

Notes on

Gravitational Waves

v0.1

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About

These are semiprivate notes sketching topics and calculations discussed in about 15 lectures of 1.5 hours each at Jena FSU. They are not meant to substitute books. Please visit

<http://sbernuzzi.gitpages.tpi.uni-jena.de/gw/>

for an updated list of books, references and other material, including the exercise sheets distributed each week.

I welcome constructive feedbacks. **Red text is work in progr...**

Conventions. The spacetime and the metric are indicated as $(\mathcal{M}, g_{ab})$ and the notation mostly follows Wald's book: signature convention $(-, +, +, +)$, $a, b, \dots$ indexes in abstract notation, $\alpha, \beta, \dots$ indexes of tensor components, $i, j, \dots$ spatial coordinate indexes, etc. Vector and tensors are sometimes indicated as $\boldsymbol{v}$ and $\boldsymbol{g}$, i.e. in geometrical instead of index notation. Coordinate basis of the tangent vector space $T_p(M)$ are indicated as e_μ ; the natural basis of partial derivatives is $e_\mu = \partial_\mu$. The dual basis $e^{*\nu}$ ($e_\mu e^{*\nu} = \delta_\mu^\nu$) is constructed by the gradients of the coordinates is $e^{*\mu} = dx^\mu$. The exterior derivative of an n -form is indicated with $\mathbf{d}$; applied to scalars it reduces to the gradient (1-form) $\mathbf{d}f = df = \text{grad}(f)$ with components $(\mathbf{d}f)_\mu = (df)_\mu = \partial_\mu f$. Covariant derivatives (Levi-Civita connection) are indicated with ∇ . ∇ applied to scalars reduces to the gradient $\nabla f = df$ (components $\nabla_\mu f = (df)_\mu = \partial_\mu f$) and it is consistent with the concept of tangent vector $v(f) = v^\mu \nabla_\mu f$. The symbol $:=$ is an assignment, while $\equiv$ an identity.

Units are $c = G = 1$ if not specified. $GM_\odot = 4.925490947\mu\text{s}$ or 1.476625038km . The dimensionless frequency $\hat{\omega} = GM\omega$ relates to the frequency in Hz

$$f = \frac{\omega}{2\pi} = 32.3125 \hat{\omega} \frac{M_\odot}{M} \text{ kHz}. \quad (1)$$

1. Introduction

Gravitational waves (GWs) are a key prediction of Einstein's general relativity (GR). Einstein discusses these solutions of the theory already in 1916, but a rigorous understanding of gravitational radiation is available only starting from the 50s in connection to fundamental studies on the asymptotic structure of spacetime (Bondi, Goldberg, Newman, Penrose, Pirani, Robison, Sachs, Trautman and many others.)

Observational evidence for GWs exists since 1974 with the measurements of radio signals from the pulsar PSR B1913+16. The source is a binary system in our galaxy at 21000 light years (6400 pc) made of two neutron stars, one of those being a pulsar that emits period radio pulses. The variation in the time arrival of the pulses allows to identify the source is a binary system and to precise measure its orbital period (~ 7.75 hrs) and the masses. The observations performed during the decades have proven that the period of the orbit decay as predicted by GR due to the emission of GWs Fig. (1.1). Several of these systems have been identified to date and they provide stringent tests of GR.

Direct measurements of GW travelling through Earth are possible since 2015 using gravitational-wave interferometric techniques. The first detection was obtained by the LIGO experiment on September 14th 2015. The source is identified as the collision (merger) of black holes in circularized orbits as predicted by GR. The signal corresponds to the one calculated by means of numerical relativity simulations and consist of a chirp signal followed by the final black hole's ringdown. The former is predicted by the post-Newtonian formalism, the latter by black hole perturbation theory. The binary was a distance of ~ 440 Mpc (redshift 0.09) and the collision of the two masses of $\sim 35 + 30 M_{\odot}$ formed a black hole of $\sim 62 M_{\odot}$ emitting in GWs an energy of $\sim 3 M_{\odot} c^2$. Several GW detections from black hole mergers and two from neutron star binary mergers have been reported since then by the LIGO-Virgo collaboration.

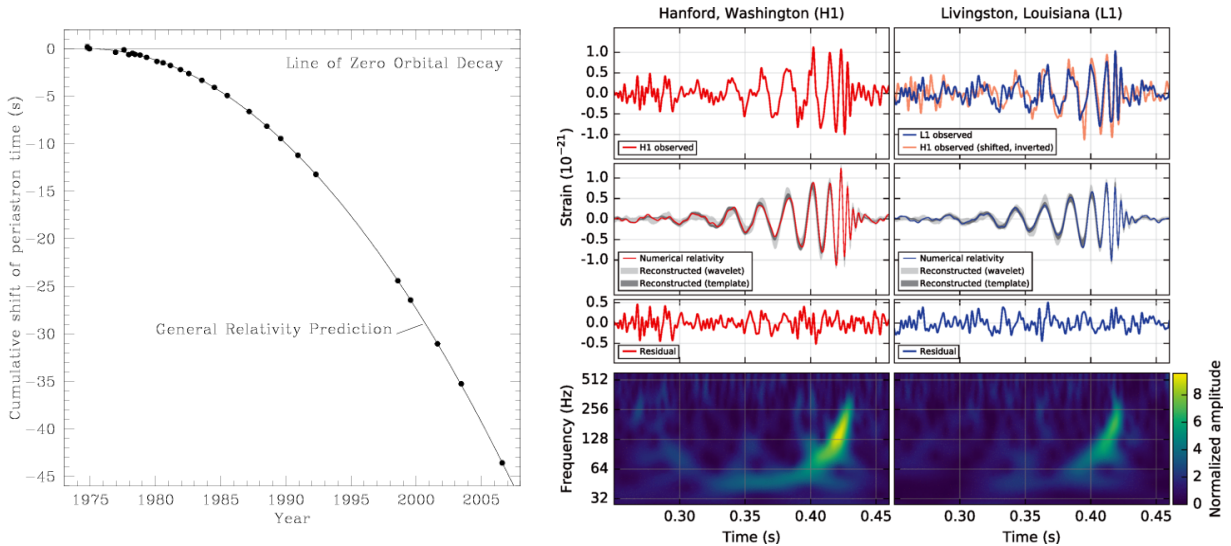


Figure 1.1: Observations of GWs. Left: The decay of the orbital period of the binary system observed in PSR B1913+16; Right: The binary black hole merger waveform GW150914 observed by LIGO in 2015.

The course introduces the basic theoretical concepts necessary to understand the two observations above. The linearized theory (Chap. 2,3) is complemented with a discussion on the short-wave approximation and the Isaacson tensor (Chap. 4). Expanding on the quadrupole formula, the systematic multipolar expansion is discussed both in terms of STF tensors and tensor spherical harmonics (Chap. 5). Those tensors are key tools for computing solutions in the post-Newtonian (PN) formalism and in perturbation theory of spherical spacetimes. The post-Newtonian formalism is summarized in Chap. 6. Black-hole linear perturbations are discussed with focus on the results of Regge-Wheeler-Zerilli and on the black-hole ringdown (Chap. 7). The post-Newtonian and spherical perturbation formalisms combines in the most advanced framework to compute the GR two-body dynamics and GW from binary black holes (The effective-one-body formalism, Chap.8). Modern techniques and tools for GW data-analysis of compact binaries are finally introduced and complemented by hands-on tutorials (Chap. 9.)

2. Weak field GR

Weak field limit of GR and the Einstein Field Equations (EFE) linearized around Mikowski spacetime. The linearized EFE equations apply for the description of Solar-system phenomena, gravitoelectric/magnetic phenomena and gravitational waves.

Suggested readings. *Chap. 4 of Wald (1984); Chap. 4 of Carroll (1997); Chap. 7-8 of Schutz (1985); Chap. 1,3 of Maggiore (2007).*

2.1 Weak field GR

In the regime of weak gravity one assumes that there exists a global inertial Cartesian frame in which the metric can be written as

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \quad \text{with} \quad |h_{\mu\nu}| \ll |\eta_{\mu\nu}| \sim 1. \quad (2.1)$$

Since the component of the perturbation of Mikowski spacetime are “small” (in the sense above), the GR equations can be linearized at linear order in h . Linearized equations apply, for example, to the Solar system where

$$|h_{\mu\nu}| \sim \frac{\phi}{c^2} \lesssim \frac{GM_\odot}{c^2 R_\odot} \sim 10^{-6}, \quad (2.2)$$

and in general describe

- Newtonian gravity;
- Gravitoelectric and gravitomagnetic phenomena;
- Propagation of gravitational waves.

Formally, the linearized theory can be regarded as a field theory in which

- $\eta_{\mu\nu}$ is a background metric;
- The grav.field generated by the matter does not backreact on the source;
- $h_{\mu\nu}$ is the main field and transforms as a tensor on flat spacetime under Lorentz transformation (Lorentz covariance). Consider a Lorentz transformation of coordinates ($\Lambda^T \eta \Lambda = \eta$):

$$x^\mu = \Lambda^\mu_{\nu'} (x')^{\nu'} = \frac{\partial x^\mu}{\partial x^{\nu'}} x^{\nu'} \Rightarrow g_{\mu'\nu'} = \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^\nu}{\partial x^{\nu'}} g_{\mu\nu} = \Lambda^\mu_{\mu'} \Lambda^\nu_{\nu'} g_{\mu\nu} = \Lambda^\mu_{\mu'} \Lambda^\nu_{\nu'} (\eta_{\mu\nu} + h_{\mu\nu}) = \quad (2.3a)$$

$$= \underbrace{\Lambda^\mu_{\mu'} \Lambda^\nu_{\nu'} \eta_{\mu\nu}}_{=\eta_{\mu'\nu'}} + \Lambda^\mu_{\mu'} \Lambda^\nu_{\nu'} h_{\mu\nu} = \eta_{\mu'\nu'} + \Lambda^\mu_{\mu'} \Lambda^\nu_{\nu'} h_{\mu\nu} \quad (2.3b)$$

that implies

$$h_{\mu\nu} \rightarrow h_{\mu'\nu'} = \Lambda^\mu_{\mu'} \Lambda^\nu_{\nu'} h_{\mu\nu}. \quad (2.4)$$

2.2 Infinitesimal diffeomorphism invariance

Symmetry of linearized GR. Consider an infinitesimal coordinate transformation:

$$x^\mu \mapsto x^{\mu'} = x^\mu + \xi^\mu(x^\alpha) \quad \text{with} \quad |\partial_\beta \xi^\alpha| \sim |h_{\mu\nu}| \ll 1 \quad (2.5a)$$

$$\frac{\partial x^{\mu'}}{\partial x^\mu} = \delta^\mu_{\mu'} + \partial_\mu \xi^{\mu'} \quad (2.5b)$$

$$\frac{\partial x^\mu}{\partial x^{\mu'}} = \delta^\mu_{\mu'} - \partial_{\mu'} \xi^\mu + \mathcal{O}(|\partial \xi|^2) \quad (2.5c)$$

where one uses the Taylor expansion of a matrix $A = (1 + \delta\xi)$ for the inverse $(1 + \delta\xi)^{-1} \approx 1 - \delta\xi$. Note prime indexes refer to tensor components in primed coordinates x' , while the unprimed indexes refer to tensor component in unprime coordinates x . Indexes on r.h.s. and l.h.s. take the same values, although they do not “match” on the two side of the

equation. For example, $g_{\mu'\nu'} = \eta_{\mu\nu} + h_{\mu\nu} = \eta_{\mu'\nu'} + h_{\mu\nu}$ really means $g_{\mu\nu}(x') = \eta_{\mu\nu} + h_{\mu\nu}(x)$. Using this notation is useful to keep formulas compact. To linear order in h and in $\partial\xi$ the metric change is:

$$g_{\mu'\nu'} = \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^\nu}{\partial x^{\nu'}} g_{\mu\nu} = (\delta_{\mu'}^\mu - \partial_{\mu'} \xi^\mu)(\delta_{\nu'}^\nu - \partial_{\nu'} \xi^\nu)(\eta_{\mu\nu} + h_{\mu\nu}) = \quad (2.6a)$$

$$= (\delta_{\mu'}^\mu \delta_{\nu'}^\nu - \partial_{\mu'} \xi^\mu \delta_{\nu'}^\nu - \partial_{\nu'} \xi^\nu \delta_{\mu'}^\mu + \partial_{\mu'} \xi^\mu \partial_{\nu'} \xi^\nu)(\eta_{\mu\nu} + h_{\mu\nu}) = \quad (2.6b)$$

$$= \underbrace{\delta_{\mu'}^\mu \delta_{\nu'}^\nu \eta_{\mu\nu}}_{=\eta_{\mu'\nu'}} - \underbrace{\partial_{\mu'} \xi^\mu \delta_{\nu'}^\nu \eta_{\mu\nu}}_{=\partial_{\mu'} \xi_{\nu'}} - \partial_{\nu'} \xi^\nu \delta_{\mu'}^\mu \eta_{\mu\nu} + \underbrace{\delta_{\mu'}^\mu \delta_{\nu'}^\nu h_{\mu\nu}}_{h_{\mu'\nu'}} - \underbrace{\partial_{\mu'} \xi^\mu \delta_{\nu'}^\nu h_{\mu\nu} - \partial_{\nu'} \xi^\nu \delta_{\mu'}^\mu h_{\mu\nu} + \partial_{\mu'} \xi^\mu \partial_{\nu'} \xi^\nu h_{\mu\nu}}_{=\mathcal{O}(h^2)} = \quad (2.6c)$$

$$= \eta_{\mu'\nu'} + h_{\mu'\nu'} - 2\partial_{(\mu'} \xi_{\nu')} = \quad (2.6d)$$

$$= \eta_{\mu\nu} + h_{\mu'\nu'} - 2\partial_{(\mu'} \xi_{\nu')} \quad (2.6e)$$

and any infinitesimal coordinate transformation that maps the perturbed metric field into

$$h_{\mu\nu} \mapsto h_{\mu'\nu'} = h_{\mu\nu} + 2\partial_{(\mu} \xi_{\nu)} , \quad (2.7)$$

leaves the metric invariant. The weak metric is represented by the equivalence classes of metrics linked by the infinitesimal coordinate transformations above. Note the linearized Einstein tensor is invariant w.r.t the above transformation [exercise].

Hence, if one considers ξ^α as the components of a vector, then the transformation can be written in terms of the Lie derivatives, and the infinitesimal coordinate transformation is then interpreted as an *infinitesimal diffeomorphism* generated by ξ :

$$2\partial_{(\mu} \xi_{\nu)} = \mathcal{L}_\xi \eta_{\mu\nu} \Rightarrow h_{\mu\nu} \mapsto h_{\mu\nu} + \mathcal{L}_\xi \eta_{\mu\nu} . \quad (2.8)$$

Weak field GR is thus *invariant under infinitesimal diffeomorphisms*.

