Gravitational waves — Exercise sheet n.5

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23.06.2022

Exercise 5.1: Stationary phase approximation

Let us consider an inspiralling compact binary as source of gravitational radiation in the Newtonian limit, i.e. 0th post-Newtonian (0PN) order, such as $\omega^2 \gg \dot\omega$ where ω is the orbital frequency of the source. The two objects are assumed to be point-particles with different masses, m_1 and m_2. In such approximation, the quadrupole formula gives us the analytic expression for the radiated GW strain. By assuming the energy balance between the gravitational power emitted in the quadrupole approximation and the energy loss in presence of a Newtonian gravitational potential (see previous exercises), we are able to recover an analytical expression for the (instantaneous) orbital frequency $\omega \equiv \omega(t)$. Then, we define the evolution of the GW phase (which is 2×the orbital phase) as

$$\Phi(t) = 2\int_{t_0}^{t} \omega(t')\, dt' = -2\left(\frac{5G\mathcal{M}}{c^3}\right)^{-5/8} \tau^{5/8} + \phi_0\,, \tag{1}$$

where $\tau = t_{\rm coal} - t$ encodes the time dependency, $t_{\rm coal}$ is the coalescence time, $\mathcal{M}$ is the chirp mass and ϕ_0 is a reference phase value. In these conditions, the emitted GW strain can be written as

$$h_+(t_{\rm ret}) = A(t_{\rm ret})\cos\Phi(t_{\rm ret})\,, \quad h_\times(t_{\rm ret}) = A(t_{\rm ret})\sin\Phi(t_{\rm ret})\,, \tag{2}$$

where

$$A(t) = \frac{1}{r}\left(\frac{G\mathcal{M}}{c^3}\right)^{5/4}\left(\frac{5}{c\tau}\right)^{1/4} D_{+,\times}(\iota)\,, \tag{3}$$

and $D_{+,\times}(\iota)$ is a scale factor that depends on the inclination of the source.

- Compute the frequency-domain strain $\tilde h(f)$ for the same source (i.e. the Fourier transform) by employing the *stationary phase approximation* (SPA) [Hint: expand the exponent to leading (nonvanishing) order in $t - t_s(f)$, where $t_s(f)$ is the stationary point with respect to the Fourier variable f, defined through the equation $2\pi f = \dot\Phi(t_s)$. Note that the Fourier variable f and the instantaneous GW frequency $F = \dot\Phi/2\pi$ are in principle not the same thing!]

- What is the frequency dependence of the amplitude $|\tilde{h}(f)|^2$? Is this behaviour intuitively correct and why?

Solution ??

In the Exercise Sheet n.1, we saw that, assuming that GWs are the only energy loss of the system, it is possible to write a solution for the orbital frequency of the binary imposing the energy balance

$$\frac{dE}{dt} = -P_{\rm gw} \,. \tag{4}$$

This solution leads to the dynamical evolution of the phase of the emitted GWs at Newtonian order of approximation (0PN),

$$f(\tau) = \frac{1}{\pi}\left(\frac{5}{256}\frac{1}{\tau}\right)^{3/8}\left(\frac{G\mathcal{M}}{c^3}\right)^{-5/8} \,, \tag{5}$$

where f is the frequency of the GW signal, and then the phase,

$$\Phi(t) = -2\left(\frac{5G\mathcal{M}}{c^3}\right)^{-5/8}\tau^{5/8} + \phi_0 \,, \tag{6}$$

where $\tau = t_{\rm coal} - t$ encodes the time dependency, $\mathcal{M}$ is the chirp mass, $t_{\rm coal}$ is the coalescence time and ϕ_0 is a reference phase value. This equation represents the leading order term of the post-Newtonian expansion of the GW solution for a binary system expressed as a function of the time.

