2} \in T_pM$ — $T(\dots, av_1 + bv_2, \dots) = aT(\dots, v_1, \dots) + bT(\dots, v_2, \dots)$

$a, b \in \mathbb{R}$

<u>Examples</u>

- $(0,0)$ tensor = scalar
- $(0,1)$ tensor = dual vector
- $(1,0)$ tensor = vector

<u>Examples</u> (Physics)

- Faraday/Maxwell tensor is a $(0,2)$ tensor
- Stress-energy tensor for particles, matter, etc

→ <u>Tensor fields</u>

<u>Properties</u>

- Tensors form a vector space of dim $= n^{k+\ell}$, $\mathcal{T}(k,\ell)$
- <u>Tensor product</u>

$T_1 \in \mathcal{T}(k,\ell)$

$T_2 \in \mathcal{T}(k',\ell')$

$$T_1 \otimes T_2 : \mathcal{T}(k,\ell) \times \mathcal{T}(k',\ell') \longrightarrow \mathcal{T}(k+k', \ell+\ell')$$

$$T_1 \otimes T_2 (\omega_1 \dots \omega_{k+k'}, v_1 \dots v_{\ell+\ell'}) = T_1(\omega_1 \dots \omega_k, v_1 \dots v_\ell)\, T_2(\omega_1 \dots \omega_{k'}, v_1 \dots v_{\ell'})$$

Note that $T_1 \otimes T_2 \neq T_2 \otimes T_1$.

- Tensor basis

$\{e_\mu\}$ basis of T_pM , $\{e^{*\mu}\}$ basis of T_p^*M

Construct a basis for $\mathcal{T}(k,l)$.

Action of the tensor:

$$T(\omega_1, \dots, \omega_k, v_1, \dots, v_l) =$$

$$= T(\dots, \underbrace{\omega_{\mu_j} e^{*\mu_j}}_{j^{th}\text{ position}}, \dots, \underbrace{v^{\nu_i} e_{\nu_i}}_{i^{th}\text{ position}}, \dots) =$$

$$\underset{\text{linearity}}{=} \sum_{\mu_1} \dots \sum_{\nu_1} \dots \omega_{\mu_1} \cdots \omega_{\mu_k} v^{\nu_1} \cdots v^{\nu_l} \underbrace{T(\dots, e^{*\mu_j}, \dots, e_{\nu_i}, \dots)}_{\text{Tensor components } \equiv T^{\mu_1 \dots \mu_k}{}_{\nu_1 \dots \nu_l}} =$$

$$= \omega_{\mu_1} \cdots \omega_{\mu_k} v^{\nu_1} \cdots v^{\nu_l} \; T^{\mu_1 \dots \mu_k}{}_{\nu_1 \dots \nu_l}$$

claim: $e_{\mu_1} \otimes \dots \otimes e_{\mu_k} \otimes e^{*\nu_1} \otimes \dots \otimes e^{*\nu_l}$ is a basis of $\mathcal{T}(k,l)$.

verify:

$$T(\omega_1, \dots, \omega_k, v_1, \dots, v_l) = T^{\mu_1 \dots \mu_k}{}_{\nu_1 \dots \nu_l} \; e_{\mu_1} \otimes \dots \otimes e^{\nu_1} \otimes \dots \otimes e^{\nu_l} (\omega_1 \dots v_l) =$$

$$= T^{\mu_1 \dots \mu_k}{}_{\nu_1 \dots \nu_l} \; e_{\mu_1} \otimes \dots \otimes e^{*\nu_l} (\dots, \omega_{\sigma_j} e^{*\sigma_j}, \dots, v^{\alpha_i} e_{\alpha_i}, \dots) =$$

$$= T^{\mu_1 \dots \mu_k}{}_{\nu_1 \dots \nu_l} \; \omega_{\sigma_1} \cdots \omega_{\sigma_k} v^{\alpha_1} \cdots v^{\alpha_l} \; e_{\mu_1} \otimes \dots \otimes e^{*\nu_l} (e^{*\sigma_1}, \dots, e_{\alpha_l}) =$$

$$= T^{\mu_1 \dots \mu_k}{}_{\nu_1 \dots \nu_l} \; \omega_{\sigma_1} \cdots v^{\alpha_l} \; e_{\mu_1}(e^{*\sigma_1}) e_{\mu_2}(e^{*\sigma_2}) \dots e^{*\nu_1}(e_{\alpha_1}) \dots e^{*\nu_l}(e_{\alpha_l}) =$$

$$e_{\mu_j}(e^{*\sigma_j}) = \delta^{\sigma_j}{}_{\mu_j} \quad ; \quad e^{*\nu_i}(e_{\alpha_i}) = \delta^{\nu_i}{}_{\alpha_i} \quad ;$$

$$= T^{\mu_1 \dots \mu_k}{}_{\nu_1 \dots \nu_l} \, \omega_{\mu_1} \dots \omega_{\mu_k} v^{\nu_1} \dots v^{\nu_l} \qquad \square$$

Note: in terms of components the tensor product is:

$$(T_1 \otimes T_2)^{\mu_1 \dots \mu_k \mu_{k+1} \dots \mu_{k+k'}}{}_{\nu_1 \dots \nu_l \nu_{l+1} \dots \nu_{l+l'}} = T_1{}^{\mu_1 \dots \mu_k}{}_{\nu_1 \dots \nu_l} \, T^{\mu_{k+1} \dots \mu_{k+k'}}{}_{\nu_{l+1} \dots \nu_{l+l'}}$$

- <u>Tensor contraction</u>

$$C_{(ij)} : \mathcal{T}(k,l) \longrightarrow \mathcal{T}(k-1, l-1)$$

$$C_{(ij)} T \equiv \sum_{\sigma=1}^{m} T(\dots, \underset{\substack{\uparrow \\ i^{th} \text{ position}}}{e^{*\sigma}}, \dots, \underset{\substack{\uparrow \\ j^{th} \text{ position}}}{e_\sigma}, \dots)$$

In terms of components:

$$(C_{(ij)} T)^{\mu_1 \dots \mu_k}{}_{\nu_1 \dots \nu_l} = \sum_{\sigma} T^{\mu_1 \dots \sigma \dots \mu_k}{}_{\nu_1 \dots \sigma \dots \nu_l}$$

- <u>Change of coordinates</u>

$$e_\mu = \partial_\mu \quad , \quad e^* = dx^\mu$$

$$\partial_{\mu'} = \frac{\partial x^\mu}{\partial x^{\mu'}} \partial_\mu \quad , \quad dx^{\mu'} = \frac{\partial x^{\mu'}}{\partial x^\mu} dx^\mu \quad \longrightarrow \text{basis transformation}$$

Components:

$$T^{\mu_1' \dots \mu_k'}{}_{\nu_1' \dots \nu_l'} = \frac{\partial x^{\mu_1'}}{\partial x^{\mu_1}} \dots \frac{\partial x^{\mu_k'}}{\partial x^{\mu_k}} \frac{\partial x^{\nu_1}}{\partial x^{\nu_1'}} \dots \frac{\partial x^{\nu_l}}{\partial x^{\nu_l'}} T^{\mu_1 \dots \mu_k}{}_{\nu_1 \dots \nu_l}$$

- Abstract notation

- $T^{\mu_1 \dots \mu_k}{}_{\nu_1 \dots \nu_\ell}$ components of $T \in \mathcal{T}(k,\ell)$ w.r.t. basis, say $\{\partial_\mu, dx^\mu\}$

  use latin letters $\mu, \nu, \alpha, \beta, \rho, \sigma, \dots$

- "abstract" $T^{a_1 \dots a_k}{}_{b_1 \dots b_\ell} \rightarrow$ symbol to indicate $T \in \mathcal{T}(k,\ell)$
  No association to a basis

  - indicate "where" indexes of components would go
  - keep notation compact, avoid to introduce other symbols for contractions

  Examples

  - $T^{abc}{}_{de} \in \mathcal{T}(3,2)$
  - $T^{abc}{}_{be} \in \mathcal{T}(2,1)$ obtained by the contraction $C_{(24)}T$
  - $T^{abc}{}_{de}\, S^f{}_g \in \mathcal{T}(4,3)$ obtained by the tensor product
    $(T^{abc}{}_{de}) \otimes (S^f{}_g)$

