Gravitational waves — Exercise sheet n.1 Solutions

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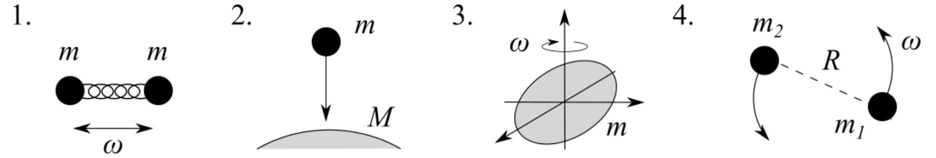
Exercise 1.1: Quadrupole approximation



Consider the following sources of gravitational radiation (see Fig. 1):

1. Two point particles with mass m oscillating with pulsation ω along a fixed axis;
2. Free-falling point-particle with mass m in a Newtonian gravitational field (of mass M);
3. Ellipsoid (with semi-axes a, b, c) rotating around one of its principal axis with frequency ω;
4. Two point particles (with different masses m_1, m_2) in Newtonian circular orbit.

For these cases, compute:

- Inertia tensor of the source,

$$Q^{ij}(t) \equiv I^{ij} - \frac{1}{3}\delta^{ij}\, I^{kk} = \int d^3x\, \rho(t,\vec{x}) \left(x^i x^j - \frac{1}{3} r^2 \delta^{ij} \right) . \tag{1}$$

  Note that I_{ij} is the standard inertia tensor, while Q_{ij} is the trace-free inertia tensor.

- Gravitational wave emitted in quadrupole approximation in the TT gauge,

$$h_{ij}^{\rm TT}(t,\vec{x}) = \frac{2G}{r\,c^4}\,\Lambda_{ij,mn}(\theta,\phi)\,\ddot{Q}_{mn}(t-r/c)\,, \tag{2}$$

  where $\Lambda_{ij,mn}$ is the TT projector.

Solution ??

1. Oscillating particles:
   - In the center of mass frame, the equation of motion can be reduced to the dynamics of a single particle of mass $\mu = m/2$. Then, fixing the oscillation along the z axis, we can write

     $$\ddot{\delta} + \omega^2 \delta = 0 \,, \tag{3}$$

     where $\delta = z_1 - z_2$ is the distance between the two particles. From Eq. (3), it follows the equation of motion for $\delta(t) \propto \cos(\omega t)$. Then, the inertia tensor of the source in the center of mass frame can be computed as

     $$I_{ij} = \mu x_i(t) x_j(t) \,. \tag{4}$$

     The only non-zero component of I_{ij} correspond to $i = j = 3$ since the motion is constrained on the z axis (by construction). This form can be easily mapped into its trace-free version, which takes a diagonal form, $Q_{11} = Q_{22} = -(1/3)m\delta^2(t)$ and $Q_{33} = (2/3)m\delta^2(t)$.
   - From this result, we can derive the gravitational strain h_{ij}^{TT} in the TT gauge as

     $$h_+ = -\frac{G}{rc^4} \sin^2\theta \, \ddot{I}_{11}(t) \qquad h_\times = 0 \,. \tag{5}$$

     Note that the radiate GW will be a monochromatic signal with frequency $\omega_{gw} = 2\omega$ only if the rest separation between the particle is zero. Otherwise, the GW spectrum will be characterized by two peaks located at frequency ω and 2ω.

2. Free-falling particle:
   - Taking the z-axis parallel to the velocity of the particle, In the Newtonian approximation, we can write

     $$\frac{1}{2} m\dot{z}^2 - \frac{GMm}{z} = 0 \,, \tag{6}$$

     where $\dot{z}$ represent the derivative of z with respect to time (i.e. the velocity). Then it follows,

     $$\dot{z} = c\sqrt{\frac{R_s}{z}} \quad \Rightarrow \quad z(t) \propto t^{2/3} \,, \tag{7}$$

     where $R_s = 2GM/c^2$ is the Schwarzschild radius of the central object. Now we can compute the inertia tensor for the free-falling particle as

     $$I_{ij} = m x_i(t) x_j(t) \,, \tag{8}$$

     where the only non-vanishing term is $I_{33} = mz^2(t)$. This form can be easily mapped into its trace-free version, which takes a diagonal form, $Q_{11} = Q_{22} = -(1/3)mz^2(t)$ and $Q_{33} = (2/3)mz^2(t)$.