Remark 2.2.1. *The above transformation is the analogous of gauge transformation for the potentials in electromagnetism:*

$$A_\alpha \mapsto A_\alpha + \partial_\alpha \chi . \quad (2.9)$$

2.3 Weak field equations

Calculation of the Einstein tensor linearized in h . Indexes are raised on lowered with the flat metric η ; for example the trace of the perturbation is $h := h^\alpha_\alpha = \eta^{\alpha\beta} h_{\alpha\beta}$. It is left as [exercise] to show:

$$g^{\mu\nu} = \eta^{\mu\nu} + h^{\mu\nu} + \mathcal{O}(h^2) \quad (2.10a)$$

$$\Gamma_{\alpha\beta}^\mu = \frac{1}{2} \eta^{\mu\lambda} (\partial_\alpha h_{\lambda\beta} + \partial_\beta h_{\lambda\alpha} - \partial_\lambda h_{\alpha\beta}) + \mathcal{O}(h^2) \quad (2.10b)$$

$$R_{\mu\nu} = \partial\Gamma - \partial\Gamma + \underbrace{\Gamma\Gamma - \Gamma\Gamma}_{\mathcal{O}(h^2)} = \partial_\alpha \Gamma_{\mu\nu}^\alpha - \partial_\mu \Gamma_{\alpha\nu}^\alpha + \mathcal{O}(h^2) \approx \partial^\alpha \partial_{(\mu} h_{\nu)\alpha} - \frac{1}{2} \partial_\lambda \partial^\lambda h_{\mu\nu} - \frac{1}{2} \partial_\mu \partial_\nu h \quad (2.10c)$$

$$R = \eta^{\mu\nu} R_{\mu\nu} = \frac{1}{2} (\eta^{\mu\nu} \partial^\alpha \partial_\mu h_{\nu\alpha} + \eta^{\mu\nu} \partial^\alpha \partial_\nu h_{\mu\alpha}) - \frac{1}{2} \partial_\lambda \partial^\lambda (\underbrace{\eta^{\mu\nu} h_{\mu\nu}}_{=h}) - \frac{1}{2} \underbrace{\eta^{\mu\nu} \partial_\mu \partial_\nu h}_{\partial_\lambda \partial^\lambda h} = \frac{1}{2} \partial^\alpha \partial^\nu h_{\nu\alpha} - \partial_\lambda \partial^\lambda h \quad (2.10d)$$

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} R = \partial^\alpha \partial_{(\mu} h_{\nu)\alpha} - \frac{1}{2} \partial_\lambda \partial^\lambda h_{\mu\nu} - \frac{1}{2} \partial_\mu \partial_\nu h - \frac{1}{2} \eta_{\mu\nu} (\partial^\alpha \partial^\beta h_{\alpha\beta} - \partial_\lambda \partial^\lambda h) . \quad (2.10e)$$

The linearized Einstein tensor

$$G_{\mu\nu} = \underbrace{\partial^\alpha \partial_{(\mu} h_{\nu)\alpha}}_{\text{I.}} - \underbrace{\frac{1}{2} \partial_\lambda \partial^\lambda h_{\mu\nu}}_{\text{II.}} - \underbrace{\frac{1}{2} \partial_\mu \partial_\nu h}_{\text{III.}} - \underbrace{\frac{1}{2} \eta_{\mu\nu} \partial^\alpha \partial^\beta h_{\alpha\beta}}_{\text{IV.}} + \underbrace{\frac{1}{2} \eta_{\mu\nu} \partial_\lambda \partial^\lambda h}_{\text{V.}} , \quad (2.11)$$

can be written in a simpler form considering the trace reverse metric

$$\bar{h}_{\mu\nu} := h_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} h . \quad (2.12)$$

Note that

$$\bar{h} = \eta^{\mu\nu} \bar{h}_{\mu\nu} = \underbrace{\eta^{\mu\nu} h_{\mu\nu}}_{=h} - \frac{1}{2} \underbrace{\eta^{\mu\nu} \eta_{\mu\nu}}_{=4} h = -h . \quad (2.13)$$

Calculate each term:

$$I = \partial^\alpha \partial_{(\mu} \bar{h}_{\nu)\alpha} + \frac{1}{4} \partial^\alpha \partial_\mu (\eta_{\nu\alpha} h) + \frac{1}{4} \partial^\alpha \partial_\nu (\eta_{\mu\alpha} h) = \underbrace{\partial^\alpha \partial_{(\mu} \bar{h}_{\nu)\alpha}}_{\text{Ia}} + \underbrace{\frac{1}{2} \partial_{(\mu} \partial_{\nu)} h}_{\text{Ib}} \quad (2.14a)$$

$$II = -\underbrace{\frac{1}{2} \eta_{\alpha\beta} \partial^\alpha \partial^\beta \bar{h}_{\mu\nu}}_{\text{IIa}} - \underbrace{\frac{1}{4} \eta_{\mu\nu} \eta_{\alpha\beta} \partial^\alpha \partial^\beta h}_{\text{IIb}} \quad (2.14b)$$

$$IV = -\frac{1}{2} \eta_{\mu\nu} \partial^\alpha \partial^\beta (\bar{h}_{\alpha\beta} + \frac{1}{2} \eta_{\alpha\beta} h) = -\underbrace{\frac{1}{2} \eta_{\mu\nu} \partial^\alpha \partial^\beta \bar{h}_{\alpha\beta}}_{\text{IVa}} - \underbrace{\frac{1}{4} \eta_{\mu\nu} \eta_{\alpha\beta} \partial^\alpha \partial^\beta h}_{\text{IVb}} \quad (2.14c)$$

$$0 = V + \text{IIb} + \text{IVb} \quad (2.14d)$$

$$0 = \text{III} + \text{Ib} \quad (2.14e)$$

$$G_{\mu\nu} = \text{IIa} + \text{Ia} + \text{IVa} = -\frac{1}{2} \eta_{\alpha\beta} \partial^\alpha \partial^\beta \bar{h}_{\mu\nu} + \partial^\alpha \partial_{(\mu} \bar{h}_{\nu)\alpha} - \frac{1}{2} \eta_{\mu\nu} \partial^\alpha \partial^\beta \bar{h}_{\alpha\beta} \quad (2.14f)$$

From the above expression one sees that the last two terms contain the divergence of the metric. Imposing the *Hilbert gauge* (or Lorentz)

$$\partial^\alpha \bar{h}_{\mu\alpha} = 0, \quad (\text{Hilbert gauge}) \quad (2.15)$$

leads to the following equations for linearized GR:

$$\square_\eta \bar{h}_{\mu\nu} = -\frac{16\pi G}{c^4} T_{\mu\nu}. \quad (2.16)$$

Observations

- By making an infinitesimal coordinate transformation, it is always possible to reduce to Hilbert gauge. The set of 4 functions ξ^μ is given by the solution of 4 inhomogeneous wave equations (Cf. Lorentz gauge in electrodynamics):

$$h_{\mu\nu} \mapsto h_{\mu\nu} + 2\partial_{(\mu} \xi_{\nu)} \quad (2.17a)$$

$$h \mapsto h + \eta^{\alpha\beta} \partial_\alpha \xi_\beta + \eta^{\alpha\beta} \partial_\beta \xi_\alpha = h + 2\partial_\mu \xi^\mu \quad (2.17b)$$

$$\bar{h}_{\mu\nu} \mapsto \bar{h}_{\mu\nu} + 2\partial_{(\mu} \xi_{\nu)} - \eta_{\mu\nu} \partial_\alpha \xi^\alpha \quad (2.17c)$$

$$\partial^\alpha \bar{h}_{\mu\alpha} \mapsto \partial^\alpha \bar{h}_{\mu\alpha} + \square \xi_\mu + \cancel{\partial^\mu \partial_\nu \xi_\mu} - \cancel{\partial_\nu \partial_\lambda \xi^\lambda} \Rightarrow \square \xi_\mu = -\partial^\alpha \bar{h}_{\mu\alpha} =: V_\mu \neq 0 \quad (2.17d)$$

- Eq. (2.16) is a linear wave equation for the components of $\mathbf{h}$. At linear order in $\mathbf{h}$, the stress energy tensor does **not** depend on $\mathbf{h}$, so in linear GR one can specify the matter source and solve for the metric. For example, it is possible to calculate solutions using Green functions as in electrodynamics (see below).
- The Bianchi identity in the weak field simplifies and it is written as the partial derivative of the Einstein tensor since $\boldsymbol{\partial}$ is the connection associated to $\boldsymbol{\eta}$. Hence, the weak field equations imply the conservation of the stress-energy tensor on flat background, and that matter does not backreact on the curvature:

$$\partial_\nu G^{\mu\nu} = 0 \Rightarrow \partial_\nu T^{\mu\nu} = 0. \quad (2.18)$$

- In vacuum, linearized EFEs are the equations for a massless spin-2 field propagating in flat spacetime (Chap. 13 Wald (1984)).

2.4 Weak field solutions

Formal solutions of Eq. (2.16) for static and stationary matter distributions.

2.4.1 Static source

A static matter distribution is modeled by a time-independent stress-energy tensor in the form

$$T_{\mu\nu} = \rho t_\mu t_\nu \quad \text{i.e.} \quad T_{00} = \rho(x^i), \quad T_{0i} = T_{ij} = 0, \quad (2.19)$$

where $t^\mu = (\partial_t)^\mu$ is the vector along the time direction of the global inertial coordinates. With this prescription, the r.h.s. of Eq. (2.16) is time-independent, thus also the grav. field must be time independent

$$\partial_t \bar{h}_{\mu\nu} = 0, \quad (2.20)$$

and the linearized EFE reduce to Poisson equations for the components of the metric field:

$$\begin{cases} \Delta \bar{h}_{\mu\nu} = -16\pi\rho & , \quad \mu = \nu = 0 \\ \Delta \bar{h}_{\mu\nu} = 0 & , \quad \text{otherwise} . \end{cases} \quad (2.21)$$

The Poisson equations have boundary values $\bar{h}_{ij}|_{r \rightarrow \infty} = 0$. Hence, the solution of the 00 equation is immediately given in terms of the Newton grav. potential by formally comparing to Newton's $\Delta\phi = 4\pi\rho$. The solution of the other equations (without source terms) is simply zero:

$$\begin{cases} \bar{h}_{\mu\nu} = -4\phi , & \mu = \nu = 0 \\ \bar{h}_{\mu\nu} = 0 , & \text{otherwise} \end{cases} \Rightarrow \bar{h}_{\mu\nu} = -4\phi t_\mu t_\nu . \quad (2.22)$$

Reversing the trace:

$$\bar{h} = \eta^{\mu\nu} \bar{h}_{\mu\nu} = \eta^{00} \bar{h}_{00} = +4\phi \quad (2.23a)$$

$$h_{\mu\nu} = \bar{h}_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} \bar{h} = -4\phi t_\mu t_\nu - \frac{1}{2} \eta_{\mu\nu} (4\phi) \quad (2.23b)$$

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} = \eta_{\mu\nu} + \bar{h}_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} \bar{h} = \eta_{\mu\nu} (1 - \frac{\bar{h}}{2}) + \bar{h}_{\mu\nu} = \eta_{\mu\nu} (1 - 2\phi) - 4\phi t_\mu t_\nu \quad (2.23c)$$

or

$$g = - \left(1 + \frac{2\phi}{c^2}\right) c^2 dt^2 + \left(1 - \frac{2\phi}{c^2}\right) \delta_{ij} dx^i dx^j \approx -(1 + 2\phi) c^2 dt^2 + \delta_{ij} dx^i dx^j , \quad (2.24)$$

where the second line highlights the leading order terms in a $1/c$ expansion.

Note that far from a source of mass M the multipolar expansion of the grav. potential starts with $\phi \approx -\frac{M}{r} + \mathcal{O}(1/r^2)$, so the distant metric is fully specified by the source mass and reduces to Mikowski at $r \rightarrow \infty$ if $M \neq 0$, or everywhere if $M \equiv 0$. It is remarkable that this metric describes the motion of particles (geodesic motion) in both Newtonian gravity and SR (using the appropriate limits). Note, however, a subtle point: the geodesic equations on the weak metric (weak gravity field) imply

$$\frac{d^2 x^i}{dt^2} = -\partial_i \phi , \quad (\text{weak gravity}) \quad (2.25)$$

but they are **not** consistent with the equation $\partial_\mu T^{\mu\nu} = 0$. The latter implies the geodesics on flat metric (Mikowski spacetime, unaccelerated motion, no gravity)

$$\frac{d^2 x^i}{dt^2} = 0 , \quad (\text{no gravity}) . \quad (2.26)$$

This illustrates some of the difficulties/inconsistencies in predicting the EOM of matter from linearized EFE as an expansion on η . Eq. (2.26) is the EOM of the matter that determines curvature, Eq. (2.25) is the EOM of matter in the resulting slightly curved spacetime.

Example 2.4.1. Deflection of light (Einstein's 1915 calculation). *Consider a photon moving in a weak and static gravitational field generated by a mass M . The weak metric is fully defined by Eq. (2.24) and the grav. potential $\phi = GM/rc^2$. The photon moves in the $z = 0$ plane in direction x with impact parameter $y = b$. Setting $d\ell^2 = \delta_{ij} dx^i dx^j$ and interpreting the d as differential, the coordinate speed of the photon can be quickly computed from the condition $g = 0$,*

$$v = \frac{d\ell}{dt} = c \left(\frac{1 - 2\phi}{1 + 2\phi} \right)^{1/2} \approx c(1 - 2\phi) = c \left(1 - \frac{2GM}{c^2 r} \right) , \quad (2.27)$$

where the square root was expanded in $\phi \ll 1$. The speed of light measured in these non-inertial coordinates decreases the closer the photon approach to the mass. This is analogous to a wave front passing through a medium in which the speed of the wave varies with position. Hence, a beam of light rays is bent towards the mass the closer is to the mass. The deflection can be calculated using Huygens principle in analogous way to the refraction angle of waves in a medium,

$$\frac{d\theta}{dx} = \frac{1}{c} \frac{dv}{dy} = \frac{2GM}{c^2} \frac{y}{(x^2 + y^2)^{3/2}} \Rightarrow \theta = \frac{2GM}{c^2} \int \frac{b dx}{(x^2 + b^2)^{3/2}} = \frac{4GM}{c^2 b} . \quad (2.28)$$

Details and calculations are left as [exercise]. The same solution can be found considering null geodesics, e.g. (Carroll, 1997), and also by starting from the Schwarzschild metric in isotropic coordinates. Note that one can also perform a calculation using Newton gravity (the acceleration does not depend on the photon mass...) and find that GR result is twice the Newtonian prediction.

Observations conducted by Eddington in 1916 and others later, indicate that the Sun deflects photons of an angle $\theta_d \approx 8.5 \times 10^{-6}$ radians (1.75 arcsec) in agreement with the GR formula above ($M_\odot = 1.98847 \times 10^{33}$ g and $R_\odot = 6.960 \times 10^{10}$ cm). Accurate measurements of light deflection are also available from the 60s using radio interferometers and astrophysical sources called blazars (gravitational lensing).

2.4.2 No-stresses source

A matter distribution with mass-energy density current vector J^μ and no stresses is modeled by a stress-energy tensor in the form

$$T_{\mu\nu} = 2J_{(\mu}t_{\nu)} - 2\rho t_\mu t_\nu \quad \text{i.e.} \quad T_{0\mu} = J_\mu, \quad T_{ij} = 0. \quad (2.29)$$

Note that $J^\mu = \rho u^\mu = \rho(W, Wv^i/c)$ and that $T_{ij} = \mathcal{O}(v^2/c^2)$, thus (i) the static case considered above is equivalent to consider the slow velocity limit $v/c \ll 1$ for the (nonrelativistic) source, and (ii) taking $T_{ij} \equiv 0$ is equivalent to neglect velocity terms $\mathcal{O}(1/c^2)$ in the source motion.