- Symmetry & Anti-symmetry

- $S \in \mathcal{T}(0,2)$ is symmetric iff $S(v_1, v_2) = S(v_2, v_1)$ $\quad \forall\, v_{1,2} \in T_pM$

  Abstract notation: $S_{ab} = S_{ba}$, because in components:

  $$S_{\mu\nu} v_1^\mu v_2^\nu = S_{\nu\mu} v_2^\nu v_1^\mu \Rightarrow S_{\mu\nu} = S_{\nu\mu}.$$

  to symmetrize a tensor $(0,2)$: $\quad S(v_1, v_2) \equiv \frac{1}{2}\left(T(v_1,v_2) + T(v_2,v_1)\right)$

  or: $\quad S_{ab} = T_{(ab)} \equiv \frac{1}{2}\left(T_{ab} + T_{ba}\right).$

- $A \in \mathcal{T}(0,2)$ is <u>antisymmetric</u> iff $A(v_1,v_2) = -A(v_2,v_1)$

Abstract notation: $A_{ab} = -A_{ba}$

Components: $A_{\mu\nu} v_1^{\mu} v_2^{\nu} = -A_{\nu\mu} v_2^{\nu} v_1^{\mu} \rightarrow A_{\mu\nu} = -A_{\nu\mu}$

To antisymmetrize a $(0,2)$ tensor: $A(v_1,v_2) = \frac{1}{2}\left(T(v_1,v_2) - T(v_2,v_1)\right)$

or $A_{ab} = T_{[ab]} \equiv \frac{1}{2}\left(T_{ab} - T_{ba}\right)$.

- Generic totally symmetric/antisymm. tensors:

$$T_{(a_1 \dots a_m)} \equiv \frac{1}{m!} \sum_{\pi} T_{a_{\pi(1)} \dots \pi(m)}$$

$$T_{[a_1 \dots a_m]} \equiv \frac{1}{m!} \sum_{\pi} \sigma_{\pi} T_{a_{\pi(1)} \dots \pi(m)}$$

π = permutations

$\sigma_\pi = \pm 1$ even/odd permutation

- One can symmetrise/antisymm. upper indexes or groups of indexes; ex.

$$T^{(ab)c}{}_{de} = \frac{1}{2}\left(T^{abc}{}_{de} + T^{bac}{}_{de}\right)$$

$$T^{abc}{}_{[de]} = \frac{1}{2}\left(T^{abc}{}_{de} - T^{abc}{}_{ed}\right)$$

$$T^{(ab)c}{}_{[de]} = \frac{1}{2}\left(T^{abc}{}_{[de]} + T^{bac}{}_{[de]}\right) =$$

$$= \frac{1}{2}\cdot\frac{1}{2}\left(T^{abc}{}_{de} - T^{abc}{}_{ed} + T^{bac}{}_{de} - T^{bac}{}_{ed}\right)$$

Example: Stress-energy tensor

Matter and fields are described by $T \in \mathcal{T}(0,2)$ symmetric tensor field.

u, vector field = velocity tangent to the worldline of an observer $\mathcal{O}$
Timelike vector*

- $T(u,u) \equiv$ energy density measured by $\mathcal{O}$
- $-\frac{1}{c} T(e_i, u) \equiv$ impulse density measured by $\mathcal{O}$

where: e_i = vectors $\perp$ to u

$\{e_i\}$ $i=1,2,3$ basis of the plane $\perp$ worldline

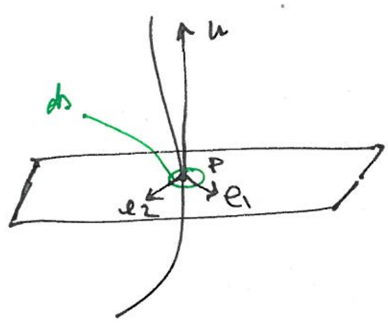
$p^i \equiv -\frac{1}{c} T(e_i,u)$, 3 numbers at each point

$p \equiv p^i e_i$, IMPULSE VECTOR OF THE MATTER.

- $-c\,T(u,e_i) \equiv$ energy flux measured by $\mathcal{O}$

$\varphi^i \equiv -c\,T(u,e_i)$

$\varphi \equiv \varphi^i e_i$, ENERGY per unit of time through a surface element



$$\frac{dE}{dt} = \varphi^i n_i \, ds$$

---

* Meaning of "timelike" is clear in SR.
We will see in general the def. later: $g(u,u) = g_{\mu\nu} u^\mu u^\nu < 0$.

Observation :

$$\text{symmetry}^{*} \Rightarrow \quad T(u, e_i) = T(e_i, u)$$

$$\varphi^i = c^2 p^i \quad \rightarrow \quad \varphi = c^2 p$$

energy flux $= c^2 \times$ impulse density

$\sim E = mc^2$ in SR ...

- $T(e_i, e_j) \equiv$ force exerted by the matter in direction e_i on the unit surface identified by e_j

Example in $n=4$ and SR

$S_{ij} \equiv T(e_i, e_j)$ is the 3D stress energy tensor of matter

For a perfect fluid : $\quad S_{ij} = \mathrm{diag}(P, P, P)$

pressure of the fluid

Observation : For the matter

(i) $E \equiv T(u,u) > 0 \quad \forall u$ timelike $\Leftrightarrow$ WEAK ENERGY CONDITION (WEC)

(ii) P is non-spacelike $\Rightarrow E^2 \geq c^2 p^2 \Leftrightarrow$ DOMINAN ENERGY CONDITION (DEC)

$p^i p_i \leq 0$

- DEC $\Rightarrow$ WEC
- "Standard" forms of matter including EM fields satisfy DEC.

$*$ For the moment we <u>assume</u> symmetry.
Justification will be given by Einstein equations.

METRIC

Given a point $p \in M$, we want to introduce an object that gives the "infinitesimal squared distance" associated to an "infinitesimal displacement".

The idea is of course to generalize the concepts:

$$dl^2 = \sum_{i=1}^{N} (dx^i)^2 \qquad \text{in Euclidean } \mathbb{R}^N$$

$$ds^2 = \sum_{\mu=0}^{3} \sum_{\nu=0}^{3} \eta_{\mu\nu}\, dx^\mu dx^\nu \qquad \text{in Minkowski/Lorentz } \mathbb{R}^4$$

to the case of a generic manifold.

"infinitesimal displacement" $\longleftrightarrow$ T_pM

$ds^2 \longleftrightarrow$ quadratic expression ...

Def: Metric :
- tensor $(0,2)$ $\quad g: T_pM \times T_pM \longrightarrow \mathbb{R}$
- Symmetric $\quad g(v,u) = g(u,v)$
- Non-degenerate $\quad g(v,v_1) = 0 \ \forall v \Rightarrow v_1 = 0$

Introduce a coordinate basis:

$$\underbrace{ds^2}_{\text{"line element"}} = g = \sum_{\mu,\nu} g_{\mu\nu} \underbrace{(dx^\mu)}_{\text{1-form basis}} \otimes (dx^\nu) = g_{\mu\nu}\, dx^\mu dx^\nu$$

Properties

1. It is always possible to introduce special coordinates called normal coords such that

$$g(e_\mu, e_\nu) = \begin{cases} 0 & \mu \neq \nu \\ \pm 1 & \mu = \nu \end{cases}$$

i.e. the component of the metric in normal coordinates are:

$$g_{\mu\nu} = \mathrm{diag}\,(-1,-1,\dots,-1,+1,+1,\dots,+1)$$

Physically these coordinates (observers) must exist in GR because of the EEP.

$\rightarrow$ local (at point p) Lorentz frame of SR!

Mathematically it can be proven in general (later). But note that:

The number of basis vector giving "+1" and those giving "-1" is independent on the specific choice of the basis.

Def: SIGNATURE of g = # of "+" and "−"

2. g is said:

- Euclidean or Riemannian iff $\mathrm{sign}(g)$ has all "+"

$\rightarrow$ metric is positive definite.

