Gravitational waves — Exercise sheet n.4

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Exercise 4.1: Binary dynamics

We saw that it is possible to define a stress-energy tensor for GWs, which can be written in the case of flat background as

$$\Theta_{\mu\nu} = \frac{c^4}{32\pi G}\left\langle \partial_\mu h_{\alpha\beta}\partial_\nu h^{\alpha\beta} \right\rangle\,, \tag{1}$$

where $\langle \dots \rangle$ denotes the average over a certain time-scale (much larger than the GWs time-scale). From this result, we can write the emitted power for an inspiralling compact binary system in quasi-circular orbit (i.e. $\omega_{\rm src}^2 \gg \dot\omega_{\rm src}$, where $\omega_{\rm src}$ is the orbital frequency), at the leading order of approximation, as

$$P_{\rm gw} = \frac{32}{5\,G\,c^5}\,(G\mathcal{M}\omega_{\rm src})^{10/3}\,. \tag{2}$$

Suppose that the gravitational radiation is the only source of energy loss in the system.

- Write the energy balance equation

$$\frac{dE_{\rm tot}}{dt} = -P_{\rm gw}\,, \tag{3}$$

  where $E_{\rm tot} = E_{\rm kin} + E_{\rm grav} = -\frac{Gm_1m_2}{2R}$ and compute the the evolution of the separation $R(t)$ between the two objects and the evolution of the orbital frequency $\omega_{\rm src}(t)$ using Newtonian approximation. Relate this results with the phase of the emitted GW. [Hint: it is useful to define the time to coalescence $\tau = t_{\rm coal} - t$]

- Compute the time to coalescence for the following binaries:
  1. Hulse-Taylor binary system: $M_1 = 1.441\ M_\odot$, $M_2 = 1.387\ M_\odot$, $T = 7.75$ hours
  2. Earth-Sun system: $M_\odot = 1.9891\times 10^{30}$ kg, $M_{\rm Earth} = 5.972\times 10^{24}$ kg, $T = 365$ days