In order to perform the Fourier transform of this signal, we focus on the + polarization, which can be written as

$$h_+(t_{\rm ret}) = A(t_{\rm ret})\cos\Phi(t_{\rm ret}) \,, \tag{7}$$

with

$$A(t_{\rm ret}) = \frac{1}{r}\left(\frac{G\mathcal{M}}{c^3}\right)^{5/4}\left(\frac{5}{c(t_0 - t_{\rm ret})}\right)^{1/4}\left(\frac{1+\cos^2\iota}{2}\right) \,, \tag{8}$$

where t_0 is the retarded reference time and ι is the inclination angle of the binary with respect to the line of sight. The Fourier transform can be written as

$$\begin{aligned}\tilde{h}_+(f) &= \int dt\, A(t_{\rm ret})\cos\Phi(t_{\rm ret})\, e^{2i\pi f t} \\ &= \frac{1}{2}e^{2i\pi f r/c}\int dt_{\rm ret}\, A(t_{\rm ret})\left(e^{i\Phi(t_{\rm ret})} + e^{-i\Phi(t_{\rm ret})}\right) e^{2i\pi f t_{\rm ret}} \,.\end{aligned} \tag{9}$$

Now in the last integral we can rename the integration variable $t_{\rm ret} \to t$. We are going to compute this integral on a saddle stationary point, then we can avoid to write the integration boundaries. All we need is that the stationary point lies within the range $t < t_0$. The stationary point is the value of the time $t_*(f)$ that satisfies the condition $2\pi f = \dot{\Phi}(t_*)$. This equation obviously express the relation between the phase and the frequency of the wave, but now we are imposing also that the Fourier variable must satisfy the same identity.

We restrict our analysis to positive frequencies $f > 0$, since for a generic Fourier function we can write $\tilde{h}(-f) = \tilde{h}^*(f)$. Then, observing that $\dot{\Phi} = \omega > 0 \; \forall \, t$, we can see that only the contribution proportional to $e^{-i\Phi + i2\pi f t}$ has a stationary point, while term $e^{i\Phi + i2\pi f t}$ is always oscillating. Therefore,

$$\tilde{h}_+(f) \simeq \frac{1}{2} e^{2i\pi f r/c} \int dt \, A(t) \, e^{-i\Phi(t) + 2i\pi f t} \, . \tag{10}$$

Expanding to the second order the exponential around the stationary point, we get

$$\tilde{h}_+(f) \simeq \frac{1}{2} A(t_*) \, e^{i[2\pi f(t_* - r/c) - \Phi(t_*)]} \left(\frac{2}{\ddot{\Phi}(t_*)} \right)^{1/2} \int_{-\infty}^{+\infty} dx \, e^{ix^2} \, . \tag{11}$$

The integral can be solved using the Fresnel formulae, and we get

$$\int_{-\infty}^{+\infty} dx \, e^{ix^2} = \sqrt{\pi} \, e^{-i\pi/4} \, , \tag{12}$$

and then

$$\tilde{h}_+(f) \simeq \frac{1}{2} A(t_*) \, e^{i\Psi_+} \left(\frac{2}{\ddot{\Phi}(t_*)} \right)^{1/2} \, , \tag{13}$$

where

$$\Psi_+ = 2\pi f(t_* + r/c) - \Phi(t_*) - \pi/4 \, . \tag{14}$$

Combining Eq. (6) and Eq. (8), we get

$$A(t_*) \left(\frac{2}{\ddot{\Phi}(t_*)} \right)^{1/2} = \frac{1}{\pi^{2/3}} \sqrt{\frac{5}{24}} \frac{c}{r} \left(\frac{G\mathcal{M}}{c^3} \right)^{5/6} f^{-7/6} \left(\frac{1 + \cos^2 \iota}{2} \right) \, , \tag{15}$$

and analogously

$$\Psi_+(f) = 2\pi f(t_* - r/c) - \phi_0 - \frac{\pi}{4} + \frac{3}{128\nu} \left(\pi \frac{GM}{c^3} \right)^{-5/3} f^{-5/3} \, , \tag{16}$$

where $\nu = m_1 m_2 / M^2$.