- Lorentzian iff $\mathrm{sign}(g)$ has a "−".

3. g establish a natural correspondence between T_pM and T_p^*M.

Given $v \in T_pM$, one associate a unique $\omega \in T_p^*M$ by:

$$v \longrightarrow g(\cdot, v)$$

i.e.

$$\omega \in T_p^*M \longleftrightarrow u \in T_pM \quad : \quad \omega(v) = g(u,v) \qquad \forall\, v \in T_pM.$$

Isomorphism (bijective and continuous).

$\rightarrow$ Property 3. allow us to "raise-lower" indexes!

$$g(v,u) = \underbrace{g_{\mu\nu} v^\mu}_{v_\nu} u^\nu = v_\nu u^\nu = v^\mu u_\mu = v^\mu \underbrace{g_{\mu\nu} u^\nu}_{u_\mu}$$

Remark :

In Newtonian or SR physics the metric is always given (and simple)
Vectors and duals are naturally identified and connected by simple expressions.
In GR the metric is not given and it is important to distinguish them.

Example : Euclidean 3-space

Cartesian coords : $dl^2 = \sum_{i=1}^{3} (dx^i)^2 = dx^2 + dy^2 + dz^2$

$g_{\mu\nu} = \mathrm{diag}(+1, +1, +1)$

change coordinates : $(x, y, z) \to (r, \theta, \varphi)$: $\begin{cases} x = r\sin\theta\cos\varphi \\ y = r\sin\theta\sin\varphi \\ z = r\cos\theta \end{cases}$

$g_{\mu\nu} = \mathrm{diag}(1, r^2, r^2\sin^2\theta)$

but the distance is always :

$$dl^2 = g_{\mu\nu}\, dx^\mu dx^\nu = dx^2 + dy^2 + dz^2 = dr^2 + r^2 d\theta^2 + r^2\sin^2 d\varphi^2$$

- components are different
- lenght is the same

Example : 2-sphere

$d\Omega^2 = d\theta^2 + \sin^2\theta\, d\varphi^2$: lenght on unit $(r=1)$ sphere

$g_{\mu\nu} = \mathrm{diag}(1, \sin^2\theta)$

<u>Example</u> : Minkowski spacetime

$$ds^2 = -dt^2 + dl^2 \qquad (c=1)$$

$$= \eta_{\mu\nu}\, dx^\mu dx^\nu \qquad \eta_{\mu\nu} = \mathrm{diag}\,(-1,+1,+1,+1) \quad : \text{Lorentzian metric}$$

<u>Example</u> : Number of independent components of a <u>symm</u> (0,2) tensor

$M = 3$

$$\begin{array}{ccc} \times & \circ & \circ \\ \times & \times & \circ \\ \times & \times & \times \end{array} \qquad \underbrace{3}_{\text{diag}} + \underbrace{(2+1)}_{\text{lower triangular}} = 6$$

$M = 4$

$$\begin{array}{cccc} \times & \circ & \circ & \circ \\ \times & \times & \circ & \circ \\ \times & \times & \times & \circ \\ \times & \times & \times & \times \end{array} \qquad 4 + (3+2+1) = 10$$

M

M - diag elements

$M^2 - M$ remaining $= M(M-1)$

$\dfrac{M(M-1)}{2}$ upper (lower) triangular

$$M + \frac{M(M-1)}{2} = \frac{2M + M^2 - M}{2} = \frac{M^2 + M}{2} = \boxed{\frac{M(M+1)}{2}}$$

<u>Example</u> : Number of indep. components of <u>antisymm.</u> (0,2) tensor

- Diagonal is "0" $\rightarrow$ only upper (lower) triangular part :

$$\boxed{\frac{M(M-1)}{2}}$$

Example: Lorentzian geometry of an expanding Universe

Consider a 2D Universe with metric $g = -dt^2 + a^2(t)\,dx^2$ $\quad x^\mu = (t, x)$

$a(t) = t^q \quad 0<q<1 \quad t \in (0, +\infty)$ SCALE FACTOR

$\hookrightarrow$ prevent problems at $a = 0 \ldots$

→ Compute the $\underbrace{\text{causal structure}}$.

= light cones = paths (curves) with tangent vector null

SR : null vector $\quad \eta_{\alpha\beta} V^\alpha V^\beta = 0$

Generalize definition: $\quad \eta \to g$

$$\eta(V,V) = 0 \longrightarrow g(V,V) = 0$$

these are 1-forms!

$$0 = g(V,V) = -dt \otimes dt\,(V,V) + a^2(t)\, dx \otimes dx\,(V,V) =$$

$$= -dt(V)\,dt(V) + a^2 dx(V)\,dx(V) =$$

$$= -dt(\dot{x}^\mu \partial_\mu)\,dt(\dot{x}^\mu \partial_\mu) + a\;dx(\dot{x}^\mu\partial_\mu)\,dx(\dot{x}^\mu\partial_\mu) =$$

$V = \frac{dx^\mu}{d\lambda}\partial_\mu$

$$= -\left[\dot{x}^\mu\,dt(\partial_\mu)\right]\left[\dot{x}^\nu\,dt(\partial_\nu)\right] + a^2\left[\dot{x}^\mu\,dx(\partial_\mu)\right]\left[\dot{x}^\nu\,dx(\partial_\nu)\right] =$$

$$= -\left[\dot{x}^\mu \frac{\partial t}{\partial x^\mu}\right]^2 + a^2\left[\dot{x}^\mu\frac{\partial x}{\partial x^\mu}\right]^2 =$$

these are now differentials!

$$= -\left(\frac{dt}{d\lambda}\right)^2 + a^2\left(\frac{dx}{d\lambda}\right)^2$$

$$= -\left(\frac{dt}{d\lambda}\right)^2 + a^2\left(\frac{dx}{dt}\frac{dt}{dx}\right)^2 \rightarrow 1\cancel{\left(\frac{dt}{d\lambda}\right)^2} = a\left(\frac{dx}{dt}\right)^2\cancel{\left(\frac{dt}{dx}\right)^2} \rightarrow$$

$$\boxed{\frac{dx}{dt} = \pm t^{-q}} \Rightarrow \boxed{t = (1-q)^{\frac{1}{1-q}}\left(\pm\, x - x_0\right)^{\frac{1}{1-q}}}$$

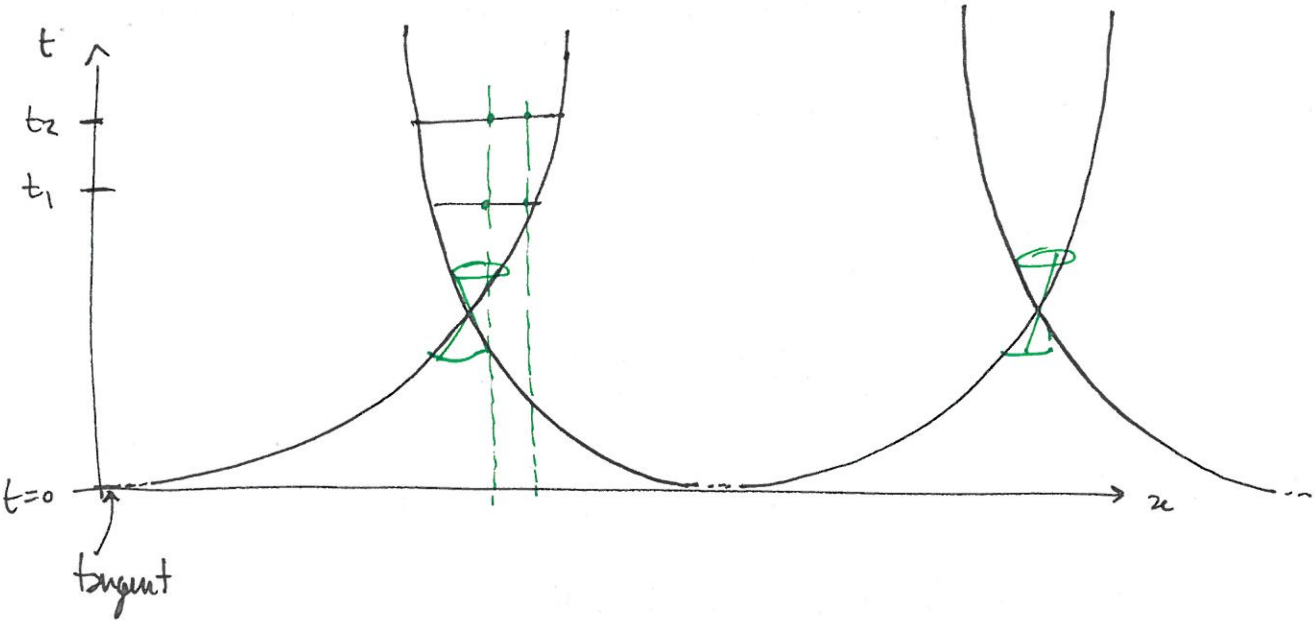
Remark :

