Gravitational waves — Exercise sheet n.3

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Exercise 3.1: STF Projector

Consider the STF projector $\Lambda_{ij,kl}$, defined as

$$\Lambda_{ij,kl}(\boldsymbol{n}) = P_{ik}P_{jl} - \frac{1}{2}P_{ij}P_{kl}\,, \tag{1}$$

where $P_{ij}(\boldsymbol{n}) = \delta_{ij} - n_i n_j$ and $\boldsymbol{n}$ is a unitary vector. Show that:

- P_{ij} is symmetric;
- P_{ij} is transverse, $n^i P_{ij} = 0$;
- P_{ij} is a projector, $P_{ik}P_{kj} = P_{ij}$;
- P_{ij} has trace equal to 2, $P_{kk} = 2$;
- $\Lambda_{ij,kl}$ is a projector, $\Lambda_{ij,mn}\Lambda_{mn,kl} = \Lambda_{ij,kl}$;
- $\Lambda_{ij,kl}$ is transverse in all indexes;
- $\Lambda_{ij,kl}$ is trace-less in ij and in kl, $\Lambda_{ii,kl} = \Lambda_{ij,kk} = 0$;
- $\Lambda_{ij,kl}$ is symmetric for exchange of $ij \leftrightarrow kl$.

Exercise 3.2: No-stress sourse

Consider a matter distribution characterized by a current vector $J^\mu = \rho u^\mu$ and by a vanishing stress tensor $T_{ij} = 0$. For this source, we can write the stress-energy tensor as

$$T_{\mu\nu} = \begin{pmatrix} J_0 & J_j \\ J_i & 0_{ij} \end{pmatrix}\,. \tag{2}$$

- Compute the metric $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ described by the source in linearized GR assuming stationarity, i.e. $\partial_t h_{\mu\nu} = 0$;
- Compute the Lagrangian function for a particle of mass m moving in the metric $g_{\mu\nu}$ in the small velocity limit, highlighting the gravito-electric and gravito-magnetic contributions.