Gravitational waves — Exercise sheet n.8

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Exercise 8.1: STF multipole decomposition of scalar field

Consider a scalar field ϕ satisfying the relativistic wave equation,

$$\Box\phi = -4\pi\rho\,, \tag{1}$$

where $\rho(t,\vec{x})$ is the source term and it has a compact support. The reader can view this as the Newtonian gravitational field generated by a mass distribution. Prove that in the region outside the source (say $|\vec{x}| > d$), the most general exact solution can be written as a multipole decomposition,

$$\phi(t,\vec{x}) = \sum_{\ell=0}^{\infty} \frac{(-1)^\ell}{\ell!}\,\partial_L\left[\frac{F_L(t-r/c)}{r}\right]\,, \tag{2}$$

where the multi-index notation means $L \equiv i_1 i_2 \dots i_\ell$ and F_L are symmetric-trace-free tensors that are only functions of the retarded time. [Hint: expand the source term using the STF decomposition $\rho(t,\vec{x}) = \sum_{\ell\geq 0} n_L(\theta,\phi) S_L(r,t)$ and match the solution from multipolar expansion]