We would have obtained the same equation by doing:

$$0 = -dt^2 + a^2 dx^2 \rightarrow 0 = -\frac{dt^2}{dx^2} + a^2 \cancel{dx^2} .$$

This works! But it is not rigorous: the meaning of the dt^2, dx^2 in "g" is not differentials.

Plot the solution:



- the light cones are tangent to $t=0$
- the light cones of 2 events do not necessarily intersect (different from Minkowski)
- the metric could describe an expanding universe :

$t = t_2 > t_1$

proper lenght : $dl^2(t_1) = a^2(t_1)\, dx^2 < a^2(t_2)\, dx^2 = dl^2(t_2)$

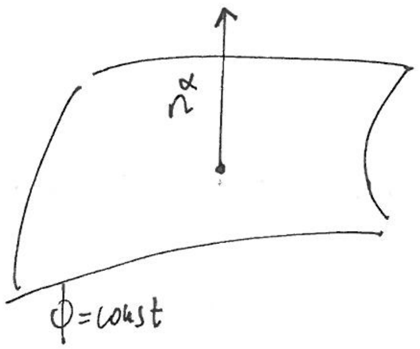
- In this universe there exist worldlines that are causally disconnected

→ example of how "curvature" can generate "horizons" ...

Example : surfaces and normal to surfaces are naturally described by 1-forms

Consider SR : $g_{\alpha\beta} = \eta_{\alpha\beta} = \mathrm{diag}(-1,+1,+1,+1)$

Consider a scalar field $\phi(X^\mu)$ and the surface $\phi = \text{const}$.



$$d\phi = \mathrm{grad}(\phi) = \left(\frac{\partial\phi}{\partial t}, \frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}\right) \quad \text{1-form}$$

Normal vector to $\phi = \text{const}$ is :

$$n^\alpha = \left(-\frac{\partial\phi}{\partial t}, \frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}\right)$$

as determined by the identification $v \longrightarrow g(\cdot, v)$:

$$d\phi(v) = g(n, v) \qquad \forall v \in \text{"}\phi = \text{const"}$$

$$\rightarrow (d\phi)_\alpha = g_{\alpha\beta}\, n^\beta$$

One can equivalently consider $d\phi$ or n to characterize $\phi = \text{const}$, but:

- Defining $\hat{n}$ requires a metric ;
- Using $d\phi = 0$ is metric independent .

$\rightarrow$ Normal surfaces are naturally described by 1-forms.

Example : Quantum Mechanics

$\psi(x) : \mathbb{R} \to \mathbb{C}$ WAVE FUNCTION , $|\psi\rangle \in \mathcal{H}$ VECTOR in Hilbert space.

The scalar product : $\int \phi^*(x)\psi(x)\,dx = \langle\phi|\psi\rangle$

we could write : $\langle\phi|\psi\rangle = \phi_\alpha \psi^\alpha \in \mathbb{R}$

and think about ϕ^* (complex conj.) or $\langle\phi|$ as a <u>1-form</u> .

Example : Perfect fluid stress-energy tensor in generic manifold

$T \in \mathcal{T}(0,2)$ symmetric

$$\boxed{T \equiv (\rho c^2 + p)\, \underline{u} \otimes \underline{u} + p\, g}$$

where :

- ρ, p scalar fields ; energy density and pressure of the fluid as measured by an observer comoving with the fluid.

- $\underline{u}$ 1-form ; dual of the fluid's 4-velocity field u

$$\underline{u} : T_pM \to \mathbb{R}$$

$$\underline{u}(v) = g(u, v) \quad \forall\, v \in T_pM$$

$$= \underbrace{g_{ab} u^a}_{\substack{\text{components of} \\ \text{the 1-form}}} v^b = u_b v^b$$

$$\underline{u} = u_\mu e^{*\mu} .$$

- g (0,2) tensor ; spacetime metric.

Abstract notation:

$$T_{ab} = (\rho c^2 + p)\, u_a u_b + g_{ab} p .$$

Justification of the definition based on the general definition of T given at p. 11.

Consider an observer with arbitrary velocity V.

The energy measured by the observer is :

$$E = T(V,V) = (\rho c^2 + p)\, \underline{u}(V) \otimes \underline{u}(V) + p\, g(V,V) =$$

$$\underset{\substack{\uparrow \\ g(v,v)=-1}}{=} (\rho c^2 + p)\, \underline{u}(v) \otimes \underline{u}(v) - p =$$

V is timelike

$$\underset{\uparrow}{=} (\rho c^2 + p)\,(u_b v^b)^2 - p = (\rho c^2 + p) W^2 - p$$

$\underline{u}(v) = g(u,v) = u_b v^b = -W$ Lorentz factor between the observer and the fluid (X) p.17 →

if $v = u$, then $\begin{cases} W = 1 \\ \mathcal{E} = \rho c^2 \end{cases}$

Remark: there is a "W^2" and not a "W" as one would expect from SR's $E = W m c^2$, because $\mathcal{E}$ is an energy density $\left(\mathcal{E} = \frac{E}{\text{Volume}}\right)$ the additional "W" comes from the lenght contraction along the direction of movement (volume = W.volume prop.) that reduces the volume and increases the energy density.

Similarly one finds the momentum/impulse density in direction "i":

$$p^i = -\frac{1}{c}\, g T(v, e_i) = -\frac{1}{c} \underbrace{(u_a v^a)}_{=-W} \underbrace{(u_a e_i^a)}_{=W\frac{v^i}{c}} (\rho c^2 + p) - \frac{p}{c} \underbrace{g(v, e_i)}_{=0}$$

$$= W^2 \left(\rho + \frac{p}{c^2}\right) v^i \qquad v^i = \text{relative velocity in direction "}i\text{"}.$$

Finally:

$$S_{ij} = T(e_i, e_j) = (\rho c^2 + p) \underbrace{(u_a e_i^a)}_{W\frac{v^i}{c}} (u_b e_j^b) + p \underbrace{g(e_i, e_j)}_{\delta_{ij}} =$$

$$= W^2 \left(\rho + \frac{p}{c^2}\right) V_i V_j + p\, \delta_{ij}.$$

if $V = u$, then $\begin{cases} V_i = 0 \\ S_{ij} = p\, \delta_{ij} \end{cases}$

$$\left.\begin{array}{ll} \mathcal{E} \geq 0 & \text{WEC} \\ P^i P_i c^2 \leq \mathcal{E} & \text{DEC} \end{array}\right\} \Rightarrow \begin{cases} \rho \geq 0 \ \wedge \ \rho c^2 + p \geq 0 \\ \rho c^2 \geq |p| \end{cases}$$

(x) Remember from SR: $u^a = (W, W\vec{v})$

$V^a = (1, 0, 0, 0)$: Ref. system.

