Gravitational waves — Exercise sheet n.4
Solutions

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Exercise 4.1: Binary dynamics

We saw that it is possible to define a stress-energy tensor for GWs, which can be written in the case of flat background as

$$\Theta_{\mu\nu} = \frac{c^4}{32\pi G} \left\langle \partial_\mu h_{\alpha\beta} \partial_\nu h^{\alpha\beta} \right\rangle \,, \tag{1}$$

where $\langle \dots \rangle$ denotes the average over a certain time-scale (much larger than the GWs time-scale). From this result, we can write the emitted power for an inspiralling compact binary system in quasi-circular orbit (i.e. $\omega_{\rm src}^2 \gg \dot{\omega}_{\rm src}$, where $\omega_{\rm src}$ is the orbital frequency), at the leading order of approximation, as

$$P_{\rm gw} = \frac{32}{5\, G\, c^5} \left(G\mathcal{M}\omega_{\rm src}\right)^{10/3} \,. \tag{2}$$

Suppose that the gravitational radiation is the only source of energy loss in the system.

- Write the energy balance equation

$$\frac{dE_{\rm tot}}{dt} = -P_{\rm gw} \,, \tag{3}$$

  where $E_{\rm tot} = E_{\rm kin} + E_{\rm grav} = -\frac{Gm_1m_2}{2R}$ and compute the the evolution of the separation $R(t)$ between the two objects and the evolution of the orbital frequency $\omega_{\rm src}(t)$ using Newtonian approximation. Relate this results with the phase of the emitted GW. [Hint: it is useful to define the time to coalescence $\tau = t_{\rm coal} - t$]

- Compute the time to coalescence for the following binaries:
  1. Hulse-Taylor binary system: $M_1 = 1.441\ M_\odot$, $M_2 = 1.387\ M_\odot$, $T = 7.75$ hours
  2. Earth-Sun system: $M_\odot = 1.9891 \times 10^{30}$ kg, $M_{\rm Earth} = 5.972 \times 10^{24}$ kg, $T = 365$ days

Solution ??

- In the case of inspiralling binaries in quasi-circular regime, as long as $\omega^2 \gg \dot{\omega}$, we are allowed to apply the Kepler law and we can rewrite the total energy of the system in terms of $\omega_{\rm src}$ instead of using R,

$$E_{\rm tot} = E_{\rm kin} + E_{\rm grav} = -\frac{G m_1 m_2}{R} = -\sqrt[3]{\frac{G^2 \mathcal{M}^5 \omega_{\rm src}^2}{8}}\,, \tag{4}$$

where in the second step we used the virial theorem and in the third we applied the Kepler law $\omega_{\rm src}^2 = GMR^{-3}$, where $M = m_1 + m_2$ is the total mass of the binary and $\mathcal{M} = (m_1 m_2)^{3/5} M^{-1/5}$ is the chirp mass.

Recalling that we can write the emitted power for inspiralling binaries (in the slow-motion and weak-field approximation) as

$$P_{\rm gw} = \frac{32}{5\,G\,c^5}\,(G\mathcal{M}\omega_{\rm src})^{10/3}\,, \tag{5}$$

we can impose the equality of Eq. (3) and find

$$\dot{\omega}_{\rm gw} = \frac{12}{5}\sqrt[3]{2}\left(\frac{G\mathcal{M}}{c^3}\right)^{5/3}\omega_{\rm gw}^{11/3}\,, \tag{6}$$

where $\omega_{\rm gw} = 2\omega_{\rm src}$ in the quadrupole approximation. Then defining $\tau = t_{\rm coal} - t$ (which is the time to coalescence and it is negative in the inspiral phase), we can integrate the previous expression and get

$$\omega_{\rm gw}(\tau) = 2\left(\frac{G\mathcal{M}}{c^3}\right)^{-5/8}\left(\frac{5}{256}\,\frac{1}{\tau}\right)^{3/8}\,. \tag{7}$$

We can observe that, $\dot{\omega}_{\rm gw}$ is never lower than zero, and this means that the frequency always increase. Furthermore, the frequency evolution can be used to compute the phase of the emitted gravitational radiation as

$$\Phi(t) = \int_{t_0}^{t} \omega_{\rm gw}(t')\,dt'\,. \tag{8}$$

In this case, we get

$$\Phi(t) = -2\left(\frac{5G\mathcal{M}}{c^3}\right)^{-5/8}\tau^{5/8} + \phi_0\,, \tag{9}$$

where ϕ_0 is the value of the phase at t_0.

Now, still using the Kepler law and the previous solution, we note that

$$\frac{\dot{R}}{R} = -\frac{2}{3}\frac{\dot{\omega}_{\rm gw}}{\omega_{\rm gw}} = -\frac{1}{4\tau}\,. \tag{10}$$

Integrating on the time τ we get

$$R(\tau) = R_0 \left(\frac{\tau}{\tau_0}\right)^{1/4} , \tag{11}$$

which show that the distance between the two objects decrease while the frequency increase, since they are spiralling around each other.

- Putting together the solutions of Eq. (7) and Eq. (11), we are able to estimate the time to coalescence of a binary,

$$\tau_0 = \frac{5}{256} \frac{c^5 R_0^4}{G^3 M^2 \mu} \tag{12}$$

where $\mu = m_1 m_2 / M$ is the reduced mass. Now we are able to compute R_0 using the Kepler law and we can compute the times of collapse of the listed binaries:

1. Hulse-Taylor binary: $\tau_0 = 5.16 \times 10^{16}$ s $= 1.64 \times 10^9$ yr,
2. Earth-Sun binary: $\tau_0 = 3.37 \times 10^{30}$ s $= 1.07 \times 10^{23}$ yr.