The linearized equations read

$$\begin{cases} \square \bar{h}_{0\mu} &= -16\pi T_{0\mu}, \quad \mu = 0, \dots, 3 \\ \square \bar{h}_{ij} &= 0, \quad i, j = 1, \dots, 3. \end{cases} \quad (2.30)$$

If the spatial component of the metric are assumed time-independent, then they are solution of the boundary value problem with the Poisson equation and boundary values $\bar{h}_{ij}|_{r \rightarrow \infty} = 0$. This implies they are zero:

$$\partial_t \bar{h}_{ij} = 0 \quad \Rightarrow \quad \begin{cases} \Delta \bar{h}_{ij} = 0 \\ \bar{h}_{ij}|_{r \rightarrow \infty} = 0 \end{cases} \quad \Rightarrow \quad \bar{h}_{ij} = 0. \quad (2.31)$$

The linearized equations reduce to only those for $\bar{h}_{0\mu}$, that are formally equivalent to the Maxwell equations in Lorentz gauge for the field

$$A_\mu := -\frac{1}{4}\bar{h}_{0\mu} = -\frac{1}{4}\bar{h}_{\mu\nu}t^\nu. \quad (2.32)$$

Once a solution is found, the metric is given by [exercise]

$$g_{00} = -1 + 2A_0, \quad g_{0i} = 4A_i, \quad g_{ij} = 1 + 2A_0\delta_{ij}. \quad (2.33)$$

If one further assumes that $\bar{h}_{0\mu}$ is time-independent a formal solution can be obtained with the usual Green function method for the Poisson equation

$$\partial_t \bar{h}_{0\mu} = 0 \quad \Rightarrow \quad \begin{cases} A_0 &= -\phi \\ A_i &= \int d^3x' \frac{T_{0i}(x')}{|\vec{x} - \vec{x}'|}. \end{cases} \quad (2.34)$$

Reintroducing the factors c , the metric reads

$$g = -\left(1 + \frac{2\phi}{c^2}\right) c^2 dt^2 + 4A_i d(ct) dx^i + \left(1 - \frac{2\phi}{c^2}\right) \delta_{ij} dx^i dx^j. \quad (2.35)$$

Using the same formulas as in electrodynamics, one then defines from A_μ the *gravitoelectric* and *gravitomagnetic* fields and the geodesics EOM in the small velocity limit reduce to those of a particle subject to the (gravitational) Lorentz force. Consider the Lagrangian for a particle in the weak metric Eq. (2.35) and expand in v/c ¹:

$$L = -mc \sqrt{-g_{\mu\nu} \frac{dx^\mu}{dt} \frac{dx^\nu}{dt}} = -mc^2 \sqrt{-g_{00} - 2g_{0i} \frac{v^i}{c} - g_{ij} \frac{v^i v^j}{c^2}} \quad (2.36a)$$

$$= -mc^2 \sqrt{1 - 2A_0 - 8A_i \frac{v^i}{c} - \frac{v^j v_j}{c^2} + 2A_0 \frac{v^j v_j}{c^2}} \quad (2.36b)$$

$$\approx -mc^2 + \frac{m}{2}v^2 + m\phi + 4mcA_i v^i. \quad (2.36c)$$

The above equation implies an EOM with the (gravitational) Lorentz force,

$$\ddot{\vec{x}} = \vec{E} + 4\vec{v} \times \vec{B}. \quad (2.37)$$

The difference w.r.t. the EOM for a charge particle are that (i) there is no charge; (ii) there is a factor “4” in front of the gravitomagnetic field.

Example 2.4.2. Lense-Thirring effect. *The spacetime of a “weakly” gravitating planet or star in slow rotation is described by the metric of Eq. (2.33). The grav. field is stationary and the motion of a test body in such field is precisely described by the gravitoelectric/magnetic equations. For example, the precession motion of a gyroscope due to the gravitomagnetic field of the rotating object is precisely given by the spin-precession formula of electromagnetism*

$$\frac{d\vec{s}}{dt} = \vec{\mu} \times \vec{B} = \frac{q}{2m} \vec{s} \times \vec{B} = \vec{s} \times \vec{\Omega}, \quad (2.38)$$

¹Recall that $x^0 = ct$ and $v^i = dx^i/dt$.

where $\vec{\Omega} = -q/m\vec{B}$, q is the particle charge and $\vec{\mu}$ the magnetic moment. The formal substitution

$$q \mapsto m, \quad \vec{B} \mapsto 4\vec{B}_g, \quad (2.39)$$

maps the eletroagnetic problem top the gravitomagnetic one. The precession frequency in the latter is thus $\Omega_g = -2B_g$. In the grav. field of Earth one obtains that a free-falling body at distance r acquires an angular velocity

$$\Omega = \dot{\varphi} \sim 0.22''/\text{yr} \left(\frac{R_\oplus}{r} \right)^3. \quad (2.40)$$

This effect is called also frame dragging, and it has been measured by the satellite mission Gravity Probe B in 2004 with 20% confidence (Everitt et al., 2011) ².

Example 2.4.3. An extreme frame dragging phenomenon happens around rotating black holes. But in this case it is not a weak field effect and Eq. (2.33) do not strictly apply. Particles close to the black hole horizon are dragged around at a speed comparable to the hole's rotation $\Omega \sim \Omega_{BH}$. The Lense-Thirring effect in strong field play an important role to understand high-energy particle emission from matter accreting onto black holes.

2.5 Relativistic precession

To conclude this Chapter we briefly review relativistic precession.

A simple way to derive Eq. (2.24) is to start from the Schwarzschild metric in isotropic coordinates

$$g = - \left(1 - \frac{M}{2r} \right)^2 \Psi^2(r) dt^2 + \Psi^4(r) (dr^2 + r^2 d\Omega^2), \quad (2.41)$$

where r the isotropic radius ³ and $\Psi(r) = 1 + M/(2r)$ is the conformal factor that for $\phi = -M/r \ll 1$ is

$$\Psi \approx 1 - \frac{\phi}{2c^2}. \quad (2.42)$$

The conformal spatial metric is flat and can be thus written in Carterian coordinates

$$dr^2 + r^2 d\Omega^2 = \delta_{ij} dx^i dx^j. \quad (2.43)$$

Using this coordinates and taking the limit to large radii one gets Eq. (2.24).

The simplest way to analyze relativistic precession is to consider test bodies in the Schwarzschild metric in standard coordinates, e.g. (Wald, 1984). Starting from the Schwarzschild radial geodesic and (i) restricting to equatorial plane, (ii) including the constant of motions, (iii) multiplying by $(\dot{\phi})^{-2}$, (iv) changing variable to $u = 1/r = GM/c^2 r$ (from now on r is the Schwarzschild radius ⁴), one obtains [exercise]

$$\frac{d^2 u}{d\phi^2} + u = \frac{M}{\ell^2} + \underbrace{3Mu^2}_{\text{GR term}}. \quad (2.44)$$

The above equation is similar to the Newtonian equations for elliptic orbits but includes the “GR term”. Without the GR term one has the orbit equation

$$u_N = M\ell^2(1 + \varepsilon \cos \phi), \quad (2.45)$$

describing the Kepler ellipses and where the eccentricity ε is fixed by the initial condition. The GR term can be treated as a perturbation when compared to the Newtonian term at the r.h.s. if $3Mu^2 \ll M\ell^{-2}$. For example, this condition is verified for Mercury's orbit around the Sun since the ratio of the two terms reduced to the tangential velocity that is small,

$$\frac{3Mu^2}{M\ell^{-2}} = 3u^2\ell^2 = 3r^{-2}(r^2\dot{\phi})^2 \simeq 3\left(r\frac{d\phi}{dt}\right)^2 = 3\left(\frac{v_\perp}{c}\right)^2 \approx 10^{-7}. \quad (2.46)$$

Let $u = u_N + v$ and find a linear equation in v :

$$\underbrace{\frac{d^2 u_N}{d\phi^2} + u_N - \frac{M}{\ell^2}}_{=0} + \frac{d^2 v}{d\phi^2} + v = 3M(u_N^2 + 2u_N v + v^2) \approx 3Mu_N^2. \quad (2.47)$$

This is an equation for a forced oscillator; the solution is given by the general solution of the homogeneous equations plus a particular solution of the complete equation. A particular solution is given by

$$v = 3M^2\ell^{-4} \left[1 + \underbrace{\varepsilon\phi \sin \phi}_{\text{secular term}} + \varepsilon^2 \left(\frac{1}{2} - \frac{1}{6} \cos(2\phi) \right) \right], \quad (2.48)$$

²Summary webpage: <http://einstein.stanford.edu/highlights/status1.html>

³The isotropic radius is defined from the Schwarzschild radial coordinate as $R = r(1 + M/(2r))^2$.

⁴To switch from/to geometric units is often sufficient to remember that $M \leftrightarrow GM/c^2$.

which is the combination of a constant term, a secular term $\propto \phi$ and an oscillating term. An approximate solution to the perturbation problem is obtained by just picking the secular term,

$$u \approx u_N + v_{\text{secular}} = M\ell^{-2}(1 + \varepsilon \cos \phi) + 3M^2\ell^{-4}\varepsilon\phi \sin \phi \simeq M\ell^{-2} (1 + \varepsilon \cos(\phi - 3M^2\ell^{-2}\phi)) , \quad (2.49)$$

where in the first passage we approximate $3M^2\ell^{-2}\phi \sim \sin(3M^2\ell^{-2}\phi)$ for small arguments of the $\sin(\cdot)$, and trigonometric identities were used in the second passage. The approximate solution above shows that if $\varepsilon \neq 0$, then the orbit is not periodic of 2π and it is not an ellipses. For a revolution of $\phi = 2\pi$, the periastron (perihelion) shift is

$$2\pi(1 - 3M^2\ell^{-2}) = 2\pi - 6M^2\ell^{-2} \Rightarrow \Delta\phi = 6\pi 3M^2\ell^{-2} . \quad (2.50)$$

The angular momentum in the formula above is difficult to measure from astronomical observations and it is best to replace it with the semi-major axis a . This is done by comparing $u_N(\phi)$ to the generic equation for an ellipsis $u(\phi)$:

$$ua(1 - \varepsilon^2) = (1 + \varepsilon \cos \phi) , \quad (2.51)$$

which implies $M\ell^2 = a(1 - \varepsilon^2)$. Plugging in the angular momentum expression in terms of the semi-major axis, the final result for the perihelion shift is

$$\Delta\phi = \frac{6\pi M}{a(1 - \varepsilon^2)} = \frac{6\pi GM}{a(1 - \varepsilon^2)c^2} . \quad (2.52)$$

Example 2.5.1. Mercury perihelion shift. *Mercury data are $GM_\odot/c^2 \simeq 1.48 \cdot 10^5$ cm, $a \simeq 5.79 \cdot 10^{12}$ cm, $\varepsilon \simeq 0.20$, $T \simeq 88$ days, resulting in $\Delta\phi \simeq 0.103''/\text{orbit} \simeq 43''/100$ yrs. This is a famous and successful experimental test of GR.*

Example 2.5.2. PSR 1913+16 is a relativistic binary system that shows precession. This precession is not described by the Schwarzschild geodesic because the binary has components of comparable masses. However, it can be studied using the post-Newtonian formalism Damour and Deruelle (1986). PSR 1913+16 has a precession of $\Delta\phi \sim 4.2^\circ/100$ yrs, which is $\sim 270\times$ the one of Mercury and indicate the extreme gravity of that binary system. Differently from Mercury, PSR 1913+16 cannot be used this way to verify GR from measured precession because the masses of PSR are not known. The measurement is in fact used to estimate the masses, as we shall see.

3. GWs in linear GR

Gravitational-wave in linearized GR: propagation properties, effect on test masses, and the quadrupole formula.

Suggested readings. *Chap. 1,3 of Maggiore (2007). Chap. 4 of Wald (1984); Chap. 7 of Carroll (1997); Chap. 7-8 of Schutz (1985);*

3.1 Gravitational waves (GWs) propagation

The linearized EFE in vacuum are homogeneous wave equations for each component of the metric,

$$0 = \square_\eta \bar{h}_{\mu\nu} = \eta^{\alpha\beta} \partial_\alpha \partial_\beta \bar{h}_{\mu\nu} . \quad (3.1)$$

Solutions to the above equation can be constructed by superposition of plane waves with (constant) wave vector $k^\mu = (\omega, \vec{k})$ ¹ and amplitudes $A_{\mu\nu}$:

$$\bar{h}_{\mu\nu} = A_{\mu\nu} \exp(ik_\mu x^\mu) = A_{\mu\nu} \exp[i(-k_0 x^0 + \vec{k} \cdot \vec{x})] = A_{\mu\nu} \exp[i(-\omega t + \vec{k} \cdot \vec{x})] \quad (3.2a)$$

$$\partial_\mu \bar{h}_{\alpha\beta} = A_{\alpha\beta} \partial_\mu (\exp(ik_\rho x^\rho)) = \bar{h}_{\alpha\beta} \partial_\mu (ik_\rho x^\rho) = i \bar{h}_{\alpha\beta} k_\rho \delta_\mu^\rho \quad (3.2b)$$

Substituting the plane-wave ansatz into the wave equations, one finds immediately that the *wave vector is null* (Minkowski metric):

$$0 = \square \bar{h}_{\alpha\beta} \stackrel{p.w.}{=} -\eta^{\mu\nu} k_\mu k_\nu \bar{h}_{\alpha\beta} \Rightarrow \eta^{\mu\nu} k_\mu k_\nu = k_\mu k^\mu = 0 \Rightarrow \omega^2 = |\vec{k}|^2 c^2 . \quad (3.3)$$

The last equation is the *dispersion relation* for GWs and indicates that GWs propagate at the speed of light. Another way to see this is to consider the worldline of a photon moving along k^μ , and observe that it moves in phase with the wave's phase $\varphi = k_\mu x^\mu$:

$$x^\mu(\lambda) = k^\mu \lambda + x^\mu(0) \Rightarrow k_\mu x^\mu(\lambda) = \underbrace{k_\mu k^\mu}_{=0} \lambda + k_\mu x^\mu(0) = k_\mu x^\mu(0) = \text{const} . \quad (3.4)$$

The Hilbert gauge Eq. (2.15) translates into the 4 equations that imply the waves are *transverse to the direction of propagation*:

$$0 = -\partial^\alpha \bar{h}_{\mu\alpha} = ik^\mu A_{\mu\nu} \exp[i(k_\rho x^\rho)] = ik^\mu \bar{h}_{\mu\alpha} \Rightarrow k^\mu A_{\mu\nu} = 0 . \quad (3.5)$$

Transverse-traceless (TT) gauge and physical degrees of freedom. Linearized EFE in vacuum are 10 equations. Imposing the Hilbert gauge (4 equations) reduces the problem to $10 - 4 = 6$ equations, but there remains freedom in the gauge choice. Looking at Eq. (2.17), it is immediate that the Hilbert gauge is defined up to 4 harmonic functions ξ^μ : any infinitesimal transformation such that $\square \xi_\nu = 0$ maintains the Hilbert gauge $-\partial^\alpha \bar{h}_{\mu\alpha} = 0$. Let us further specify the gauge choice.

The additional gauge freedom can be fixed by

- (i) writing the harmonic function as (note the following expression clearly solves $\square \xi_\nu = 0$)

$$\xi^\mu = B^\mu \exp[i(k_\rho x^\rho)] , \quad (3.6)$$

such that the transformation of the trace reverse metric translates into the following transformation of the plane-wave amplitude,

$$\bar{h}_{\mu\nu} \mapsto \bar{h}_{\mu\nu} + 2\partial_{(\mu} \xi_{\nu)} - \eta_{\mu\nu} \partial_\alpha \xi^\alpha \Rightarrow A_{\mu\nu} \mapsto A_{\mu\nu} - i2k_{(\mu} B_{\nu)} + i\eta_{\mu\nu} k_\rho B^\rho . \quad (3.7)$$

¹Note that in general the wave frequency measured by an observer of 4-velocity u^μ is $\omega = k_\mu u^\mu$.