$\Rightarrow$ the Lorentz factor between V^a and u^a is:

$$u^a V_a = \eta_{ab}\, u^a V^b = -W.$$

DIFFERENTIAL OR P-FORMS

p-form = totally <u>antisymmetric</u> $(0,p)$ tensor

Λ_p vector space of p-forms

Abstract notation $\quad \omega = \omega_{[a_1 \dots a_p]} \in \Lambda_p$

<u>Properties of Λ_p</u>

- $\dim M = m \implies \dim \Lambda_p = \dfrac{m!}{p!\,(m-p)!}$

- No p-forms for $p > m$:

Scalar · dual vector (m-components) · "2x2 antisymm. matrix" $\begin{pmatrix} 0 & x \\ -x & 0 \end{pmatrix}$

$m=2$

p	0	1	2
$\dim \Lambda_p$	1	2	1

$m=3$

p	0	1	2	3
$\dim \Lambda_p$	1	3	3	1

$m=4$

p	0	1	2	3	4
$\dim \Lambda_p$	1	4	6	4	1

- Given $\{e_\mu\}$ basis of T_pM and $\{e^{*\mu}\}$ basis of T_p^*M,

a $(0,p)$ tensor basis would be:

$$e^{*\mu_1} \otimes \dots \otimes e^{*\mu_p}$$

but note that for an antisymm. tensor most of the components would be redundant...

Wedge product

$\omega, \eta \in \Lambda_1$

$$\wedge \ : \ \Lambda_1 \times \Lambda_1 \longrightarrow \Lambda_2$$

$$(\omega \wedge \eta) := \omega \otimes \eta - \eta \otimes \omega$$

- check it is a 2-form:

$$(\omega \wedge \eta)(u,v) = \qquad\qquad u, v \in T_pM$$

$$= \omega \otimes \eta \, (u,v) - \eta \otimes \omega (u,v) \overset{\text{def. of } \otimes}{=} \omega(u)\eta(v) - \eta(u)\omega(v) =$$

$$\overset{\text{product can be commuted}}{=} \eta(v)\omega(u) - \omega(v)\eta(u) = -[-\eta(v)\omega(u) + \omega(v)\eta(u)] =$$

$$\overset{\text{switch the order ... (product of functions)}}{=} -[\omega(v)\eta(u) - \eta(v)\omega(u)] \overset{\text{def. of } \otimes}{=} -(\omega \otimes \eta \, (v,u) - \eta \otimes \omega (v,u))$$

$$\overset{\text{def. of } \wedge}{=} -(\omega \wedge \eta)(v,u)$$

- Given $\{e_\mu\}$ basis of T_pM
  $\{e^{*\mu}\}$ " " T_p^*M $\qquad (e^{*\mu}(e_\nu) = \delta^\mu{}_\nu)$,

$\{e^{*\mu} \wedge e^{*\nu}\}$ is a basis for Λ_2.

$$\alpha \in \Lambda_2 : \begin{cases} \alpha = \alpha_{\mu\nu} \, e^{*\mu} \otimes e^{*\nu} \\ \alpha_{\mu\nu} = -\alpha_{\nu\mu} \end{cases}$$

- Verify components are antisymmetric; first recall that:

$$\alpha(e_\mu, e_\nu) = \alpha_{\sigma\rho}\, e^{*\sigma} \otimes e^{*\rho} (e_\mu, e_\nu) =$$

$$= \alpha_{\sigma\rho}\, e^{*\sigma}(e_\mu) \otimes e^{*\rho}(e_\nu) =$$

$$= \alpha_{\sigma\rho}\, \delta^\sigma_{\ \mu}\, \delta^\rho_{\ \nu} = \alpha_{\mu\nu} .$$

Then it follows immediately: $\alpha(e_\mu, e_\nu) = -\alpha(e_\nu, e_\mu) \Rightarrow \alpha_{\mu\nu} = -\alpha_{\nu\mu}$.

- Verify $e^{*\mu} \wedge e^{*\nu} = e^{*\mu} \otimes e^{*\nu} - e^{*\nu} \otimes e^{*\mu}$ is a basis:

- $$e^{*\rho} \wedge e^{*\sigma} (e_\mu, e_\nu) = e^{*\rho}(e_\mu)\, e^{*\sigma}(e_\nu) - e^{*\sigma}(e_\mu)\, e^{*\rho}(e_\nu) =$$

$$= \delta^\rho_{\ \mu}\, \delta^\sigma_{\ \nu} - \delta^\sigma_{\ \mu}\, \delta^\rho_{\ \nu} \qquad (i)$$

- $$e^{*\rho} \wedge e^{*\sigma} (e_\nu, e_\mu) = \delta^\rho_{\ \nu}\, \delta^\sigma_{\ \mu} - \delta^\sigma_{\ \nu}\, \delta^\rho_{\ \mu} \qquad (ii)$$

- So the linear combinations:

$$\alpha_{\rho\sigma}\, e^{*\rho} \wedge e^{*\sigma} (e_\nu, e_\mu) \overset{(ii)}{=} \alpha_{\nu\mu} - \alpha_{\mu\nu}$$

$$\alpha_{\rho\sigma}\, e^{*\rho} \wedge e^{*\sigma} (e_\mu, e_\nu) \overset{(i)}{=} \underbrace{\alpha_{\mu\nu} - \alpha_{\nu\mu}}_{\text{these are the definitions of}} = -(\alpha_{\nu\mu} - \alpha_{\mu\nu})$$

$$= 2\, \alpha_{[\mu\nu]}$$

$\Rightarrow$ the linear combination is a 2-form and span Λ_2.

- the wedge product is <u>associative</u> $[\ (a\wedge b)\wedge c = a\wedge b\wedge c\ ]$

$\Rightarrow$ it can be trivially extended to arbitrary p-forms :

$$\wedge : \Lambda_p \times \Lambda_q \longrightarrow \Lambda_{p+q}$$

$$(\omega,\eta) \longmapsto \omega\wedge\eta = \text{Antisymm combination } (\omega\otimes\eta)$$

In abstract notation:

$$(\omega\wedge\eta)_{a_1\cdots a_p a_{p+1}\cdots a_p} \equiv \frac{(p+q)!}{p!\,q!}\,\omega_{[a_1\cdots a_p}\eta_{a_{p+1}\cdots a_{p+q}]}$$

<u>Properties</u>

- $\omega\in\Lambda_p \quad \eta\in\Lambda_q$

$$(\omega\wedge\eta) = (-1)^{pq}\,(\eta\wedge\omega)$$

- <u>Basis of Λ_p</u>

$\omega\in\Lambda_p$ $$\omega = \underbrace{\omega_{\mu_1\cdots\mu_p}}_{\substack{\text{components.}\\ \text{functions } \mathcal{O}\to\mathbb{R}}}\ \underbrace{e^{*\mu_1}\wedge e^{*\mu_2}\wedge\ldots\wedge e^{*\mu_p}}_{\text{basis}}$$

if $\begin{cases} e_\mu = \partial_\mu \\ e^{*\mu} = dx^\mu \end{cases}$, then the "natural basis" is $dx^{\mu_1}\wedge\ldots\wedge dx^{\mu_p}$

i.e. $\omega = \omega_{\mu_1\cdots\mu_p}\, dx^{\mu_1}\wedge\ldots\wedge dx^{\mu_p}$

- The components transform as expected from $dx^{\mu'} = \frac{\partial x^{\mu'}}{\partial x^\mu}dx^\mu$:

$$\omega_{\mu_1'\cdots\mu_p'} = \frac{\partial x^{\mu_1}}{\partial x^{\mu_1'}}\cdots\frac{\partial x^{\mu_p}}{\partial x^{\mu_p'}}\, dx^{\mu_1}\cdots dx^{\mu_p}.$$

n-form on a n-manifold

A very special case of p-forms is when $p = n = \dim M$.

In this case: $\dim \Lambda_n = 1 \Rightarrow$

$$\omega \in \Lambda_n \qquad \omega = a \, dx^1 \wedge \dots \wedge dx^n \qquad (\underline{\text{no sum}})$$

The (single) component $a = a(x^\mu)$ is $\underline{\text{a}}$ function.

Under a coordinate transformation:

$$x^\mu \to x^{\mu'} \Rightarrow a(x^\mu) \longrightarrow a(x^{\mu'}) \underbrace{\left| \frac{\partial x^\mu}{\partial x^{\mu'}} \right|}_{\text{determinant of the coords. transformation.}}$$

(The result follows from $dx^{\mu'} = \frac{\partial x^{\mu'}}{\partial x^\mu} dx^\mu$ and the wedge product of dx^μ ...)

