Gravitational waves — Exercise sheet n.3 Solutions

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Exercise 3.1: Spherical tensor components

In order to introduce to describe any traceless rank-2 symmetric tensor, we need to introduce the spherical tensor harmonics. Let us recall the standard spherical harmonics $Y^{\ell m}(\theta,\phi)$ for $\ell = 2$,

$$\begin{aligned} Y^{22}(\theta,\phi) &= \sqrt{\frac{15}{32\pi}}\left(e^{i\phi}\sin\theta\right)^2\,, \\ Y^{21}(\theta,\phi) &= \sqrt{\frac{15}{8\pi}}e^{i\phi}\sin\theta\cos\theta\,, \\ Y^{20}(\theta,\phi) &= \sqrt{\frac{5}{16\pi}}\left(3\cos^2\theta-1\right)\,, \end{aligned} \tag{1}$$

and the harmonics with $m < 0$ are given by $Y^{\ell\,-m} = (-1)^m Y^{\ell m *}$. This definition assumes that the unit radial vector can be written as $\mathbf{n} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$. Then, we can introduce the tensor spherical harmonics $\mathcal{Y}^{\ell m}_{i_1\dots i_\ell}$ as

$$Y^{\ell m}(\theta,\phi) = \mathcal{Y}^{\ell m}_{i_1\dots i_\ell}\, n_{i_1}\dots n_{i_\ell}\,. \tag{2}$$

- Compute the tensor spherical harmonics $\mathcal{Y}^{\ell m}_{ij}$ for $\ell = 2$.
- Show that $\mathcal{Y}^{\ell m}_{ij}$ are an orthogonal basis for the traceless symmetric rank-2 tensors.
- Given a generic traceless symmetric rank-2 tensor Q_{ij}, compute the components Q_m such that

$$Q_{ij} = \sum_m Q_m \mathcal{Y}^{\ell m}_{ij}\,. \tag{3}$$

Solution ??

- Noticing that

$$e^{i\phi}\sin\theta = n_1 + in_2\,, \qquad \cos\theta = n_3\,, \tag{4}$$

we can rewrite Y^{2m} as

$$\begin{aligned} Y^{22}(\theta,\phi) &= \sqrt{\frac{15}{32\pi}}\,(n_1+in_2)^2\ , \\ Y^{21}(\theta,\phi) &= \sqrt{\frac{15}{8\pi}}(n_1+in_2)n_3\,, \\ Y^{20}(\theta,\phi) &= \sqrt{\frac{5}{16\pi}}\left(3n_3^2 - n_in_i\right)\ . \end{aligned} \tag{5}$$

From these results it follows,

$$\mathcal{Y}^{22}_{ij} = \sqrt{\frac{15}{32\pi}}\begin{pmatrix} 1 & i & 0 \\ i & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}_{ij} \tag{6}$$

$$\mathcal{Y}^{21}_{ij} = -\sqrt{\frac{15}{32\pi}}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & i \\ 1 & i & 0 \end{pmatrix}_{ij} \tag{7}$$

$$\mathcal{Y}^{20}_{ij} = \sqrt{\frac{5}{16\pi}}\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}_{ij} \tag{8}$$

and the harmonics with $m < 0$ are given by $\mathcal{Y}^{\ell\,-m}_{ij} = (-1)^m \mathcal{Y}^{\ell\,-m\,*}_{ij}$.

- The harmonics $\mathcal{Y}^{\ell m}_{ij}$ are 5 elements. The same is the cardinality of the space of traceless symmetric rank-2 tensor. Thus, from previous results, we can easily see that $\mathcal{Y}^{2m}_{ij}$ are orthogonal, i.e.

$$\sum_{mm'} \mathcal{Y}^{2m}_{ij}\mathcal{Y}^{2m*}_{ij} = \frac{15}{8\pi}\delta^{mm'}\,. \tag{9}$$

This allows us to write any traceless symmetric rank-2 tensor as decomposition of $\ell = 2$ spherical tensor harmonics, i.e.

$$Q_{ij} = \sum_{m=-2}^{+2} Q_m \mathcal{Y}^{2m}_{ij}\,, \tag{10}$$

where Q_m are called spherical components. Note that

$$Q_{ij}n_in_j = \sum_{m=-2}^{+2} Q_m Y^{2m}\,. \tag{11}$$

- From the previous results, we can write $Q_m = (-1)^m Q_{-m}$; thus, we can limit to compute the spherical components for $m \geq 0$. From Eq. (9) and Eq. (10), we can write
$$Q_m = \frac{8}{15\pi} Q_{ij} \mathcal{Y}_{ij}^{2m*} \,, \tag{12}$$
from which we obtain the explicit expressions for the spherical components,
$$\begin{aligned} Q_{\pm 2} &= \sqrt{\frac{2\pi}{15}} \left(Q_{11} - Q_{22} \mp 2iQ_{12} \right) \,, \\ Q_{\pm 1} &= \mp \sqrt{\frac{8\pi}{15}} \left(Q_{13} \mp iQ_{23} \right) \,, \\ Q_0 &= -\sqrt{\frac{4\pi}{5}} \left(Q_{11} + Q_{22} \right) \,. \end{aligned} \tag{13}$$