(ii) and then fixing the functions B^μ requiring the additional 4 conditions

$$\begin{cases} \bar{h} = 0 = A^\mu_\mu & \text{Traceless condition} \\ \bar{h}_{0\mu} = 0 = A_{0\mu} & \text{Transverse condition} \end{cases} \quad (3.8)$$

The above conditions give a linear algebraic system for B^μ , which can be inverted to find the solution. The gauge above is called *traceless $\mathcal{E}$ transverse (TT) gauge*. The remaining degrees of freedom are $10 - 4 - 4 = 2$, that represent the two physical states of gravity waves.

Let us explicitly consider a wave propagating along the $\hat{z}$ -direction,

$$k^\mu = (\omega, 0, 0, k_3) . \quad (3.9)$$

Then

1. Null condition, $k_\mu k^\mu = 0 \Rightarrow -k_3 = \omega$;
2. Phase, $k_\rho x^\rho = \omega(t - z)$;
3. Hilbert gauge, $k^\mu A_{\mu\nu} = 0 \Rightarrow k^0 A_{0\nu} + k^3 A_{3\nu} = \omega A_{0\nu} - \omega A_{3\nu} = 0$ or $A_{0\nu} = A_{3\nu}$;
4. Transverse condition, $A_{0\mu} = 0 \Rightarrow A_{3\mu} = 0$;
5. Traceless condition, $-A_{00} + A_{11} + A_{22} + A_{33} = 0$;

From (3.-4.) one gets the first and last row/col of the amplitude matrix components are zero, and (note $A_{\mu\nu}$ is symmetric)

$$A_{\mu\nu} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & A_{11} & A_{12} & 0 \\ 0 & A_{12} & A_{22} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (3.10)$$

The trace condition (5.) then gives $A_{22} = -A_{11}$. Setting $A_+ := A_{11}$ and $A_\times := A_{12}$, the plane-wave solution in TT gauge can be written

$$h_{\mu\nu}^{\text{TT}} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & A_+ & A_\times & 0 \\ 0 & A_\times & -A_+ & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \exp \left[i\omega \left(t - \frac{z}{c} \right) \right] . \quad (3.11)$$

The two *polarization of the GW* are indicated as “+” and “ $\times$ ”

$$h_+(t - z/c) = A_+ \exp[i\omega(t - z/c)] , \quad h_\times(t - z/c) = A_\times \exp[i\omega(t - z/c)] . \quad (3.12)$$

Observations.

- In the TT gauge $h_{\mu\nu} = \bar{h}_{\mu\nu}$.
- The TT gauge can be defined only in vacuum, because in case matter is present $\square \bar{h}_{\mu\nu} \neq 0$ and, while there is still the freedom to rescale the $\bar{h}_{\mu\nu}$ with an infinitesimal coordinate transformation generated by four harmonic functions, we cannot set to zero the components $\bar{h}_{\mu\nu}$ inside the source.
- The *metric* in the TT gauge reads

$$g = -dt^2 + dz^2 + (1 + h_+)dx^2 + (1 - h_+)dy^2 + 2h_\times dx dy \quad (3.13)$$

$$= -dt^2 + (\delta_{ij} + h_{ij}^{\text{TT}})dx^i dx^j , \quad (3.14)$$

where the first expression holds for a GW along the $\hat{z}$ -direction and the second is general.

- Given a solution $\bar{h}_{\mu\nu}$ of linearized EFE in vacuum in Hilbert gauge propagating in direction $\hat{n}$, it is possible to obtain the solution in TT outside the source by means of the following projection operator (below summation on repeated indexes is understood although they are not raised)

$$\bar{h}_{\mu\nu}^{\text{TT}} = \Lambda_{\mu\nu, \alpha\beta} \bar{h}_{\alpha\beta} \quad (3.15a)$$

$$\Lambda_{\mu\nu, \alpha\beta}(\hat{n}) := P_{\mu\alpha} P_{\nu\beta} - \frac{1}{2} P_{\mu\nu} P_{\alpha\beta} \quad (3.15b)$$

$$P_{\mu\nu}(\hat{n}) := \delta_{\mu\nu} - n_\mu n_\nu . \quad (3.15c)$$

The following properties also hold [exercise]

1. $P_{\mu\nu}$ is symmetric;
2. $P_{\mu\nu}$ is transverse, $n^i P_{\mu\nu} = 0$;
3. $P_{\mu\nu}$ is a projector, $P_{\mu\alpha} P_{\alpha\nu} = P_{\mu\nu}$;

4. $P_{\mu\nu}$ has trace $P_{\mu\mu} = 2$;
 5. $\Lambda_{\mu\nu,\alpha\beta}$ is a projector, $\Lambda_{\mu\nu,\alpha\beta}\Lambda_{\alpha\beta,\rho\sigma} = \Lambda_{\mu\nu,\rho\sigma}$;
 6. $\Lambda_{\mu\nu,\alpha\beta}$ is transverse in all indexes;
 7. $\Lambda_{\mu\nu,\alpha\beta}$ is traceless in $\mu\nu$ and $\alpha\beta$, $\Lambda_{\mu\mu,\alpha\beta} = 0 = \Lambda_{\mu\nu,\alpha\alpha}$;
 8. $\Lambda_{\mu\nu,\alpha\beta}$ is symmetric in $\mu\nu - \alpha\beta$, $\Lambda_{\mu\nu,\alpha\beta} = 0 = \Lambda_{\alpha\beta,\mu\nu}$.
- More in general, for any symmetric tensor $S_{\mu\nu}$ one can obtain a *symmetric, transverse and tracefree (STF)* tensor using the $\Lambda_{\mu\nu,\alpha\beta}$ projector. STF tensors are a powerful tool to develop multipolar expansions of tensor satisfying wave equations, thus generalizing the multipolar expansion of the Newtonian and electrostatic potentials and scalar wave equations (Thorne, 1980; Maggiore, 2007) (Chap. 5.)

3.2 Effect of GWs on test masses

A simple argument to understand the effect of GW on test masses is given by the following

Example 3.2.1. Distance measurement with the radar method. *Consider two masses initially at rest and at distance L_0 in absence of GW. Denote the two spacetime points (at $t = 0$) as p, q . The distance L can be measured by sending a light pulse from the first mass (p) to the second, and then back from the second to the first. A time measurement with the clock at the first mass allows one to calculate L . Let q', p' and p'' the events in which, respectively, the second mass receives the signal, the simultaneous event at the position of the first mass, and the first mass receives the signal. The length measurement with this radar procedure is*

$$L = \frac{1}{2}c(t_{p''} - t_p) . \quad (3.16a)$$

Let us now take spatial coordinate of the two masses at $t = 0$ as $x_p^i = (0, 0, 0)$ and $x_q^i = L_0 n^i$, where n^i ($\eta_{ij}n^i n^j = 1$) is the direction between the masses, so that $L_0^2 = \delta_{ij}x_q^i x_q^j$. Calculate the variation of the distance measured at a time $t_{p'} = t_{q'}$ when the second mass receives the light ray in the case a GW is present:

$$L^2 = g_{\mu\nu}(x_{q'}^\mu - x_{p'}^\mu)(x_{q'}^\nu - x_{p'}^\nu) \stackrel{t_{p'}=t_{q'}}{=} g_{ij}(x_{q'}^i - x_{p'}^i)(x_{q'}^j - x_{p'}^j) \stackrel{x_{p'}^i=0}{=} g_{ij}x_{q'}^i x_{q'}^j = (\delta_{ij} + h_{ij}^{TT})L_0^2 n^i n^j , \quad (3.16b)$$

that implies the relative variation of the length

$$\frac{\delta L}{L_0} = \frac{L}{L_0} - 1 = \sqrt{1 + h_{ij}^{TT}n^i n^j} - 1 \approx \frac{1}{2}h_{ij}^{TT}n^i n^j . \quad (3.16c)$$

The distance's relative variation due to the GW is proportional to the GW amplitude.

A more formal study the effect of GW on test masses employs the geodesic deviation equation

$$u^\mu \nabla_\mu (u^\nu \nabla_\nu s^\alpha) = R^\alpha_{\nu\rho\sigma} u^\nu u^\rho s^\sigma , \quad (3.17)$$

where u is the tangent to the particles worldline and s is the displacement vector between worldlines. Here, one considers the “relative motion” of nearby particles at the passage of the GW using tensorial equations.

In linearized GR, the Riemann tensor is proportional to $h_{\mu\nu}$ and a particle initially at rest (in the global inertial reference system of the background metric) acquires a velocity due to the perturbation

$$\frac{dx^0}{d\tau} = 1 + \mathcal{O}(h) , \quad \frac{dx^i}{d\tau} = \mathcal{O}(h) \Rightarrow u^\mu = c \frac{dx^\mu}{d\tau} \simeq (1, 0, 0, 0) + \mathcal{O}(h) . \quad (3.18)$$

At leading order, coordinate and proper time coincide. Using this velocity and coordinate time instead of proper time, the geodesic equation at lowest order and in TT gauge reduces to

$$\frac{d^2 s_\alpha}{dt^2} = R_{\alpha 00\mu} s^\mu , \quad \text{with} \quad R_{\mu 00\nu} = \frac{1}{2} \frac{\partial^2 h_{\mu\nu}^{TT}}{\partial t^2} . \quad (3.19)$$

The formula above shows that the two degrees of freedom of the GW are physical: they cannot be gauged away. Restricting to the spatial indexes and calling $s_0^i = s^i(t=0)$ the deviation vector before the GW arrives, the formula above reduces to

$$\frac{d^2 s^i}{dt^2} = \frac{1}{2} \ddot{h}_{ij}^{TT} s_0^j + \mathcal{O}(h^2) , \quad (3.20)$$

where terms $\propto \ddot{h}_{ij}^{TT} s^i(t)$ can be discarded if one works at leading order in the perturbation. Because the particle are initially at rest, $\dot{s}^i(0) = 0$, the solution for the deviation vector is simply

$$s^i(t) = (\delta_{ij} + \frac{1}{2} h_{ij}^{TT}) s_0^j . \quad (3.21)$$

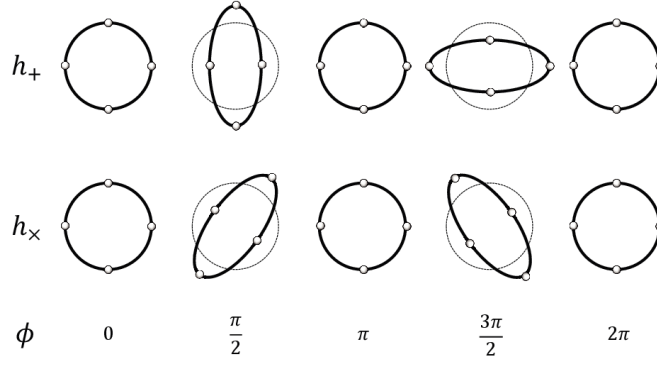


Figure 3.1: Effect on a ring of test masses on the $x - y$ plane at the passage of a GW in perpendicular direction.

Specify for a GW propagating in $\hat{z}$ -direction and setting $s_0^i = (x_0, y_0, z_0)$, one gets

$$\begin{cases} \delta x(t) &= x_0 + \frac{1}{2}(h_+(t)x_0 + h_\times(t)y_0) \\ \delta y(t) &= y_0 + \frac{1}{2}(-h_+(t)y_0 + h_\times(t)x_0) \\ \delta z(t) &= z_0 . \end{cases} \quad (3.22)$$

The test masses oscillates in the x - y plane, transverse to the direction of propagation of the GW. The effect of the two polarizations is clearly visualized by considering a ring of test masses places in the plane, Fig. (3.1). If initially $(x_0, y_0) = r_0(\cos \phi, \sin \phi)$, the effect of the *plus* polarization is

$$\begin{cases} \delta x(t) &= x_0 + \frac{1}{2}h_+(t)x_0 = r_0(1 + h_+(t)) \cos \phi \\ \delta y(t) &= y_0 - \frac{1}{2}h_+(t)y_0 = r_0(1 - h_+(t)) \sin \phi . \end{cases} \quad (3.23)$$

Take the square of the equations and sum them up:

$$1 = \cos^2 \phi + \sin^2 \phi = \frac{\delta x(t)^2}{r_0^2(1 + h_+(t))^2} + \frac{\delta y(t)^2}{r_0^2(1 - h_+(t))^2} , \quad (3.24)$$

this is an ellipsis of semi-axes $a_\pm := r_0(1 \pm h_+)$, and because $h_+(t)$ is an oscillating function with period $T = 2\pi/\omega$, one axis gets shorter and the other longer with the period T .

Similarly, for the *cross* polarization

$$\begin{cases} \delta x(t) &= x_0 + \frac{1}{2}h_\times(t)y_0 \\ \delta y(t) &= y_0 + \frac{1}{2}h_\times(t)x_0 , \end{cases} \quad (3.25)$$

one can proceed by diagonalizing the r.h.s. matrix with a rotation of $\pi/4$

$$\begin{bmatrix} 1 & h_\times/2 \\ h_\times/2 & 1 \end{bmatrix} \Rightarrow \lambda_\pm = \pm \frac{1}{2}h_\times , R = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} \cos \alpha & \sin \alpha \\ -\sin \alpha & \cos \alpha \end{bmatrix} \Big|_{\alpha=\pi/4} . \quad (3.26)$$

In the rotated frame, the equations are identical to Eq. (3.23). Thus, the cross polarization moves the ring in the same way as the plus but for a phase of $\pi/4$.

3.3 Sources of GWs

A formal solution of the linearized EFE

$$\square \bar{h}_{\mu\nu} = -16\pi T_{\mu\nu} , \quad (3.27)$$

is given in terms of the Green functions with retarded time $u := t - |\vec{x} - \vec{x}'|/c$ (See e.g. Jackson (1975))

$$\bar{h}_{\mu\nu}(t, \vec{x}) = -16\pi \int G_R(x^\mu - x^{\mu'}) T_{\mu\nu} d^4 x' = 4 \int \frac{T_{\mu\nu}(u, \vec{x}')}{|\vec{x} - \vec{x}'|} d^3 x' , \quad (3.28)$$

where in the last passage G_R is given by the retarded time solution of

$$\square_{(x)} G(x^\mu, x^{\mu'}) = \delta^{(4)}(x^\mu - x^{\mu'}) \Rightarrow G_R(x^\mu, x^{\mu'}) = -\frac{1}{4\pi} \frac{1}{|\vec{x} - \vec{x}'|} \delta(u - t') . \quad (3.29)$$

The physics picture is that the solution at point p (time t and location x^i) is determined by the events in the past lightcone. Note that since these are weak field equations, they apply to a source with negligible selfgravity $\sigma = 2GM/(c^2 R) \ll 1$, where M, R are the typical mass and size of the source.

3.3.1 Quadrupole formula

Let us specify the above formula under the conditions

- (i) Large distance from a compact source $r = |\vec{x}| = \sqrt{\delta_{ij}x^ix^j} \gg R$;
- (ii) Slow velocity, the source motion is slow $v \sim |T_{0i}|/|T_{00}| \ll c$ and $T_{00} \approx \rho c^2$.

