Gravitational waves — Exercise sheet n.5

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Exercise 5.1: Stationary phase approximation

Let us consider an inspiralling compact binary as source of gravitational radiation in the Newtonian limit, i.e. 0th post-Newtonian (0PN) order, such as $\omega^2 \gg \dot{\omega}$ where ω is the orbital frequency of the source. The two objects are assumed to be point-particles with different masses, m_1 and m_2. In such approximation, the quadrupole formula gives us the analytic expression for the radiated GW strain. By assuming the energy balance between the gravitational power emitted in the quadrupole approximation and the energy loss in presence of a Newtonian gravitational potential (see previous exercises), we are able to recover an analytical expression for the (instantaneous) orbital frequency $\omega \equiv \omega(t)$. Then, we define the evolution of the GW phase (which is 2×the orbital phase) as

$$\Phi(t) = 2\int_{t_0}^{t} \omega(t')\, dt' = -2\left(\frac{5G\mathcal{M}}{c^3}\right)^{-5/8} \tau^{5/8} + \phi_0\,, \tag{1}$$

where $\tau = t_{\rm coal} - t$ encodes the time dependency, $t_{\rm coal}$ is the coalescence time, $\mathcal{M}$ is the chirp mass and ϕ_0 is a reference phase value. In these conditions, the emitted GW strain can be written as

$$h_+(t_{\rm ret}) = A(t_{\rm ret})\cos\Phi(t_{\rm ret})\,, \quad h_\times(t_{\rm ret}) = A(t_{\rm ret})\sin\Phi(t_{\rm ret})\,, \tag{2}$$

where

$$A(t) = \frac{1}{r}\left(\frac{G\mathcal{M}}{c^3}\right)^{5/4}\left(\frac{5}{c\tau}\right)^{1/4} D_{+,\times}(\iota)\,, \tag{3}$$

and $D_{+,\times}(\iota)$ is a scale factor that depends on the inclination of the source.

- Compute the frequency-domain strain $\tilde{h}(f)$ for the same source (i.e. the Fourier transform) by employing the *stationary phase approximation* (SPA) [Hint: expand the exponent to leading (nonvanishing) order in $t - t_s(f)$, where $t_s(f)$ is the stationary point with respect to the Fourier variable f, defined through the equation $2\pi f = \dot{\Phi}(t_s)$. Note that the Fourier variable f and the instantaneous GW frequency $F = \dot{\Phi}/2\pi$ are in principle not the same thing!]

- What is the frequency dependence of the amplitude $|\tilde{h}(f)|^2$? Is this behaviour intuitively correct and why?