Example: $n = 2$

$$\omega = a(x^\mu) \, dx^1 \wedge \dots \wedge dx^n = a(x^1, x^2) \, dx^1 \wedge dx^2$$

$$= a(y^1, y^2) \, dy^1 \wedge dy^2 = a(y^1, y^2) \left(dy^1 \otimes dy^2 - dy^2 \otimes dy^1 \right) =$$

coords. transf.

$$\odot \left\{ \begin{array}{l} dy^1 = \frac{\partial y^1}{\partial x^1} dx^1 + \frac{\partial y^1}{\partial x^2} dx^2 \\ dy^2 = \frac{\partial y^2}{\partial x^1} dx^1 + \frac{\partial y^2}{\partial x^2} dx^2 \end{array} \right.$$

$$= a(y^1, y^2) \left[\left(\frac{\partial y^1}{\partial x^1} dx^1 + \frac{\partial y^1}{\partial x^2} dx^2 \right) \otimes \left(\frac{\partial y^2}{\partial x^1} dx^1 + \frac{\partial y^2}{\partial x^2} dx^2 \right) + \right.$$

$$\left. - \left(\frac{\partial y^2}{\partial x^1} dx^1 + \frac{\partial y^2}{\partial x^2} dx^2 \right) \otimes \left(\frac{\partial y^1}{\partial x^1} dx^1 + \frac{\partial y^1}{\partial x^2} dx^2 \right) \right] =$$

$$= a(y^1,y^2)\Bigg[\Bigg(\underline{\frac{\partial y^1}{\partial x^1}\frac{\partial y^2}{\partial x^1}\, dx^1 \otimes dx^1} + \frac{\partial y^1}{\partial x^1}\frac{\partial y^2}{\partial x^2}\, dx^1 \otimes dx^2 + \underline{\frac{\partial y^2}{\partial x^2}\frac{\partial y^2}{\partial x^2}\, dx^2 \otimes dx^2} +$$

$$+ \frac{\partial y^1}{\partial x^2}\frac{\partial y^2}{\partial x^1}\, dx^2 \otimes dx^1\Bigg) - \Bigg(\underline{\frac{\partial y^2}{\partial x^1}\frac{\partial y^1}{\partial x^1}\, dx^1 \otimes dx^1} + \frac{\partial y^2}{\partial x^1}\frac{\partial y^2}{\partial x^2}\, dx^1 \otimes dx^2 +$$

$$+ \frac{\partial y^2}{\partial x^2}\frac{\partial y^1}{\partial x^1}\, dx^2 \otimes dx^1 + \underline{\frac{\partial y^2}{\partial x^2}\frac{\partial y^2}{\partial x^2}\, dx^2 \otimes dx^2}\Bigg)\Bigg] =$$

: terms cancel each other, other terms group as:

$$= a(y^1,y^2)\left\{\underbrace{\left[\frac{\partial y^1}{\partial x^1}\frac{\partial y^2}{\partial x^2} - \frac{\partial y^2}{\partial x^1}\frac{\partial y^1}{\partial x^2}\right]}_{=\det\left(\frac{\partial y}{\partial x}\right)} dx^1 \otimes dx^2 - \underbrace{\left[\frac{\partial y^2}{\partial x^2}\frac{\partial y^1}{\partial x^1} - \frac{\partial y^1}{\partial x^2}\frac{\partial y^2}{\partial x^1}\right]}_{=\det\left(\frac{\partial y}{\partial x}\right)} dx^2 \otimes dx^1\right\} =$$

$$= a \det\left|\frac{\partial y}{\partial x}\right| dx^1 \wedge dx^2$$

def. of "det" of the matrix $\frac{\partial y}{\partial x}$.

INTEGRATION ON MANIFOLDS

Integrals on a manifold of dimension "m" are defined on m-forms $\omega \in \Lambda_m$.

<u>Euristic argument</u> :

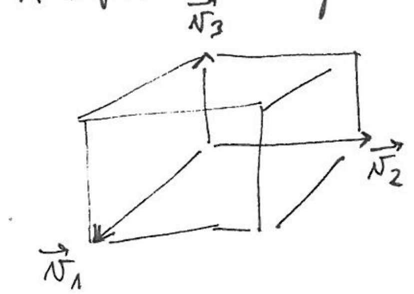
$$\int \underbrace{f(x)}_{\text{function}} \, \underbrace{d\mu}_{\text{MEASURE}}$$

SCALAR

A measure associates an infinitesimal region to a number.

$$d\mu : \text{infinitesimal region} \longrightarrow \text{number (infinitesimal volume)}$$

A infinitesimal region is identified by vectors, e.g.



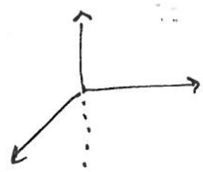
, and so $d\mu(\vec{v}_1, \vec{v}_2, \vec{v}_3)$ = volume of the region identified by v_1, v_2, v_3.

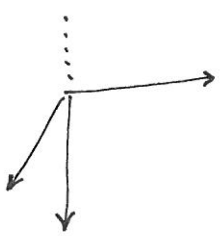
thus the <u>measure</u> is an object that :

1 – maps vectors to numbers (m vectors if $\dim M = m$)

2 – $d\mu(a\vec{v}_1, b\vec{v}_2, c\vec{v}_3) = abc \; d\mu(\vec{v}_1, \vec{v}_2, \vec{v}_3)$ $\quad a, b, c \in \mathbb{R}$

3 – is linear in $\vec{v}_i$

4 – is <u>antisymmetric</u>, because the volume of  $= d\mu$

must be  $= -d\mu$.

In other terms, there is a choice of ORIENTATION of the elementary / infinitesimal volume element; that choice implies antisymmetry.

$$\boxed{\begin{array}{lll} 1., 2., 3. & \Rightarrow & d\mu \text{ is a } (0,m) \text{ Tensor} \\ 4. & \Rightarrow & d\mu \text{ is a } m\text{-form.} \end{array}}$$

Integration of an n-form on a n-manifold

M : $\dim M = m$

$\omega \in \Lambda_m \rightarrow \omega = a(x^\mu)\, dx^1 \wedge \dots \wedge dx^m$ with $a(x^\mu)$ a function,

and x^μ a local coordinate system.

The integral of ω on $O_\alpha \subset M$ is defined as:

$$\int_{O_\alpha} \omega \equiv \underbrace{\int_{\psi_\alpha(O_\alpha)} a(x^\mu)\, dx^1 dx^2 \dots dx^m}_{\text{integral on } \mathbb{R}^N}.$$

The key observation is that: because of the transformation properties of n-forms, the definition does not depend on the particular coordinates employed.

Verify.

Coord. transformation : $x^\mu \rightarrow x^{\mu'}$

Transformation of the n-form component : $a \rightarrow a' = a \underbrace{\left| \frac{\partial x}{\partial x'} \right|}_{\text{"det" of the Jacobian}}$

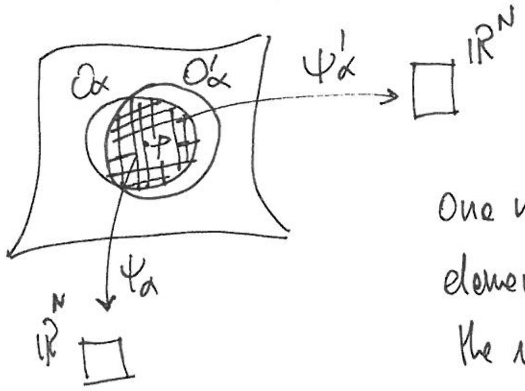
Integral remains invariant:

$$\int_{O_\alpha} \omega = \int_{\psi_\alpha(O_\alpha)} a\, dx_1 \dots dx_m = \int_{\psi_\alpha(O'_\alpha)} a'\, dx'_1 \dots dx'_m = \int_{\psi'_\alpha(O'_\alpha)} \underbrace{a(x^{\mu'}) \left| \frac{\partial x^\mu}{\partial x^{\mu'}} \right| dx'_1 \dots dx'_m}_{\text{standard "change of variables" in } \mathbb{R}^N \text{!}} =$$

$$= \int_{O'_\alpha} \omega$$

Caveats and extensions :

— the relation hold only restricted to the common elements $O_\alpha \cap O_\beta$:

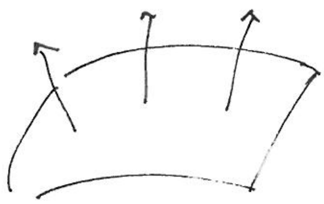


One needs to "contract" O_α and O_β to the shaded elements. This is typically possible, in particular if the manifold is:

simply connected manifold $\longleftrightarrow$ curves can be contracted to a point ("no holes")

— there is a sign ambiguity in the definition. To fix the sign one needs to orient the manifold. Intuitively, an orientation guarantees that there exists a normal to the space we want to integrate.