The hypothesis (i) implies that

$$|\vec{x} - \vec{x}'| = r|\hat{n} - \frac{\vec{x}'}{r}| = r\sqrt{(\hat{n} - \frac{\vec{x}'}{r}) \cdot (\hat{n} - \frac{\vec{x}'}{r})} = r\sqrt{1 - 2\hat{n} \cdot \frac{\vec{x}'}{r} + \left(\frac{\vec{x}'}{r}\right)^2} \approx r - \hat{n} \cdot \vec{x}' , \quad (3.30)$$

and restricting to the spatial indexes (those relevant in the TT gauge) Eq. (3.28) becomes

$$\bar{h}_{ij}(t, \vec{x}) \approx \frac{4}{r} \int T_{ij}(t_r + \frac{\hat{n} \cdot \vec{x}'}{c}, \vec{x}') d^3x' , \quad (3.31)$$

where one retains only the leading-order term at the denominator $1/|\vec{x} - \vec{x}'| \sim 1/r$, and where $u \approx t_r := t - r/c$.

The hypothesis (ii) allows one to expand the stress-energy tensor in $\hat{n} \cdot \vec{x}'/c$:

$$T_{kl}(t_r + \frac{\hat{n} \cdot \vec{x}'}{c}, \vec{x}') = T_{kl}(t_r, \vec{x}') + \frac{n_i x'^i}{c} \partial_t T_{kl}(t_r, \vec{x}') + \frac{n_i x'^i n_j x'^j}{c^2} \partial_{tt} T_{kl}(t_r, \vec{x}') + \dots , \quad (3.32)$$

A simple way to justify the above expansion from the hypothesis (ii) is to look at the expression of T_{kl} in terms of its Fourier transform,

$$T_{kl}(t_r + \frac{\hat{n} \cdot \vec{x}'}{c}, \vec{x}') = \int d^4k \tilde{T}_{kl}(\omega, \vec{k}) e^{-i\omega(t_r + \frac{\hat{n} \cdot \vec{x}'}{c}) + i\vec{k} \cdot \vec{x}'} , \quad (3.33)$$

and realize that the integral of T_{kl} is dominated by the slow characteristic frequencies of the source ($v \sim \Omega R$)

$$\omega \frac{|\vec{x}'|}{c} \sim \Omega \frac{R}{c} \ll 1 . \quad (3.34)$$

Hence, the exponential in the Fourier transform can be expanded as

$$e^{-i\omega(u + \frac{\hat{n} \cdot \vec{x}'}{c})} \simeq e^{-i\omega u} \left(1 - i\frac{\omega}{c} x'^i n_i - \frac{1}{2} i^2 \frac{\omega^2}{c^2} x'^i n_i x'^j n_j + \dots \right) , \quad (3.35)$$

and the above expansion is equivalent to Eq. (3.32) (each multiplication by ω in Fourier domain corresponds to a time derivative in the direct space). The latter equation is a multipolar expansion of the of T_{kl} in Cartesian coordinates. Truncating the expansion at leading order one gets:

$$\bar{h}_{ij}(t, \vec{x}) \approx \frac{4}{r} \int T_{ij}(t_r, \vec{x}') d^3x' . \quad (3.36)$$

Focus now on the matter distribution and derive an equation for the integral of the spatial components T_{ij} by using the conservation law on flat background,

$$0 = \partial^\mu T_{\mu\alpha} = \eta^{\mu\nu} \partial_\nu T_{\alpha\mu} = \begin{cases} -\partial_t T_{00} + \partial_i T_{0i} & \alpha = 0 \\ -\partial_t T_{0k} + \partial_i T_{ki} & \alpha = k \end{cases} . \quad (3.37a)$$

Derive the $\alpha = 0$ equation in time ∂_t and substitute the $\alpha = k$ equation to obtain

$$0 = -\partial_{tt} T_{00} + \partial_t \partial_k T_{0k} = -\partial_{tt} T_{00} + \partial_k \partial_t T_{kl} . \quad (3.37b)$$

Multiply the above equation by $x^i x^j$ and integrate:

$$\frac{d^2}{dt^2} \int T_{00} x^i x^j d^3x = \int \partial_k \partial_t T_{kl} x^i x^j d^3x = \quad (3.37c)$$

$$= \int \partial_k (\partial_t T_{kl} x^i x^j) d^3x - \int \partial_t T_{kl} \partial_k (x^i x^j) d^3x = \quad (3.37d)$$

$$= \underbrace{\oint \partial_t T_{kl} y^i y^j n^k d^2y}_{=0} - \int \partial_t T_{kl} (\delta_k^i x^j + \delta_k^j x^i) d^3x = \quad (3.37e)$$

$$= - \int (\partial_t T_{il} x^j + \partial_t T_{kl} x^i) d^3x = - \int (\partial_l (T_{il} x^j) - \underbrace{T_{il} \partial_l x^j}_{=\delta_l^j} + \partial_l (T_{kl} x^i) - \underbrace{T_{kl} \partial_l x^i}_{=\delta_l^i}) d^3x = \quad (3.37f)$$

$$= - \underbrace{\oint T_{il} x^j n^l d^2y}_{=0} - \underbrace{\oint T_{kl} x^i n^l d^2y}_{=0} + 2 \int T_{ij} d^3x \quad (3.37g)$$

The surface integrals in the third and fifth lines are zero since the matter distribution is compact and there is no matter outside a sphere of radius $r > R$. The integral of the spatial components of the stress-energy tensor is thus related to the *moment of inertia* tensor of the matter distribution ρ

$$2 \int T_{ij} d^3x = \frac{1}{c^2} \frac{d^2}{dt^2} \int T_{00} x^i x^j d^3x = \frac{d^2}{dt^2} \int \rho x^i x^j d^3x =: \frac{d^2}{dt^2} I_{ij} . \quad (3.37h)$$

Putting together Eq. (3.36) and Eq. (3.37c), one obtains

$$\bar{h}_{ij}(t, \vec{x}) = \frac{2G}{c^4 r} \ddot{I}_{ij}(t - \frac{r}{c}) . \quad (3.38)$$

Far from the source, the solution can be projected to the TT gauge to obtain the *quadrupole formula*

$$\bar{h}_{ij}^{\text{TT}}(t, \vec{x}) = \frac{2G}{c^4 r} \Lambda_{ijkl} \ddot{Q}_{kl}(t - \frac{r}{c}) , \quad (3.39)$$

where the moment of inertia is substituted by its traceless version, the *quadrupole moment*:

$$Q_{ij} = I_{ij} - \frac{1}{3} \underbrace{(\delta^{kl} I_{kl})}_{=I} \delta_{ij} = \int \rho (x^i x^j - \frac{1}{3} \vec{x} \cdot \vec{x} \delta_{ij}) d^3x . \quad (3.40)$$

Note: $\Lambda_{ijkl} \ddot{I}_{ij} = \Lambda_{ijkl} \ddot{Q}_{kl}$. The quadrupole formula gives the leading order contribution to the GW from a (spatially) compact, slowly moving and nonselfgravitating source.

Using dimensional analysis, $[\ddot{Q}] = ML^2 T^{-2}$ and one immediately finds that the GW observed at distance D from the source is

$$h \sim \left(\frac{G}{c^4 D} \right) (M v^2) = \left(\frac{R}{D} \right) \left(\frac{GM}{c^2 R} \right) \left(\frac{v}{c} \right)^2 . \quad (3.41)$$

The formula above indicates that GW are produced by physical objects that are

- very compact;
- strongly gravitating;
- rapidly moving.

Note that the quadrupole formula does not apply for those objects!

Example 3.3.1. Quadrupole formula for a binary star system. Consider two masses m_1 and m_2 separated by $\vec{r} = \vec{x}_1 - \vec{x}_2$ in Newtonian gravity. The mass density is $\rho = m_1 \delta(\vec{x} - \vec{x}_1) + m_2 \delta(\vec{x} - \vec{x}_2)$, the total mass is $m = m_1 + m_2$ and the reduced mass is $\mu = m_1 m_2 / m$. The moment of inertia reads

$$I^{ij} = \int \rho x^i x^j d^3x = m_1 x_1^i x_1^j + m_2 x_2^i x_2^j = m x_{\text{cm}}^i x_{\text{cm}}^j + \mu r^i r^j \quad (3.42)$$

where $x_{\text{cm}}^i = (m_1 \vec{x}_1 + m_2 \vec{x}_2) / m$ is the center of mass coordinate. In the center of mass frame $I^{ij} = \mu r^i r^j$ and the trace is $I = \mu \delta_{ij} r^i r^j = \mu r^2$. The quadrupole of the 2-body system is thus

$$Q^{ij} = I^{ij} - \frac{1}{3} I \delta^{ij} = \mu (r^i r^j - \frac{1}{3} r^2 \delta^{ij}) . \quad (3.43)$$

Specialize now for a circular orbit in the $z = 0$ plane, for which $\vec{r} = (x, y, 0) = R(\cos(\Omega t + \pi/2), \sin(\Omega t + \pi/2), 0)$ (the $\pi/2$ phase factors are there for later convenient) and

$$\Omega = \frac{2\pi}{T} = \left(\frac{Gm}{R^3} \right)^{1/2} \quad (3.44)$$

is the orbital frequency as given by Kepler law². The inertial moment for circular orbits is immediately calculated, the nonzero components are

$$\begin{cases} I^{11} &= \mu R^2 \cos^2(\Omega t + \pi/2) = \mu R^2 \frac{1 + \cos^2(2\Omega t + \pi)}{2} = \mu R^2 \frac{1 - \cos^2(2\Omega t)}{2} \\ I^{22} &= \mu R^2 \sin^2(\Omega t + \pi/2) = \mu R^2 \frac{1 - \cos^2(2\Omega t + \pi)}{2} = \mu R^2 \frac{1 + \cos^2(2\Omega t)}{2} \\ I^{12} &= \mu R^2 \cos(\Omega t + \pi/2) \sin(\Omega t + \pi/2) = \mu R^2 \frac{1}{2} (\sin(2\Omega t + \pi) + \sin(0)) = -\mu R^2 \sin(2\Omega t) . \end{cases} \quad (3.45)$$

Taking the derivatives

$$\begin{cases} \ddot{I}^{11} &= 2\mu R^2 \Omega^2 \cos(2\Omega t) \\ \ddot{I}^{21} &= 2\mu R^2 \Omega^2 \sin(2\Omega t) \\ \ddot{I}^{22} &= -\ddot{I}^{11} . \end{cases} \quad (3.46)$$

²Kepler law can be derived from dimensional analysis of the quantities Ω, Gm, R or equating the grav.force to the centripetal force.

From the quadrupole formula one concludes that GW are emitted at frequency $\omega = 2\Omega$. Note this comes from the term $x^i x^j \sim \cos^2(\Omega t)$, hence any monochromatic source emits at 2Ω as a consequence of the quadrupole nature of the GW. Performing the TT projection (the first expressions omit the constants) one gets (Maggiore, 2007)

$$\begin{cases} h_+ &= \frac{1}{r}(\ddot{I}_{11} - \ddot{I}_{22}) = \frac{G}{rc^2} 4\mu R^2 \Omega^2 \frac{1+\cos^2\theta}{2} \cos(2\Omega t_R + \varphi) \\ h_\times &= \frac{2}{r}\ddot{I}_{12} = \frac{G}{rc^2} 4\mu R^2 \Omega^2 \cos\theta \sin(2\Omega t_R + \varphi) , \end{cases} \quad (3.47)$$

where (θ, φ) is the sky location of the source (related to the direction $\hat{n}(\theta, \varphi)$ in the STF projector).

The formula above can be re-written using again Kepler law to eliminate R for Ω . The result is

$$\begin{cases} h_+ &= \frac{4}{r} \left(\frac{G\mathcal{M}_c}{c^2} \right)^{5/3} \left(\frac{\pi\omega}{2c} \right)^{2/3} \frac{1+\cos^2\theta}{2} \cos(\omega t_R + \varphi) \sim \nu \frac{m}{r} (m\omega)^{2/3} \\ h_\times &= \frac{4}{r} \left(\frac{G\mathcal{M}_c}{c^2} \right)^{5/3} \left(\frac{\pi\omega}{2c} \right)^{2/3} \sin(\omega t_R + \varphi) \sim \nu \frac{m}{r} (m\omega)^{2/3} \end{cases} \quad (3.48)$$

Above we have introduced key quantities: the chirp mass

$$\mathcal{M}_c := \frac{(m_1 m_2)^{3/5}}{m^{1/5}} = \mu^{3/5} m^{2/5} = m \nu^{3/5} , \quad (3.49)$$

and the symmetric mass ratio $\nu := (m_1 m_2)/m^2$. The GW (in the quadrupole formula approximation) of a binary system depends exclusively on the chirp mass, not on the individual masses. Hence, this is the mass combination that is best measured in a GW observation. Moreover, the second line of Eq. (3.48) clearly indicates that the GW amplitude scales in a trivial way with the total mass of the system. We shall see this result holds also for the phase, and it holds also for binary black holes in GR. The key parameter determining the morphology of the waveform is thus the mass ratio.

The quadrupole moment is the tensor that appears in the multipolar expansion of the Newtonian potential

$$\phi(t, \vec{x}) = -\frac{GM}{r} + \frac{3GQ_{ij}(t)n^i n^j}{2r^3} + \dots \quad (3.50)$$

The dipolar term (multipole with “one index”) is just the center of mass vector that can be removed by using the center of mass frame. The quadrupole is the lowest multipole described by a tensor with 2 indexes. A similar expansion hold for the electrostatic potential, but in that case the dipole cannot be removed and represents the next-to-leading order approximation of the charge distribution. The multipolar expansion is a key tool in classical field theories and in particular for gravitational waves.

3.4 Multipolar expansion

The derivation of the quadrupole formula used the expansion Eq. (3.32), or equivalently Eq. (3.35), at leading order which is valid far from the source and for low frequencies. The expansion starts at the quadrupolar order ($\ell = 2$) because of mass and momentum conservation. Indeed, integrating T_{00} and using the conservation law for $T_{\mu\nu}$ one finds immediately $\dot{M} = 0$ and similarly integrating $T_{00}x^i$ gives momentum conservation $\dot{P}^i = 0$ (see below).

A generic source can be however decomposed in a infinite series of multipoles. Let us sketch, in three steps, how to build a systematic multipolar formulas for the linearized GWs.

1. Starting from the multipolar expansion of the stress-energy tensor in Eq. (3.32),

$$T_{kl}(u + \frac{\hat{n} \cdot \vec{x}'}{c}, \vec{x}') = T_{kl}(u, \vec{x}') + \frac{n_i x'^i}{c} \partial_t T_{kl}(u, \vec{x}') + \frac{n_i x'^i n_j x'^j}{c^2} \partial_{tt} T_{kl}(u, \vec{x}') + \dots ,$$

define the following quantities

$$S^{ij} = \int d^3 x' T^{ij} , \quad S^{ij,k} = \int d^3 x' T^{ij} x^k , \quad S^{ij,kl} = \int d^3 x' T^{ij} x^k x^l , \quad \dots \quad (3.51a)$$

such that the Eq. (3.36) (in TT gauge) generalizes to

$$\bar{h}_{ij}^{\text{TT}} \sim \Lambda_{ij,kl} \left(S^{kl} + \frac{1}{c} n_m \dot{S}^{kl,m} + \frac{1}{c^2} n_{m_1} n_{m_2} \ddot{S}^{kl,m_1 m_2} + \dots \right) . \quad (3.52)$$

2. The next step is to express T^{ij} in favour of T^{00} in the integrals, similarly to what done for the quadrupole formula. Define two sets of quantities, the *mass multipoles*

$$M = \int d^3x' T^{00}, \quad M^i = \int d^3x' T^{00} x^i, \quad M^{ij} = \int d^3x' T^{00} x^i x^j, \quad \dots, \quad (3.53a)$$

and the *current multipoles*

$$P^i = \int d^3x' T^{0i}, \quad P^{i,j} = \int d^3x' T^{0i} x^j, \quad P^{i,jk} = \int d^3x' T^{0i} x^j x^k, \quad \dots. \quad (3.54)$$

Mass and current multipoles are connected by identities that follow from the stress-energy conservation:

$$\dot{M} = 0, \quad \dot{P}^i = 0 \quad (3.55a)$$

$$\dot{M}^i = P^i, \quad \dot{P}^{i,j} = S^{ij} \quad (3.55b)$$

$$\dot{M}^{ij} = P^{i,j} + P^{j,i}, \quad \dot{P}^{i,jk} = S^{ij,k} + S^{ik,j} \quad (3.55c)$$

$$\dot{M}^{ijk} = P^{i,jk} + P^{j,ki} + P^{k,ij}, \quad \dots \quad (3.55d)$$

The first line expresses the conservation of energy and momentum. It is left as [exercise] to prove these relations.

