Gravitational waves — Exercise sheet n.2 Solutions

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Exercise 2.1: Linearized Einstein equations

Compute the Einstein tensor $G_{\mu\nu}$ and the Riemann tensor $R^{\mu}_{\nu\rho\sigma}$ in weak field approximation,

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} + O(h^2)\,. \tag{1}$$

Exercise 2.2: Effect of GWs

Consider two point-particles test-masses in a locally-flat spacetime (typically labeled as *detector frame*). Show explicitly that the effects of GWs can be described in terms of a Newtonian force at the leading order.

Solution ??

First, let us compute the Christoffel symbols,

$$\begin{aligned}\Gamma^{\rho}_{\mu\nu} &= \frac{1}{2} g^{\rho\sigma}\left(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}\right)\\ &= \frac{1}{2}\left(\eta^{\rho\sigma} + h^{\rho\sigma}\right)\left(\partial_\mu h_{\nu\sigma} + \partial_\nu h_{\mu\sigma} - \partial_\sigma h_{\mu\nu}\right) + O(h^2)\\ &= \frac{1}{2}\left(\partial_\mu h^{\rho}_{\nu} + \partial_\nu h^{\rho}_{\mu} - \partial^{\rho} h_{\mu\nu}\right) + O(h^2)\,.\end{aligned} \tag{2}$$

Then, plugging Eq. (1) and Eq. (2) into the definition of $R^{\mu}_{\nu\rho\sigma}$, we get

$$\begin{aligned}R^{\mu}_{\nu\rho\sigma} &= \partial_\rho \Gamma^{\mu}_{\nu\sigma} - \partial_\sigma \Gamma^{\mu}_{\nu\rho} + \Gamma^{\mu}_{\alpha\rho}\Gamma^{\alpha}_{\nu\sigma} - \Gamma^{\mu}_{\alpha\sigma}\Gamma^{\alpha}_{\nu\rho}\\ &= \frac{1}{2}\left[\partial_\rho\left(\partial_\nu h^{\mu}_{\sigma} + \partial_\sigma h^{\mu}_{\nu} - \partial^{\mu} h_{\nu\sigma}\right) - \partial_\sigma\left(\partial_\nu h^{\mu}_{\rho} + \partial_\rho h^{\mu}_{\nu} - \partial^{\mu} h_{\nu\rho}\right)\right] + O(h^2)\\ &= \frac{1}{2}\left(\partial_\nu\partial_\rho h^{\mu}_{\sigma} + \partial^{\mu}\partial_\sigma h_{\nu\rho} - \partial^{\mu}\partial_\rho h_{\nu\sigma} - \partial_\nu\partial_\sigma h^{\mu}_{\rho}\right) + O(h^2)\,.\end{aligned} \tag{3}$$

This result leads to

$$R_{\mu\nu} = \frac{1}{2}\left(\partial_\nu\partial^{\alpha} h_{\mu\alpha} + \partial_\mu\partial^{\alpha} h_{\nu\alpha} - \partial^{\alpha}\partial_\alpha h_{\mu\nu} - \partial_\mu\partial_\nu h^{\alpha}_{\alpha}\right) + O(h^2)\,, \tag{4}$$

$$R = \partial^\alpha \partial^\beta h_{\alpha\beta} - \partial^\alpha \partial_\alpha h^\beta_\beta + O(h^2)\,, \tag{5}$$

$$G_{\mu\nu} = \frac{1}{2}\left[\partial_\nu \partial^\alpha h_{\mu\alpha} + \partial_\mu \partial^\alpha h_{\nu\alpha} - \partial^\alpha \partial_\alpha h_{\mu\nu} - \partial_\mu \partial_\nu h^\alpha_\alpha - \eta_{\mu\nu}\left(\partial^\alpha \partial^\beta h_{\alpha\beta} - \partial^\alpha \partial_\alpha h^\beta_\beta\right)\right] + O(h^2)\,. \tag{6}$$

Note that it is possible to further simplify the forms of $R_{\mu\nu}$ and $G_{\mu\nu}$ imposing a specific gauge condition.

Solution ??

In the proper-detector frame, the metric is locally flat around the detector, so $\Gamma^\rho_{\mu\nu}$ vanishes in this point. Furthermore, the detector is largely non-relativistic and $dx^i/d\tau$ can be neglected with respect to $dx^0/d\tau$. So, considering two neighboring geodesics separated by a space-like distance ξ^i from the geodesic deviation we get,

$$\frac{d^2\xi^i}{d\tau^2} + \xi^\lambda \partial_\lambda \Gamma^i_{00}\left(\frac{dx^0}{d\tau}\right)^2 = 0\,, \tag{7}$$

where the terms of this equation are evaluated in the location of the detector, i.e. at $x^i = 0$. Since the metric depends quadratically on the distance x^i from the detector, $g_{\mu\nu} = \eta_{\mu\nu} + O(x_i x_j)$, we have that $\partial_0 \Gamma^i_{00} = 0$ and a non-vanishing contribution come only from the term $\partial_j \Gamma^i_{00}$, which corresponds with $R^i{}_{0j0} = \partial_j \Gamma^i_{00} - \partial_0 \Gamma^i_{0j} = \partial_j \Gamma^i_{00}$. From the previous equation, we get

$$\frac{d^2\xi^i}{d\tau^2} + \xi^j R^i{}_{0j0}\left(\frac{dx^0}{d\tau}\right)^2 = 0\,. \tag{8}$$

Moreover, limiting ourselves in $O(h)$ we can write $t \approx \tau$ and then $dx^0/d\tau = c$, and Eq. 8 becomes

$$\ddot{\xi}^i = -c^2 \xi^j R^i{}_{0j0}\,, \tag{9}$$

where the dot denotes the derivative with respect to the coordinate time. Now we have to compute the Riemann's tensor due to the GW in the detector-frame. However, in linearized theory the Riemann's tensor is *invariant* (the reader can demonstrate this computing the gauge transformation of the Riemann's tensor and proving that it vanishes), so we can use the expression for $R^i{}_{0j0}$ that we prefer. Clearly, the best choice is the TT gauge and we easily find

$$R^i{}_{0j0} = R_{i0j0} = -\frac{1}{2c^2}\ddot{h}^{\mathrm{TT}}_{ij}\,. \tag{10}$$

In conclusion we get

$$\ddot{\xi}_i = \frac{1}{2}\ddot{h}^{\mathrm{TT}}_{ij}\xi^j\,, \tag{11}$$

which can be rewritten considering the effect on a point particle with mass m as a Newtonian force,

$$F_i = m\ddot{\xi}_i = \frac{m}{2}\ddot{h}^{\mathrm{TT}}_{ij}\xi^j\,. \tag{12}$$