Technically, it traslates to the requirement that there exist a continuous, non-vanishing n-form.



The orientation fixes the sign ambiguity in the definition of the integral.

Notably : simply connected manifold $\Rightarrow \exists$ an orientation.

— The extension of the integral to the entire manifold is performed as

$$\int_M \omega = \Sigma \int_{O_\alpha} \omega$$

and it is possible under suitable conditions [cf. Wald or mathematical books].

EXTERIOR DERIVATIVE

We can now integrate forms. Can we also derive forms?

<u>Def</u>: Exterior derivative $\mathrm{dl} : \Lambda_p \mapsto \Lambda_{p+1}$
such that:

1. linear $\quad \mathrm{dl}(\alpha+\beta) = \mathrm{dl}\alpha + \mathrm{dl}\beta \qquad \alpha,\beta \in \Lambda_p$
2. "Leibnitz" $\quad \mathrm{dl}(\alpha\wedge\beta) = \mathrm{dl}\alpha\wedge\beta + (-1)^p \alpha\wedge\mathrm{dl}\beta \qquad$ ANTIDERIVATIVE
3. $\mathrm{dl}(\mathrm{dl}\alpha) = 0$

<u>Observations</u>

- For 0-forms (functions) the 1-form given by gradient is an external derivative:

$$p=0 \qquad \mathrm{dl} = d : \Lambda_0 \to \Lambda_1$$
$$f \mapsto df$$

- Property 3. guarantees the exterior derivative is an antisymmetric object. Consider for example the exterior derivative of the 1-form given by the gradient:

$$\mathrm{dl}(\mathrm{dl}f) = \mathrm{dl}(df) = \mathrm{dl}\left(\frac{\partial f}{\partial x^\mu} dx^\mu\right) \approx$$ object with components made of:
$$\frac{\partial^2 f}{\partial x^\mu \partial x^\nu} \;;$$

But 2^{nd} partial derivatives are <u>symmetric</u>. Hence we must require $\mathrm{dl}(\mathrm{dl}\alpha)=0$.

- Overall, the properties 1., 2., 3. define an object formed by an <u>antisymmetric combination of derivatives</u>; in the natural basis of Λ_{p+1}, the components are:

$$(\mathrm{dl}\,\omega)_{\mu_1\cdots\mu_p\mu_{p+1}} = (p+1)\,\partial_{[\mu_1}\omega_{\mu_2\cdots\mu_{p+1}]}$$

The latter expression can be alternatively taken as definition of $dl\omega$.

One verifies that

- it is a tensor
- satisfies 1. and 2.

} Exercise.

- satisfies 3. :

$$dl^2\omega \equiv dl(dl\omega) \propto \partial_{[a}\partial_b\,\omega_{\mu_1\cdots\mu_p]} = \text{antisymm. combination of } 2^{nd} \text{ derivatives } \partial_a\partial_b - \partial_b\partial_a$$

$$= 0$$

Def: $\omega \in \Lambda_p$ is called CLOSED p-form iff $dl\omega = 0$.

Def: $\eta \in \Lambda_p$ is called EXACT p-form iff $\eta = dl\omega$, for some $\omega \in \Lambda_{p-1}$.

Example: Exterior derivative in $\mathbb{R}^3$

$M = \mathbb{R}^3$

- 0-form (function) f, $dlf = df = \mathrm{grad}(f)$ is a 1-form.
- 1-form $w = w_1 dx^1 + w_2 dx^2 + w_3 dx^3 = w_i dx^i$

Note the 1-form w can be identified with the vector $\vec{w} = (w_1, w_2, w_3)$.

$$dl\,w = dl(w_i dx^i) = \partial_j w_i \, dx^j \wedge dx^i =$$

component definition of dlw

$$= (\partial_1 w_2 - \partial_2 w_1)\, dx^1 \wedge dx^2 + (\partial_2 w_3 - \partial_3 w_2)\, dx^2 \wedge dx^3 + (\partial_3 w_1 - \partial_1 w_3) \cdot dx^3 \wedge dx^1$$

dlw can be identified with the vector: $\mathrm{curl}(\vec{w})$! (*)

$dl\omega$ is a 2-form.

— 2-form $\quad \eta = \eta_{12}\, dx^1 \wedge dx^2 + \eta_{23}\, dx^2 \wedge dx^3 + \eta_{31}\, dx^3 \wedge dx^1$

Note the 2-form η can be identified with the vector $\vec{V} = (V_1, V_2, V_3) = (\eta_{23}, \eta_{31}, \eta_{12})$

$$d\eta = \text{3-form} = \text{function} \cdot dx^1 \wedge dx^2 \wedge dx^3$$

$$= \ldots = \left(\partial_i V^i\right) dx^1 \wedge dx^2 \wedge dx^3$$

$d\eta$ can be identified with the $\operatorname{div}(\vec{V})$.

— Note that $d^2 = 0 \Rightarrow$ the usual identities of vector calculus:

$$d^2 = 0 \Rightarrow \begin{cases} d(df) = 0 \quad f \in \Lambda_0 \Rightarrow \operatorname{curl}(\operatorname{grad}(f)) = 0 \\ d(d\omega) = 0 \quad \omega \in \Lambda_1 \Rightarrow \operatorname{div}(\operatorname{curl}(\vec{V})) = 0 \end{cases}$$

(*) Formally, the identification should be done by introducing a HODGE OPERATOR

$$* : \Lambda_2 \to \Lambda_1$$

such that, given $\omega, \eta \in \Lambda_1$, one defines

$$(\omega \wedge \eta) \in \Lambda_2 \longrightarrow *(\omega \wedge \eta) \in \Lambda_1$$

$$*(\omega \wedge \eta)_i = \sum_{j,k} \epsilon_{ijk}\, \omega_j \eta_k .$$

the resulting 1-form can be then identified with the vector.

On general manifolds, one can introduce the Hodge operator:

$$* : \Lambda_p(M) \longrightarrow \Lambda_{n-p}(M)$$

in a similar way using the Levi-Civita tensor constructed from the metric.

Example: Maxwell eqs interms of the exterior derivative

Consider SR : $M = \mathbb{R}^4$ with the Minkowski metric.

Maxwell / Faraday tensor F_{ab} is a 2-form.

Some of the Maxwell equations are:

$$\partial_{[a} F_{bc]} = 0 .$$

the L.H.S. $\partial_{[a}F_{bc]}$ is the antisymmetric combination of partial derivatives, hence the eqs can be written as:

$$dF = 0$$

$\Rightarrow$ Faraday tensor is a <u>closed</u> 2-form.

It is possible to prove that in Minkowski spacetime <u>all</u> closed forms are <u>exact</u>.