3. The last step is to use the above identities to substitute the derivatives of the $S^{ij,*}$ quantities with the mass and current multipoles. The first three terms in the expansion are the second time derivative of the mass quadrupole (quadrupole formula), the third time derivatives of *mass octupole* and the second time derivatives of the *current quadrupole*:

$$S^{ij} = \frac{1}{2} \ddot{M}^{ij}, \quad (3.56a)$$

$$\dot{S}^{ij,k} = \underbrace{\frac{1}{6} \ddot{\ddot{M}}^{ijk}}_{\text{mass octupole term}} + \underbrace{\frac{1}{3} (\ddot{P}^{i,jk} + \ddot{P}^{j,ik} + 2\ddot{P}^{k,ij})}_{\text{current quadrupole term}}, \quad (3.56b)$$

$$\ddot{S}^{ij,kl} = \dots$$

The direct calculation of arbitrary high-order multipoles with the method above is clearly cumbersome. A general formalism to help with that is presented in Chap. 5.

4. GWs energy

Discussion on the concept of gravitational-wave energy in GR, the short-wave approximation and the Isaacson tensor.

Suggested readings. Chap. 1,3 of Maggiore (2007). Chap. 4 of Wald (1984); Chap. 4 of Carroll (1997); Chap. 7-8 of Schutz (1985);

4.1 Energy of GWs

The physical reality of GW has been investigated by Einstein already in 1916 and was finally established in the 60s with the fundamental work of Bondi, Goldberg, Newmann, Penrose, Pirani, Robison, Sachs, Trautman and many others. Note this was theoretical work aiming at clarifying GWs are not coordinate effects and can transport energy ¹ Experimental evidence for GWs came afterwards with the Taylor & Hulse pulsar observation starting 1974.

Remark 4.1.1. *The GR definition of energy of the grav. field and energy of the GW is rather complex.*

- In general, there is no local definition of energy (density) for the gravitational field since the metric describes the whole spacetime and cannot be decomposed into a background component (to be used as “reference”) and a dynamical component (actually “carrying the energy”).
- However a notion of total energy of the spacetime can be constructed for a class of spacetimes called asymptotically flat, and describing the spacetime of isolated systems. This is an advanced topic [Chap. 11 of (Wald, 1984)], but one might suspect that a total energy-momentum could be defined for isolated systems because far away from them there is a natural background spacetime that can be used, see Fig. (4.1).
- Indeed, the Hamiltonian formulation of GR by Arnowitt, Deser e Misner (ADM, 1959) gives a notion of energy for asymptotically flat spacetimes. The ADM energy-momentum can be thought as the energy of a spacelike hypersurface at a given “time” and associated to time translation and boost about spatial infinity (Spi group). An example of such energy is provided by the mass of the Schwarzschild spacetime.
- Similarly, a notion of energy-momentum of GW can be given considering asymptotic null hypersurfaces at a given “retarded time”. This energy-momentum is the one carried by the gravitational radiation and it is associated to the symmetries of null infinity in asymptotically flat spacetimes (Bondi-Metzner-Sachs (BMS) group).

The following discusses how this notions of energy can be defined in linearized gravity, where a flat background metric is present. Considering an isolated system such that asymptotically $g \sim \eta$, one expects

- The energy to be quadratic in the perturbation, so one must consider 2nd order perturbations;
- Any form of energy must generate curvature throughout a stress-energy tensor;
- An energy definition must be gauge invariant.

We thus look for an effective stress-energy tensor that is quadratic in the perturbation and leads to a gauge invariant energy definition. Focus on vacuum GR and push the Minkowski expansion to second order:

$$g = \eta + h^{(1)} + h^{(2)} + \mathcal{O}(3) , \quad (4.1a)$$

where the notation $h^{(n)}$ indicates that in the global inertial coordinate system the tensor components are

$$|h_{\mu\nu}^{(1)}| \approx \epsilon |\eta_{\mu\nu}| \approx \epsilon \ll 1 , \quad |h_{\mu\nu}^{(2)}| \approx \epsilon^2 , \quad \text{etc.} \quad (4.1b)$$

The Ricci tensor admits a similar expansion

$$R_{\mu\nu} = R_{\mu\nu}^{(0)} + R_{\mu\nu}^{(1)} + R_{\mu\nu}^{(2)} + \mathcal{O}(3) \quad \text{where} \quad (4.2a)$$

$$R_{\mu\nu}^{(0)} = (\text{Ric}[\eta])_{\mu\nu} \sim \eta \partial^2 \eta \quad (4.2b)$$

$$R_{\mu\nu}^{(1)} = \left(\text{Ric}^{(1)}[h^{(1)}] \right)_{\mu\nu} \sim \eta \partial^2 h^{(1)} \quad (4.2c)$$

$$R_{\mu\nu}^{(2)} = \left(\text{Ric}^{(1)}[h^{(2)}] \right)_{\mu\nu} + \left(\text{Ric}^{(2)}[h^{(1)}] \right)_{\mu\nu} \sim \eta \partial^2 h^{(2)} + h^{(1)} \partial^2 h^{(1)} , \quad (4.2d)$$

¹The famous conference Chapel Hill in 1957 where this topic was debated triggered the Weber experimental work and also led to a famous thought experiment https://en.wikipedia.org/wiki/Sticky_bead_argument.

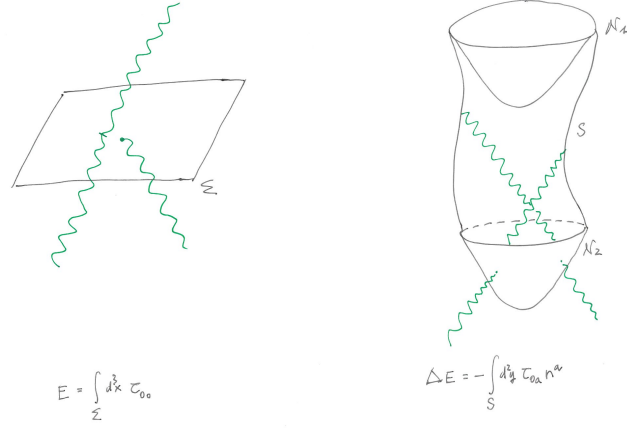


Figure 4.1: Representation of the notion of global energy of a spacelike hypersurface and of energy carried by the GW in asymptotically flat spacetime.

where $\text{Ric}[\cdot]$ indicates the full expression of the Ricci applied to its argument ², $\text{Ric}^{(1)}[\cdot]$ indicates the expression of the Ricci linearized in the metric perturbation applied to its argument, $\text{Ric}^{(2)}[\cdot]$ indicates the expression of the Ricci quadratic in the metric perturbation applied to its argument, etc. The $\mathcal{O}(2)$ term is clearly composed of a term coming from the linearized Ricci applied to the second order metric perturbation $h^{(2)}$ plus a term coming from the second order Ricci applied to the linear perturbation $h^{(1)}$. Specifically, the expression for $\text{Ric}^{(1)}$ is given by Eq. (2.10c), while

$$\left(\text{Ric}^{(2)}[h]\right)_{\mu\nu} = \frac{1}{2}h^{\rho\sigma}\partial_\mu\partial_\nu h_{\rho\sigma} - h^{\rho\sigma}\partial_\rho\partial_{(\mu}h_{\nu)\sigma} + \partial^\sigma h_\nu^\rho\partial_{[\sigma}h_{\rho]\mu} + \frac{1}{2}\partial_\sigma(h^{\rho\sigma}\partial_\rho h_{\mu\nu}) - \frac{1}{4}\partial_\rho h_{\mu\nu}\partial^\rho h - \left(\partial_\sigma h^{\rho\sigma} - \frac{1}{2}\partial^\rho h\right)\partial_{(\mu}h_{\nu)\rho}. \quad (4.3)$$

The EFE in vacuum,

$$0 = R_{\mu\nu} = R_{\mu\nu}^{(0)} + R_{\mu\nu}^{(1)} + R_{\mu\nu}^{(2)} + \mathcal{O}(3) \quad (4.4a)$$

could now be solved hierarchically order-by-order as follows

$$0 = R_{\mu\nu}^{(0)} = (\text{Ric}[\eta])_{\mu\nu} \Rightarrow \text{Trivially satisfied by } \eta \quad (4.4b)$$

$$0 = R_{\mu\nu}^{(1)} = (\text{Ric}^{(1)}[h^{(1)}])_{\mu\nu} \Rightarrow h^{(1)} \text{ (up to gauge)} \quad (4.4c)$$

$$0 = R_{\mu\nu}^{(2)} = (\text{Ric}^{(1)}[h^{(2)}] + \text{Ric}^{(2)}[h^{(1)}])_{\mu\nu} \Rightarrow h^{(2)} \text{ (up to gauge)} \quad (4.4d)$$

etc.

The last equation above can be written as “a Einstein equation” for the second-order metric by simply introducing the Einstein tensor $G_{\mu\nu}^{(2)}$ in place of the Ricci $R_{\mu\nu}^{(2)}$ and re-arranging the contributions from $h^{(2)}$ at the l.h.s and those from $h^{(1)}$ at the r.h.s. (to be treated as “known terms”):

$$G_{\mu\nu}^{(1)}[h^{(2)}] := \left(\text{Ric}^{(1)}[h^{(2)}] - \frac{1}{2}R^{(1)}[h^{(2)}]\eta\right)_{\mu\nu} = 8\pi\tau_{\mu\nu} := 8\pi(-G_{\mu\nu}^{(2)}[h^{(1)}]) = -8\pi\left(\text{Ric}^{(2)}[h^{(1)}] - \frac{1}{2}R^{(2)}[h^{(1)}]\eta\right)_{\mu\nu}, \quad (4.5)$$

where a quadratic-in- $\mathbf{h}$ stress-energy tensor constructed from the linear perturbation metric is defined. Note that $\tau_{\mu\nu}$ is

- + Symmetric;
- + Conserved on flat background $\partial_\mu\tau_{\mu\nu} = 0$ assuming that $\mathbf{h}$ satisfies the vacuum linearized EFE ;
- + Quadratic in $\mathbf{h}$, $\mathcal{O}(2)$;
- Not gauge invariant;
- Not unique, as $\tau_{\mu\nu}$ is defined up to a term $\partial^\alpha\partial^\beta U_{\mu\nu\alpha\beta}$ with $U_{\mu\nu\alpha\beta} = \mathcal{O}(2)$ such that $U_{\mu\nu\alpha\beta} = U_{\mu\alpha[\nu\beta]} = U_{[\mu\alpha]\nu\beta} = U_{\nu\beta\mu\alpha}$.

²Note that the bold notation $\mathbf{h}$ is often dropped in favour of the lighter h , where the latter notation is not ambiguous.

Fundamental result on the effective stress-energy tensor $\tau_{\mu\nu}$. For an asymptotically flat metric satisfying the conditions

$$h^{(1)} \sim \mathcal{O}(1/r) , \quad \partial h^{(1)} \sim \mathcal{O}(1/r^2) , \quad \partial\partial h^{(1)} \sim \mathcal{O}(1/r^3) \quad \text{for } r \rightarrow \infty , \quad (4.6)$$

it can be proven that the quantity

$$E := \int_{\Sigma} d^3x \tau_{00} \quad (4.7)$$

is gauge invariant under infinitesimal transformations preserving the asymptotically flat conditions,

$$E[h_{\mu\nu}^{(1)}] = E[h_{\mu\nu}^{(1)} + 2\partial_{(\mu}\xi_{\nu)}] . \quad (4.8)$$

It is also unique in the sense that it does not change if a term $\partial^\alpha\partial^\beta U_{\mu\nu\alpha\beta}$ is added to the definition of τ_{00} . Note Σ is the 3D spatial hypersurface defined by $t = \text{const}$, Fig. (4.1), and that the asymptotic flat conditions guarantee the existence of the integral. Hence, the quantity E can be taken as the total energy associated to the linearly perturbed metric.

The energy radiated in GWs can be defined as follows. Consider a situation in which the spacetime is initially time-independent, then go through a time-dependent phase, say between $t_1 < t < t_2$, and then is again time-independent. In the stationary phases $t < t_1$ and $t > t_2$ one takes two 3D surfaces that are asymptotically null and call them $\mathcal{N}_1$ and $\mathcal{N}_2$. If the radiation is measured on $\mathcal{N}_1$ but not on $\mathcal{N}_2$, then it must have escaped at null infinity. Hence, the energy radiated between the two stationary regime is defined considering the integral of the “density flux” $-\tau_{0\mu}$ over the asymptotically *timelike* surface $\mathcal{S}$ between $\mathcal{N}_1$ and $\mathcal{N}_2$ (see Fig. (4.1))

$$\Delta E := - \int_{\mathcal{S}} d^2y \tau_{a0} n^a . \quad (4.9)$$

The asymptotically flat conditions Eq. (4.6) are now required on these null hypersurfaces³. They guarantee that the integral exists and that the definition is meaningful. Note that the asymptotically flat condition **cannot** be imposed in the nonstationary phase because a wave behaves as $\sim h(t-r)/r$ and its derivative is $\partial_i h \sim \mathcal{O}(1/r)$. The integrand on the r.h.s. of the above equation gives the power (luminosity) for the GW since $\Delta E = \int \dot{E} dt$ (see below).

Remark 4.1.2. The Landau-Lifshitz pseudotensor can be defined from the “gothic metric” tensor density

$$\mathfrak{g}^{\alpha\beta} := \sqrt{-g} g^{\alpha\beta} , \quad (4.10)$$

and the related tensor density $\Lambda^{\alpha\mu\beta\nu} := \mathfrak{g}^{\alpha\beta} \mathfrak{g}^{\mu\nu} - \mathfrak{g}^{\alpha\nu} \mathfrak{g}^{\beta\mu}$. The latter has the same symmetries as the Riemann, in particular it is antisymmetric in α and μ . Taking two derivatives, one obtains the Einstein tensor multiplied by the metric determinant plus another symmetric term,

$$\partial_\mu \partial_\nu \Lambda^{\alpha\mu\beta\nu} = 2(-g)G^{\alpha\beta} + 16\pi(-g)\tau_{\text{LL}}^{\alpha\beta} = 16\pi(-g) \left(T^{\alpha\beta} + \tau_{\text{LL}}^{\alpha\beta} \right) . \quad (4.11)$$

The above equation defines the Landau-Lifshitz pseudotensor. Working in vacuum, the antisymmetry of $\Lambda^{\alpha\mu\beta\nu}$ implies the conservation law for the Landau-Lifshitz pseudotensor,

$$0 = \partial_\alpha \partial_\mu \partial_\nu \Lambda^{\alpha\mu\beta\nu} = \partial_\alpha \tau_{\text{LL}}^{\alpha\beta} . \quad (4.12)$$

By specifying these results for the weak field limit, one can find an alternative definition of the effective stress-energy tensor for the gravitational field. A stress-energy tensor for the gravitational field can be derived from a field theoretical formulation of the general relativity (as opposed to the standard geometrical picture; see also Sec. 6.4), in which gravity is treated as a nonlinear tensor field on flat spacetime, e.g. Babak and Grishchuk (2000); Maggiore (2007). Interestingly, the numerical values of the components of this tensor in the simplest gauge where the flat metric is $\text{diag}(-1, 1, 1, 1)$ match those of the Landau-Lifshitz pseudotensor, $(-g)\tau_{\text{LL}}^{\alpha\beta}$.

