Gravitational waves — Exercise sheet n.3

Matteo Breschi
matteo.breschi@uni-jena.de

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Exercise 3.1: Spherical tensor components

In order to introduce to describe any traceless rank-2 symmetric tensor, we need to introduce the spherical tensor harmonics. Let us recall the standard spherical harmonics $Y^{\ell m}(\theta, \phi)$ for $\ell = 2$,

$$
\begin{aligned}
Y^{22}(\theta,\phi) &= \sqrt{\frac{15}{32\pi}}\left(e^{i\phi}\sin\theta\right)^2 \,, \\
Y^{21}(\theta,\phi) &= \sqrt{\frac{15}{8\pi}} e^{i\phi}\sin\theta\cos\theta \,, \\
Y^{20}(\theta,\phi) &= \sqrt{\frac{5}{16\pi}}\left(3\cos^2\theta - 1\right) \,,
\end{aligned} \tag{1}
$$

and the harmonics with $m < 0$ are given by $Y^{\ell\,-m} = (-1)^m Y^{\ell m *}$. This definition assumes that the unit radial vector can be written as $\mathbf{n} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$. Then, we can introduce the tensor spherical harmonics $\mathcal{Y}^{\ell m}_{i_1 \dots i_\ell}$ as

$$
Y^{\ell m}(\theta,\phi) = \mathcal{Y}^{\ell m}_{i_1\dots i_\ell}\, n_{i_1} \dots n_{i_\ell} \,. \tag{2}
$$

- Compute the tensor spherical harmonics $\mathcal{Y}^{\ell m}_{ij}$ for $\ell = 2$.
- Show that $\mathcal{Y}^{\ell m}_{ij}$ are an orthogonal basis for the traceless symmetric rank-2 tensors.
- Given a generic traceless symmetric rank-2 tensor Q_{ij}, compute the components Q_m such that

$$
Q_{ij} = \sum_m Q_m \mathcal{Y}^{\ell m}_{ij} \,. \tag{3}
$$