$\Rightarrow \exists$ a 1-form : $F = dA$

A is the vector potential, an exact 1-form. Using A as fundamental field, Maxwell eqs looks :

$$d^2 A = 0 .$$

The other Maxwell eq

$$\partial_\mu F^{\nu\mu} = 4\pi J^\mu$$

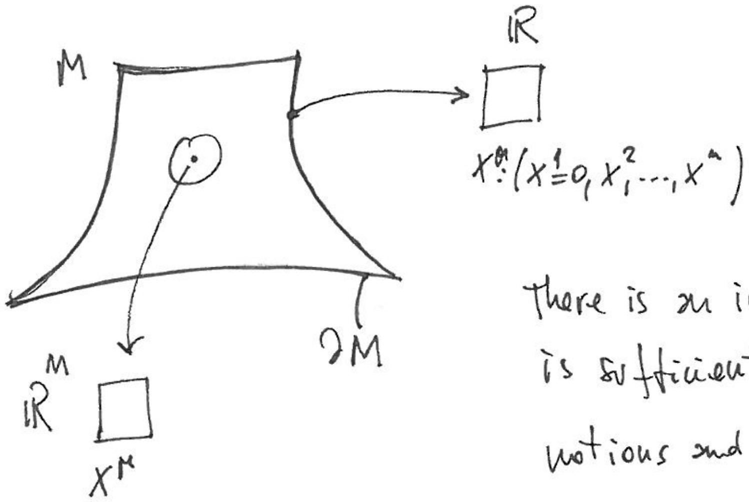
can also be written in terms of the exterior derivative. If we introduce the Hodge operator, it looks :

$$d(*F) = 4\pi(*J) .$$

STOKES THEOREM

Stokes theorem generalizes the fundamental theorem of calculus to generic manifolds.

Consider a manifold M of dimension $\dim M = n$ and a $(n-1)$ form on M $\omega \in \Lambda_{n-1}(M)$. Consider an <u>orientation</u> for M and a <u>boundary</u> ∂M:



There is an intuitive notion of boundary which is sufficient, [see Wald + math books for precise notions and def].

We assume we can extend all the definition from M to ∂M, that the orientation of M induces an orientation on ∂M, etc.

<u>Stokes theorem</u>:

$$\int_M d\omega = \int_{\partial M} \omega$$

<u>Observations</u>:

- Extends the fundamental theorem of calculus
- Includes <u>all</u> the fundamental theorems involving integrals (Divergence, circulation, etc)

Example: Stokes in $n=1$ dimension

1D manifolds:

- $\mathbb{R}$
- $\mathbb{R}_+$
- S_1 
- $I \subset \mathbb{R}$ (closed interval)

As an example of manifold with boundary let us take:

$$M = \bigcup_i I_i \qquad \text{with} \quad I_i =$$ 

$$\partial I_i = \; x_i^- \qquad x_i^+$$

Orientation $\qquad - \qquad +$

$n=1 \Rightarrow n-1=0 \qquad \omega = \text{function} = f \quad$ 0-form

$d\omega = df \quad$ 1 form

Stokes:

$$\int_M d\omega = \sum_i \int_{I_i} df = \sum_i \left(F(x_i^+) - F(x_i^-) \right) = \sum_i \int_{\partial I_i} f = \int_{\partial M} \omega$$

Example : Stokes in $\mathbb{R}^2$

$M = \mathbb{R}^2$

$v = v^i e_i$ vector

$\omega = w_i dx^i$ dual vector

} they are identified by the Euclidean scalar product : $\langle \, , \, \rangle$

$\forall\, r$ vector $\quad \omega(r) \equiv \langle v, r \rangle$, in particular :

$$w_i = \omega(e_i) = \langle v, e_i \rangle = v^j \underbrace{\langle e_j, e_i \rangle}_{\delta_{ij}} = v^i \, .$$

Exterior derivative of ω :

$$d\omega = \frac{\partial w_i}{\partial x^j} dx^j \wedge dx^i \underset{\text{Use identification}}{=} \frac{\partial v^i}{\partial x^j} dx^j \wedge dx^i = \underbrace{\left(\frac{\partial v^2}{\partial x^1} - \frac{\partial v^1}{\partial x^2} \right)}_{= \operatorname{curl}(v)} dx^1 \wedge dx^2$$

Calculate the integral on an open set of $\mathbb{R}^2$:

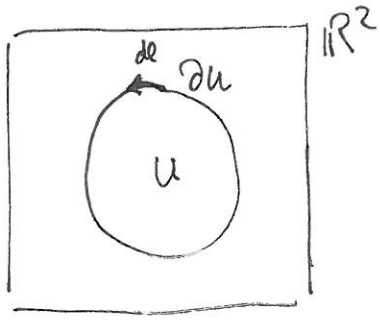
$$\int_{U \subset \mathbb{R}^2} d\omega = \int_U \operatorname{curl}(v)\, dx^1 dx^2 = \int_{\partial U} \omega = \int_{\partial U} \langle v, d\ell \rangle$$

use – def. of integral ;
– expression of $d\omega$ above.

Stokes Th.

identification $\omega \longleftrightarrow v$ from the scalar product

We obtain the "curl" integral theorem in $\mathbb{R}^2$:



$$\int_U \operatorname{curl}(\vec{v}) = \int_{\partial U} \vec{v} \cdot \vec{d\ell}$$

Example : Gauss theorem in $\mathbb{R}^3$

$M = \mathbb{R}^3$

$V = v^i e_i$ a vector. Associate the 2-form:

$$\omega = v^3 dx^1 \wedge dx^2 - v^2 dx^1 \wedge dx^3 + v^1 dx^2 \wedge dx^3$$

Exterior derivative:

$$d\omega = \left(\sum_i \frac{\partial v^i}{\partial x^i} \right) dx^1 \wedge dx^2 \wedge dx^3 ,$$

i.e. the exterior derivative is basically associated to the $\mathrm{div}(\vec{v})$.

Stokes theorem:

$$\int_M d\omega = \int_{\partial M} \omega$$

$\shortparallel$ volume integral of the $\mathrm{div}(\vec{V}) \leftrightarrow d\omega$

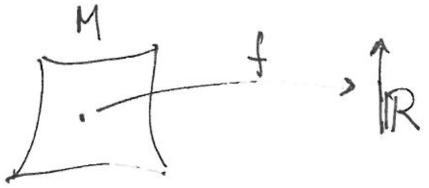
$\shortparallel$ surface integral of $\omega \leftrightarrow \vec{V}$

$$\int_V \mathrm{div}(\vec{V}) = \int_\Sigma \vec{V} \cdot \hat{n}$$

INTEGRATION OF FUNCTIONS

M manifold, $\dim M = n$

ϕ function on M



How to compute:

$$\int_M \phi = ?$$

Note:
- we do not know!
- we know only how to integrate n-forms

Tentative definition: $\int_M \phi \equiv \int_M \phi \, \epsilon$, where ϵ is a not-yet-defined n-form that gives a volume element or measure.

→ if we can find a suitable ϵ, then we can apply our previous definition of integral of forms.

Let us find ϵ.

$$\epsilon = a \, dx^1 \wedge \dots \wedge dx^n$$, must be in this form because it is an n-form.

To specify ϵ, we need to specify the function "$a(x)$".

what to use? ⟶ the metric.

the metric is the object that allow us to calculate lenghts, and so it should be used.

However, the metric is a tensor ... we need a function ...

Consider the transformation of a :

$$a \longrightarrow a' = a \left| \frac{\partial x^{\mu'}}{\partial x^{\mu}} \right|$$

We want : $a = a(\text{metric})$ such that it transform as above ...

Consider the transformation of the determinant of the metric :

$$\det(g) \longrightarrow \det(g) \left| \frac{\partial x^{\mu'}}{\partial x^{\mu}} \right|^{-2},$$

hence we can take :

$$a \equiv \sqrt{\det(g)} \,,$$

this way we obtain the correct transformation rule for a.

The <u>volume n-form</u> is thus :

$$\epsilon \equiv \sqrt{\det g}\, dx \wedge \ldots \wedge dx^{n}$$

and the integral of any function is <u>defined</u> as :

$$\int_M \phi \equiv \int_M \phi\, \epsilon = \int_M \underbrace{\phi \sqrt{\det g}\, dx^1 \wedge \ldots \wedge dx^{n}}_{\text{this is a } n\text{-form that we know how to integrate}} = \int_{\mathbb{R}^N} \phi \sqrt{\det g}\, d^n x$$