Towards the Isaacson stress-energy tensor. The above results suggest that a definition of stress-energy tensor might be still possible. Physically, one can imagine that, while it is always possible to reduce to flat spacetime (e.g. using normal coordinates) at a point and “eliminate” the energy density of the gravitational field, gravitational-waves do contribute to the curvature of a small extended region around that point, and this contribution should be encoded in a gauge invariant tensor. Indeed, as discussed in detail in Sec. 4.2, GWs can be defined in a broader context than weak-field gravity using an appropriate method to distinguish them from a generic background metric. The key

³For example, one can think of using the retarded time coordinate $u = t - r$ and taking $r \rightarrow \infty$.

conceptual step is an appropriate averaging procedure over several GW wavelengths, indicated with the symbol $\langle \dots \rangle$, that extracts the curvature contribution of the GW on the background. The explicit form of a gauge-invariant $\tau_{\mu\nu}$ is then calculated from Eq. (4.3) imposing the TT gauge and using this average procedure that allows one to integrate by parts and discard every divergence term. The result is (see e.g. Carroll (1997); Maggiore (2007))

$$\tau_{\mu\nu} := \frac{c^4}{32\pi G} \langle \partial_\mu h_{\alpha\beta}^{\text{TT}} \partial_\nu h^{\text{TT} \alpha\beta} \rangle, \quad (4.13)$$

and it is also detailed in Eq. (4.35a) below.

Using the above formula, it is possible to compute the GW luminosity. The calculation is similar to the one for the quadrupole formula and employs the conservation law for $\tau_{\mu\nu}$,

$$0 = \int d^3x (\partial_0 \tau^{00} + \partial_i \tau^{0i}) = -\dot{E} + \int d^3x \partial_i \tau^{0i} = -\dot{E} + \oint d^2y n_i \tau^{0i} = -\dot{E} + r^2 \oint d^2y n_r \tau^{0r} \quad (4.14a)$$

$$= -\dot{E} + r^2 \oint d^2y \langle \partial^0 h_{ij} \partial_r h^{ij} \rangle = -\dot{E} + r^2 \oint d^2y \langle \partial_t h_{ij} \partial_t h^{ij} \rangle, \quad (4.14b)$$

where in the surface integrals one takes a spherical surface of radius r and then uses the fact that for a wave $h(t-r/c)/r$ the spatial derivative can be written as $\partial_r h(t-r/c) = -\partial_t h(t-r)/c = +\partial^0 h(t-r)/c$. The GW luminosity is then given by

$$\frac{dE}{dt} = \frac{c^3}{32\pi G} r^2 \int d\Omega \langle \partial_t h_{ij}^{\text{TT}} \partial_t h^{\text{TT} ij} \rangle = \frac{c^3}{16\pi G} \langle \dot{h}_+^2 + \dot{h}_\times^2 \rangle = \frac{G}{5c^5} \langle \ddot{Q}_{ij} \ddot{Q}^{ij} \rangle. \quad (4.15)$$

The formulas above employ the quadrupole formulas; it is left as [exercise] to work out the integration of the Λ projector on the solid angle. It is sometimes useful to consider the power per unit solid angle, that is given by

$$\frac{dE}{dtd\Omega} = \frac{c^3}{32\pi G} r^2 \langle \partial_t h_{ij}^{\text{TT}} \partial_t h^{\text{TT} ij} \rangle = \frac{c^3}{8\pi G} \langle \ddot{Q}_{ij} \ddot{Q}^{ij} \rangle. \quad (4.16)$$

Remark 4.1.3. The two faces of GW luminosity. *The dimension analysis of the luminosity formula starts from*

$$[Q] = aML^2, \quad [\ddot{Q}] = aML^2T^{-3} \sim a\Omega^3 ML^2, \quad [G/c^5] = TE^{-1}, \quad (4.17)$$

where a dimensionless factor a and an angular frequency Ω are introduced for later convenience. From the above one notices that the numerically small factor G/c^5 in front of the formula is the inverse of a power. If $\ddot{Q}$ has typical values of laboratory experiments, then the GW luminosity generated in these experiments is ridiculously small. However, Weber (an optimist) suggested to re-express the formula in terms of

$$c^5/G \sim 10^{52} \text{ W} \quad (4.18)$$

which is an enormous luminosity factor. We use, as usual, R as the typical size of the sources, $\sigma = GM/c^2 R$ as a measure of the source's self-gravity, $M = c^2 R \sigma / G$ the mass of the source, $v = \Omega R$ the source's velocity. One gets

$$\dot{E} \sim \frac{G}{c^5} a^2 \Omega^6 M^2 R^4 = a^2 \frac{G}{c^5} \left(\frac{v}{c}\right)^6 \left(\frac{c}{R}\right)^6 \frac{c^4 R^2 \sigma^2}{G^2} R^4 = a^2 \frac{c^5}{G} \left(\frac{v}{c}\right)^6 \sigma^2 = a^2 \frac{c^5}{G} \left(\frac{v}{c}\right)^6 \left(\frac{GM}{c^2 R}\right). \quad (4.19)$$

The last formula above shows that a source with strong self-gravity $\sigma \sim 1$ and high-velocity $v \sim c$ can generate GWs corresponding to the most luminous radiation in the Universe.

Example 4.1.1. Binary stars system in circular orbit. The scaling formula $\dot{E} \sim \frac{G}{c^5} a^2 \Omega^6 M^2 R^4$ gives the exact formula for the power emitted by a binary in circular orbit if one takes

$$a = \frac{32}{5}, \quad \Omega : \text{Orbital frequency}, \quad M \mapsto \mu : \text{reduced mass}, \quad R : \text{Orbital radius}. \quad (4.20)$$

The derivation follows from the previously derived waveform formula for binaries and the general expression of the power; it is left as [exercise]. The total power is

$$\dot{E} = \frac{32}{5} \frac{G\mu^2}{c^5} R^4 \Omega^3 = \frac{1}{10} \frac{G\mu^2}{c^5} \Omega^6 \quad (4.21)$$

and by averaging over one orbit $\langle \dot{E} \rangle = 64/5 G\mu^2 / R(v/c)^5$. The total power per unit solid angle is given by

$$\frac{E}{dtd\Omega} = \frac{24\mu^2 R^4 \Omega^6}{\pi c^5} g(\theta), \quad g(\theta) = \left(\frac{1 + \cos^2 \theta}{2} \right)^2 + \cos^2 \theta. \quad (4.22)$$

It is interesting to note that the emission is never zero for any angle of view of the source.

Example 4.1.2. *How does the emission of radiation influence the motion of the binary? A simple argument for the inspiral motion. Using Kepler's law to remove the radius dependence, the GW power is*

$$P = \dot{E}_{\text{GW}} = \frac{32G}{5c^5} \mu^2 m^{3/4} \Omega^{10/3} \sim \Omega^{10/3}, \quad (4.23)$$

where the last line single out the frequency dependency specifying to equal masses $m_1 = m_2 = m/2$. Compare to the orbital energy

$$E = \frac{1}{2} \mu (\Omega R)^2 - \frac{m\mu}{R} = -\frac{1}{2} \frac{\mu m}{2R} \sim -\Omega^{2/3}, \quad (4.24)$$

where again one uses Kepler's law specifying to equal masses. The formula above indicates that if E becomes more negative due to the emission of GW, then the radius becomes smaller and the orbital frequency increases. In presence of radiation a circular orbit does not remain circular. The orbit is actually an accelerating inspiral during which the larger is the amount of radiation emitted, the larger the binary accelerates. The stars gets closer to each other and more relativistic as they radiate GW. This is the basic mechanism at the origin of the period decay of the Hulse&Taylor pulsar. The motion can only terminate with the collision of the stars and the merger of the two. GW150914 is a signal generated by the merger of two black holes.

Let us derive a simple equation describing the frequency evolution during the inspiral motion. Starting from $E \propto \Omega^{2/3}$, take the log and derive both sides

$$\frac{d}{dt} \ln E = \frac{d}{dt} \ln \Omega^{2/3} \Rightarrow \frac{2}{3} \frac{\dot{\Omega}}{\Omega} = \frac{\dot{E}}{E} = \frac{P}{E} \Rightarrow \dot{\Omega} = \frac{3}{2} \Omega \frac{\dot{E}}{E} = \frac{3}{2} \Omega \frac{P}{E} \sim \Omega^{11/3}. \quad (4.25)$$

Because the energy subtracted to the motion is the one radiated, we have used the energy balance $\dot{E} = P$. If we had kept all the factors in equations, we would have obtained for the GW frequency the expression [exercise]

$$-P = \dot{E} = \frac{dE}{d\Omega} \Omega \Rightarrow \frac{\dot{\omega}}{\omega^2} = \frac{96}{5} \left(\frac{1}{2} \frac{G\mathcal{M}_c}{c^3} \omega \right)^{5/3} = \frac{96}{5} \nu \left(\frac{1}{2} \frac{Gm}{c^3} \omega \right)^{5/3}. \quad (4.26)$$

The second expression is obtained using $\mathcal{M}_c = m\nu^{3/5}$. Note that in both sides of the equation the frequency can be substitute with the mass-rescaled quantities $\hat{\omega} := m\omega$ (the derivative is then understood as taken with $\hat{t} := t/M$.) As anticipated in Chap. 3, the GW frequency (thus the phase) scales trivially with the binary mass and the key quantity is thus the mass ratio. The l.h.s. of Eq. (4.26) is an important quantity called adiabaticity parameter,

$$Q_\omega^{-1} := \frac{\dot{\omega}}{\omega^2} = -3 \frac{\dot{R}}{R\Omega}. \quad (4.27)$$

If $Q_\omega^{-1} \ll 1$ ($Q_\omega \gg 1$), the radial velocity is much less than the tangential velocity, and the motion can be well approximated by an (adiabatic) sequence of circular orbits. It is trivial to manipulate Eq. (4.26) and express the chirp mass $\mathcal{M}_c$ in terms of the GW frequency $f = \omega/2\pi$ and its derivatives,

$$\mathcal{M}_c = \frac{c^3}{G} \left(\left(\frac{5}{96} \right)^3 \pi^{-8} f^{-11} \dot{f}^3 \right)^{1/5}. \quad (4.28)$$

Moreover, Eq. (4.26) can be integrated to obtain an explicit expression of the frequency evolution as a function of time [exercise]. The chirp signal observed in GW170817 is show in Fig. 4.2.

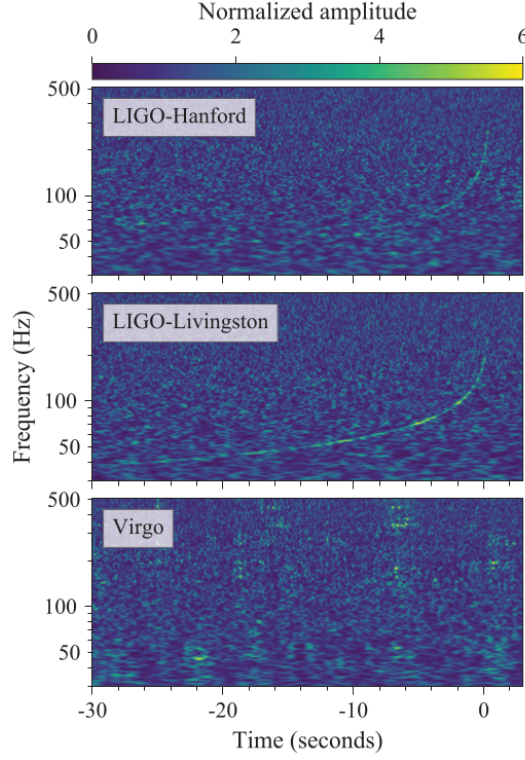


Figure 4.2: Observation of GW170817. The spectrogram in LIGO-L clearly indicates that the GW frequency and amplitude monotonically increase in time.

4.2 Short-wavelength approximation

The definition of GW energy via $\tau_{\mu\nu}$ is based on the idea that the GWs generate curvature. The latter appears in the second order perturbative expansion as an effective stress-energy tensor. Here we introduce a more general formalism to define GW on *any background* in order to better explore the concept of GW energy. As we shall see, this formalism is necessary to define second-order perturbations (see Rem. 4.2.2 below).

The above concepts of wave propagations on metric background and the definition of $\tau_{\mu\nu}$ can be generalized to an arbitrary background metric. In particular, one could be interested in defining perturbations or GWs on a background which is curved and dynamical. While in general it is not possible to split background from perturbation metric, in practise, many problems have a clear separation of scales and admit such decomposition. Consider a metric that, in some coordinate, has a typical spatial scale of variation L , then any small amplitude perturbations with reduced wavelength $\lambda \ll L$ could be clearly distinguished and separated from the background. Similarly, if the background has a temporal variation up to frequency F , then a small amplitude perturbation with frequencies $f \gg F$ could be clearly distinguished/separated.

Remark 4.2.1. GW experiments on Earth measure temporal, not length variations. *While the frequency and wavelength of a wave are related by $\lambda = c/f$, the temporal and spatial scales of variation of the background metric are **not** necessarily related. For example, consider a GW with $f \sim 10^2 - 10^3$ Hz, $\lambda \sim 500 - 50$ km and $|h| \sim 10^{-21}$ in the grav. field of Earth $\phi_\oplus \sim GM_\oplus/R_\oplus c^2 \sim 10^{-6}$. $\phi_\oplus$ is not smooth at lengthscales λ because it has variations of amplitudes $\sim 10^{-9} \gg |h|$ due to e.g. mountains; moreover the length of the laboratory apparatus to detect GW is $\ll \lambda$. It is not possible to separate the length scales of the Earth's grav. field and the GW! However, $\phi_\oplus$ is static with frequency $\ll f$. In this sense the grav. field of Earth can be separated from the GW.*

Let us address the problem of defining the propagation of the short-wavelength perturbation on the background and the generalization of the GW effective stress-energy tensor. Note there are two perturbation parameters:

- the small amplitude $\epsilon \ll 1$ (for the moment, in a vague sense...)
- the wavelength $\lambda \ll L$, or wavenumber $k = 1/\lambda \gg 1/L$, of the perturbation.

