Gravitational waves — Exercise sheet n.7

Matteo Breschi
matteo.breschi@uni-jena.de

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Exercise 7.1: Eccentric binaries

Consider two point masses m_1 and m_2 in elliptic orbit under the effect of Newtonian gravitational forces. Let us parametrize the system as it is shown in Fig. 1 and assume that the gravitational radiation is the only source of energy loss in the system. In such conditions, we can define the *eccentricity* $e = \sqrt{1 - b^2/a^2}$, which describe the shape of the elliptic orbit ($e = 0$ is the circular case) and, in Newtonian approximation, it is a constant of the motion.

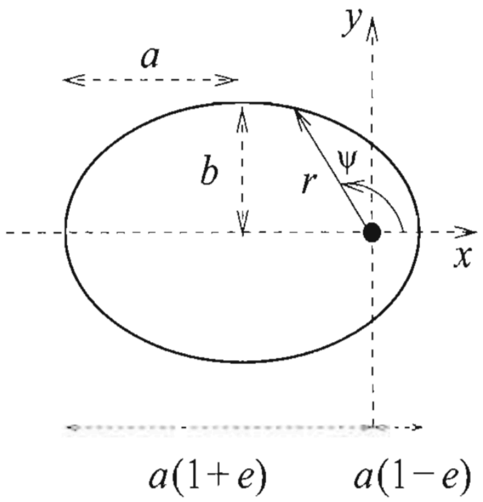

Figure 1: Geometry of the eccentric binary system. We perform the computation in the center-of-mass frame, where the central object (located in the focus) represents the total mass contribution $M = m_1 + m_2$ which generates the field, while the rotating body represents the reduced mass $\mu = m_1 m_2 / M$.

- In Newtonian approximation, prove that it is possible to write the eccentricity as

$$e = \sqrt{1 + \frac{2EL^2}{G^2 M^2 \mu^3}}\,, \tag{1}$$

where μ is the reduced mass, M is the total mass, E is the total energy and L the total angular momentum.

- Knowing that it is possible to write the emitted power due to GWs (with eccentric corrections) as,

$$P = \frac{32\,G^4\mu^2 M^3}{5\,c^5 a^5}\,\frac{1}{(1-e^2)^{7/2}}\left(1+\frac{73}{24}e^2+\frac{37}{96}e^4\right)\,, \tag{2}$$

compute the differential equation which govern the evolution of the orbital period $T(t)$. [Hint: note that in Newtonian approximation, $\dot{T}/T \propto \dot{E}/E$]

Note: It is interesting to observe that the definition $e = \sqrt{1-b^2/a^2}$ is valid only for elliptic cases, for which the eccentricity is always $0 < e < 1$. A more general and rigorous definition of eccentricity comes from the geometrical description of conic sections. A generic conic section (with focus point located in the origin of the axes) can be identified in the polar plane with the following equation:

$$r = \frac{\ell}{1+e\cos\theta}\,, \tag{3}$$

where (r,θ) are the polar coordinates, e is the eccentricity and ℓ is a length-scale (called semi-latus rectum). With this equation, it is possible to see that the values $e \geq 1$ describe open orbits, such as parabolas and hyperbolas.