- From this result, we can derive the gravitational strain h_{ij}^{TT} in the TT gauge as

$$h_+ = -\frac{G}{rc^4}\sin^2\theta\, \ddot{I}_{33}(t) \qquad h_\times = 0\,. \tag{9}$$

3. Rotating ellipsoid:

   - We can write the inertia tensor for a non-rotating ellipsoid (in a Cartesian frame with coordinates parallel to the axes of the ellipsoid) as

$$I_{11} = \frac{m}{5}\left(b^2+c^2\right)\,, \quad I_{22} = \frac{m}{5}\left(a^2+c^2\right)\,, \quad I_{33} = \frac{m}{5}\left(a^2+b^2\right)\,, \tag{10}$$

and the other components are zero. If the body rotates with angular velocity ω around the z-axis then we have

$$I'_{ij} = \mathcal{R}_{im}\mathcal{R}_{jn}I_{mn}\,, \tag{11}$$

where $\mathcal{R}_{ij}$ is the matrix representation of the rotation $\mathcal{R}(\omega t, \hat{z})$ of an angle ωt around the z-axis. Then we get

$$\begin{aligned}
I'_{11} &= I_{11}\cos^2(\omega t) + I_{22}\sin^2(\omega t) = 1 + \frac{I_{11}-I_{22}}{2}\cos(2\omega t)\,,\\
I'_{12} &= \frac{I_{11}-I_{22}}{2}\sin(2\omega t)\,,\\
I'_{22} &= I_{11}\sin^2(\omega t) + I_{22}\cos^2(\omega t) = 1 - \frac{I_{11}-I_{22}}{2}\cos(2\omega t)\,,\\
I'_{33} &= I_{33}\,,
\end{aligned} \tag{12}$$

while $I'_{13} = I'_{23} = 0$. Then, observing that the trace of I_{ij} is invariant under rotations, we can write the mass momenta as $Q_{ij} = -I'_{ij} + c_{ij}$ where c_{ij} are constant terms which will vanish after the derivation. Therefore,

$$\begin{aligned}
Q_{11} &= -\frac{I_{11}-I_{22}}{2}\cos(2\omega t) + \text{constant}\,,\\
Q_{12} &= -\frac{I_{11}-I_{22}}{2}\sin(2\omega t) + \text{constant}\,,\\
Q_{22} &= +\frac{I_{11}-I_{22}}{2}\cos(2\omega t) + \text{constant}\,,
\end{aligned} \tag{13}$$

   - Now let us compute the GW received by an observer at distance r, whose line-of-sight makes an angle θ with the direction of spin of the star (without loss of generality we set $\phi = 0$). Inserting Eq. (13) into the definition of h_{ij}^{TT} we get

$$\begin{aligned}
h_+ &= \frac{1}{r}\frac{4G\omega^2}{c^4}(I_{11}-I_{22})\,\frac{1+\cos^2\theta}{2}\cos(2\omega t)\,,\\
h_\times &= \frac{1}{r}\frac{4G\omega^2}{c^4}(I_{11}-I_{22})\cos\theta\,\sin(2\omega t)\,.
\end{aligned} \tag{14}$$

4. Binary system:

- Consider two point particles with masses m_1 and m_2 in Newtonian circular orbit at distance R. We choose the frame so that it is centered in the center of mass (CM) and the orbit lies in the xy-plane and the trajectory is given by

$$\begin{aligned} x(t) &= R\cos(\omega t + \pi/2)\,, \\ y(t) &= R\sin(\omega t + \pi/2)\,, \\ z(t) &= 0\,, \end{aligned} \tag{15}$$

  where we assume $\phi_0 = \pi/2$ (since it is a useful choice). We define the total mass of the system as $M = m_1 + m_2$ and the reduced mass of the system, $\mu = m_1 m_2/M$. From the definition of mass momenta, in the CM frame, we get

$$Q_{ij} = \mu\, x_i(t) x_j(t)\,, \tag{16}$$

  where

$$\begin{aligned} Q_{11} &= \mu R^2 \frac{1-\cos(2\omega t)}{2}\,, \\ Q_{22} &= \mu R^2 \frac{1+\cos(2\omega t)}{2}\,, \\ Q_{12} &= -\frac{1}{2}\mu R^2 \sin(2\omega t)\,, \end{aligned} \tag{17}$$

  while the other components vanish. Therefore,

$$\begin{aligned} \ddot{Q}_{11} &= -\ddot{M}_{22} = 2\mu R^2\omega^2 \cos(2\omega t)\,, \\ \ddot{Q}_{12} &= 2\mu R^2 \omega^2 \sin(2\omega t)\,. \end{aligned} \tag{18}$$

- Using Eq. (18) in the definition of h_{ij}^{TT} in the quadrupole approximation we get

$$\begin{aligned} h_+(t) &= \frac{1}{r}\frac{4G\mu R^2\omega^2}{c^4}\frac{1+\cos^2\theta}{2}\cos(2\omega t + 2\phi)\,, \\ h_\times(t) &= \frac{1}{r}\frac{4G\mu R^2\omega^2}{c^4}\cos\theta\,\sin(2\omega t + 2\phi)\,. \end{aligned} \tag{19}$$