Starting from the trace reverse EFE

$$R_{\mu\nu} = 8\pi(T_{\mu\nu} - \frac{1}{2}Tg_{\mu\nu}) =: 8\pi\bar{T}_{\mu\nu} , \quad (4.29)$$

one can repeat the formal expansion in ϵ of the Ricci tensor,

$$R_{\mu\nu} = R_{\mu\nu}^{(0)} + R_{\mu\nu}^{(1)} + R_{\mu\nu}^{(2)} + \mathcal{O}(3) , \quad (4.30a)$$

where now the background metric η is to be considered a generic one. The power counting is now

$$\eta \sim \mathcal{O}(\epsilon^0), \quad h^{(n)} \sim \mathcal{O}(\epsilon^n) = \mathcal{O}(n); \quad \partial\eta \sim \mathcal{O}(1/L), \quad \partial h^{(1)} \sim \mathcal{O}(\epsilon^1 k); \quad \partial^2\eta \sim \mathcal{O}(1/L^2), \quad \partial^2 h^{(1)} \sim \mathcal{O}(\epsilon^1 k^2), \quad (4.30b)$$

and thus

$$R_{\mu\nu}^{(0)} \sim \eta \partial^2 \eta \sim \mathcal{O}(1/L^2) \quad \text{Long wavelength (low freq.)} \quad (4.30c)$$

$$R_{\mu\nu}^{(1)} \sim \eta \partial^2 h^{(1)} \sim \mathcal{O}(\epsilon k^2) \quad \text{Short wavelength (high freq.)} \quad (4.30d)$$

$$R_{\mu\nu}^{(2)} \sim \eta \partial^2 h^{(2)} + h^{(1)} \partial^2 h^{(1)} \sim \mathcal{O}(\epsilon^2 k^2) \quad \text{Long and short wavelengths,} \quad (4.30e)$$

where the second-order Ricci can contain both long and short wavelength contributions because the short wavelength combinations $\sim h_{\alpha\beta} h_{\mu\nu}$ can result in a long wavelength mode if the two wavenumbers are comparable but have opposite signs. Hence, the expanded equations can be formally separated into a low/high frequency parts:

$$R_{\mu\nu}^{(0)} = - \left[R_{\mu\nu}^{(2)} \right]^{\text{long}} + 8\pi \left[\bar{T}_{\mu\nu} \right]^{\text{long}} \quad (4.30f)$$

$$R_{\mu\nu}^{(1)} = - \left[R_{\mu\nu}^{(2)} \right]^{\text{short}} + 8\pi \left[\bar{T}_{\mu\nu} \right]^{\text{short}} \quad (4.30g)$$

Based on the results of linearized theory discussed above, one expects

- the long wavelengths equation to describe the effect of GW on the background curvature (thus leading to the definition of the effective stress-energy tensor for GW), and
- the short wave equation to describe the propagation of the perturbation on the curved background.

Let us comment on the validity of the above equations. The latter equate terms with different powers of ϵ . To be consistent, these powers must be compensated by the other expansion parameter λ/L . Considering the long wavelength equation in vacuum $\bar{T}_{\mu\nu} = 0$ or equivalently in a situation where the curvature is dominated by GW, the power counting gives

$$\frac{1}{L^2} \sim \epsilon^2 k^2 = \frac{\epsilon^2}{\lambda^2} \Rightarrow \epsilon \sim \frac{\lambda}{L}. \quad (4.31)$$

In the opposite case, where the $\bar{T}_{\mu\nu}$ dominates over $R_{\mu\nu}^{(2)}$, one must have

$$\frac{1}{L^2} \sim \epsilon^2 k^2 + (\text{matter}) \gg \epsilon^2 k^2 \Rightarrow \epsilon \ll \frac{\lambda}{L}. \quad (4.32)$$

Remark 4.2.2. Breakdown of expansion on Mikowski and of scales separation. Eq. (4.31) indicates that the metric expansion on Mikowski cannot be pushed beyond linear order. If the background metric is Minkowski, then $1/L = 0$ (strictly zero), and no GW of finite amplitude can exist. In other terms, the expansion in powers of ϵ has no domain of validity. More in general, Eq. (4.32) indicates that if the GW amplitude becomes too large, then the hypothesis of scale separation breaks and it is not possible to define wave-like perturbation on a background. [Note the smallness of the amplitude was assumed above but not really justified.]

Long wavelengths equation & Isaacson stress-energy tensor. How to actually implement the separation of scales? The idea is to integrate on lengths that are longer than the perturbation wavelength but shorter than the background variation length, $\lambda < \ell < L$. The specific average operator was introduced by Brill&Hartle and Isaacson in 1968 and for a $(0, 2)$ tensor reads

$$\langle S_{\mu\nu} \rangle := \int d^4x \eta_{\alpha}^{\mu'}(x, x') \eta_{\beta}^{\nu'}(x, x') S_{\mu\nu}(x') f(x, x') \sqrt{|\eta(x')|}, \quad (4.33)$$

where η is the generic background metric, $\eta_{\mu}^{\mu'}(x, x')$ is called the *bivector of geodesic parallel displacement* and is an operator that transports a tensor $S_{\mu\nu}(x')$ in a neighbourhood of x' . $\eta_{\mu}^{\mu'}(x, x')$ transforms as tensor in x' in the index μ' and as a tensor in x in the index μ . The function $f(x, x')$ is a weighting function that goes rapidly to zero if the two points x and x' are separated by many wavelengths λ . [See Misner et al. (1973) 35.14 and Isaacson PhD thesis appendix for more details.] Operatively, the average operator can be used in calculations with the following rules:

- Covariant derivatives inside the average commute up to $\mathcal{O}(\lambda^2/L^2)$, $\langle h \nabla^{\alpha} \nabla^{\beta} h_{\mu\nu} \rangle = \langle h \nabla^{\alpha} \nabla^{\beta} h_{\mu\nu} \rangle$;
- Gradients are zero up to $\mathcal{O}(\lambda/L)$, $\langle \nabla^{\alpha} h_{\mu\nu} \rangle = 0$;
- One can integrate by parts, $\langle h \nabla^{\alpha} \nabla^{\beta} h_{\mu\nu} \rangle = \langle -\nabla_{\beta} h \nabla^{\alpha} h_{\mu\nu} \rangle$;
- Diverge terms average to zero, $\langle \nabla_{\rho} h_{\mu\nu}^{\rho} \rangle = 0$.

The long wavelength average of the stress-energy tensor and of the second order Ricci tensor is concretely obtained applying the operator above. In particular the average of the Ricci is calculated from an expression for $R_{\mu\nu}^{(2)}$ similar to Eq. (4.3) in which several terms are in the form of (or can be cast in, integrating by parts) divergences, and thus average to zero. The result specified for a flat background and in TT gauge (far from an isolated source) is

$$\left[R_{\mu\nu}^{(2)} \right]^{\text{long}} := \langle R_{\mu\nu}^{(2)} \rangle = -\frac{1}{4} \langle \partial_{\mu} h_{ij}^{\text{TT}} \partial_{\nu} h^{\text{TT} ij} \rangle. \quad (4.34)$$

The *Isaacson tensor* is the effective stress-energy tensor for GW:

$$\tau_{\mu\nu} := \frac{c^4}{32\pi G} \langle \partial_\mu h_{ij}^{\text{TT}} \partial_\nu h^{\text{TT} ij} \rangle \quad (4.35a)$$

$$:= \frac{c^4}{32\pi G} \langle \partial_\mu h_{\alpha\beta} \partial_\nu h^{\alpha\beta} - \frac{1}{2} \partial_\mu h \partial_\nu h - 2 \partial_\alpha h^{\alpha\beta} \partial_{(\mu} h_{\nu)\beta} \rangle . \quad (4.35b)$$

where the second expression does not specify the TT gauge.

Properties:

- The Isaacson tensor is gauge invariant under infinitesimal diffeomorphisms;
- It should be used to generalize the expressions given above for the GW energy/flux.

The long wavelength (“coarse grained”) are finally

$$R_{\mu\nu}^{(0)} = 8\pi \langle \bar{T}_{\mu\nu} \rangle + 8\pi \tau_{\mu\nu} , \quad (4.36)$$

and the Bianchi identities implies local conservation of the r.h.s. with respect to the covariant derivative of the background. Far away from the source

$$\partial^\mu \tau_{\mu\nu} = 0 . \quad (4.37)$$

Short wavelength & GW propagation in curved background. Let us examine the short-wavelength equation considering first the vacuum case and then the matter dominated case.

In vacuum Eq. (4.31) implies that there exist only one scale since $\mathcal{O}(\epsilon) \sim \mathcal{O}(\lambda/L)$ and one can use only ϵ . Inspection of the orders reveals immediately that $R_{\mu\nu}^{(2)}$ can be neglected and the equation reduces to the $1/\epsilon$ part of $R_{\mu\nu}^{(1)}$:

$$\begin{cases} R_{\mu\nu}^{(1)} & \sim \eta \partial^2 h^{(1)} \sim \mathcal{O}(\epsilon k^2) = \mathcal{O}(\frac{1}{\epsilon}) \\ R_{\mu\nu}^{(2)} & \sim \eta \partial^2 h^{(2)} + h^{(1)} \partial h^{(1)} \sim \mathcal{O}(\epsilon^2 k^2) = \mathcal{O}(1) \end{cases} . \quad (4.38)$$

The $1/\epsilon$ (leading order) part is obtained by substituting the background metric with the flat metric and the background covariant derivatives with the partial derivatives in the flat metric. Hence, this equation reduces exactly to the wave equation in flat background once the Hilbert gauge is imposed

$$0 \simeq \left[R_{\mu\nu}^{(1)} \right]_{1/\epsilon} \stackrel{H.G.}{=} \square \bar{h}_{\mu\nu} . \quad (4.39)$$

If matter dominate the curvature, the two expansion parameters are not equivalent and from Eq. (4.32) one has $\epsilon \ll \lambda/L \ll 1$. Hence, one can keep only terms linear in ϵ and expand in λ/L by keeping the leading order and next-to-leading order terms. The short wavelength part of $R_{\mu\nu}^{(2)}$ is negligible w.r.t. $R_{\mu\nu}^{(1)}$ because it contains one more power of ϵ . The short wavelength part of $T_{\mu\nu}$ must contain a short wavelength term $\mathcal{O}(\epsilon)$ because the stress-energy tensor depends in general on the metric. The trace part will have a short wavelength terms $\mathcal{O}(\epsilon)$ given by the multiplication of the (high-frequency) T with the background metric and by the multiplication of the (low frequency) T with the metric perturbation. Hence,

$$\begin{cases} R_{\mu\nu}^{(1)} & \sim \eta \partial^2 h^{(1)} \sim \mathcal{O}(\epsilon k^2) \\ \left[R_{\mu\nu}^{(2)} \right]^{\text{short}} & \sim \eta \partial^2 h^{(2)} + h^{(1)} \partial h^{(1)} \sim \mathcal{O}(\epsilon^2 k^2) \\ \left[\bar{T}_{\mu\nu} \right]^{\text{short}} & = \left[T_{\mu\nu} - \frac{1}{2} (\eta_{\mu\nu} + h_{\mu\nu}) T \right]^{\text{short}} \sim \mathcal{O}(\frac{\epsilon}{L^2}) \sim \mathcal{O}(\epsilon k^2 \frac{\lambda^2}{L^2}) \end{cases} . \quad (4.40)$$

Both the curvature and matter contributions can be discarded and the equation reduces to the linear Ricci on the curved background metric. A Hilbert gauge can be introduced demanding the covariant divergence of $\bar{h}^{(1)}$ to be zero. The result is that to $\mathcal{O}(\epsilon k^2)$ the perturbation propagates following the wave equation on the curve background:

$$0 \simeq R_{\mu\nu}^{(1)} \stackrel{H.G.}{=} \square_\eta \bar{h}_{\mu\nu} . \quad (4.41)$$

5. Multipolar expansion

Basics of systematic multipolar expansions for tensor fields on flat spacetimes. Spherical harmonics and Symmetric—Trace-Free (STF) expansion for scalar fields. Tensors spherical components. STF expansion for tensors. Tensor spherical harmonics.

Suggested readings. *Sec. 3.4-3.5 of Maggiore (2007). Sec. 12.1-12.2 of Maggiore (2018). Thorne (1980), Blanchet and Damour (1989); Damour and Iyer (1991).*

5.1 Multipolar expansion for scalar fields

5.1.1 Static potential

Consider a scalar field solution of the Poisson equation

$$\Delta\phi = 4\pi\rho, \quad (5.1)$$

where ρ is a localized source (compact support in $r < R$) and thus $\phi \rightarrow 0$ at infinity. The solution to the boundary value problem can be obtained introducing the Green function (Jackson, 1975)

$$\Delta G(x-y) = 4\pi\delta(x-y) \Rightarrow G(x-y) = \frac{1}{|x-y|}, \quad (5.2)$$

that inverts the Laplacian:

$$\phi = \Delta^{-1}\rho := \int d^3x' G(x-x')\rho(x') = \int d^3x' \frac{\rho(x')}{|x-x'|}. \quad (5.3)$$

Spherical harmonics. A way to obtain the multipolar expansion of the solution is to work in spherical coordinates and consider the expansion of the Green function in spherical harmonics (Legendre polynomials, (Jackson, 1975):

$$\frac{1}{|x-x'|} = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \frac{1}{2\ell+1} \frac{r'^{\ell}}{r^{\ell+1}} Y_{\ell m}^*(\theta', \varphi') Y_{\ell m}(\theta, \varphi), \quad (5.4)$$

where $r' = |x'|$ and $r = |x|$. The exterior solution of the Poisson equation is defined as the solution of the above problem for $r > R$. Solutions of the homogeneous equation ($\rho(r > R) = 0$) are, for example, functions in the form $r^{-\ell-1}Y_{\ell m}$, as it can be directly verified

$$\Delta\left(\frac{Y_{\ell m}}{r^{\ell+1}}\right) = \left(\frac{1}{r^2}\partial_r(r^2\partial_r) + \frac{L^2}{r^2}\right)\frac{Y_{\ell m}}{r^{\ell+1}} = \frac{Y_{\ell m}}{r^2}\frac{d}{dr}(r^2\frac{d}{dr})r^{-\ell-1} - \frac{\ell(\ell+1)}{r^2}\frac{Y_{\ell m}}{r^{\ell+1}} = \quad (5.5a)$$

$$= \frac{Y_{\ell m}}{r^2}\left(\frac{d}{dr}(r^2\frac{d}{dr}r^{-\ell-1}) - \ell(\ell+1)r^{-\ell-1}\right) = \frac{Y_{\ell m}}{r^2}\left(-(\ell+1)\frac{d}{dr}(r^2r^{-\ell-2}) - \ell(\ell+1)r^{-\ell-1}\right) = \quad (5.5b)$$

$$= \frac{Y_{\ell m}}{r^2}(\ell(\ell+1)r^{-\ell-1} - \ell(\ell+1)r^{-\ell-1}) \equiv 0 \quad (5.5c)$$

In the expression above L^2 is the Laplacian on the 2-sphere, and by definition of spherical harmonics:

$$L^2Y_{\ell m} = -\ell(\ell+1)Y_{\ell m}. \quad (5.6)$$

The general exterior solution is given by a linear combination of the function above with coefficients $Q_{\ell m}$:

$$\phi^{\text{ext}} = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \frac{Q_{\ell m}}{2\ell+1} \frac{Y_{\ell m}}{r^{\ell+1}}. \quad (5.7)$$

The coefficients $Q_{\ell m}$ can be determined by matching the exterior solution to the general solution. Plugging Eq. (5.4) into Eq. (5.3)

$$\Delta^{-1}\rho = \sum_{\ell=0} \sum_{m=-\ell}^{\ell} \int d^3x' \frac{r'^{\ell}}{2\ell+1} Y_{\ell m}^*(\theta', \varphi') \rho(r', \theta', \varphi') \frac{Y_{\ell m}(\theta, \varphi)}{r^{\ell+1}} = \sum_{m=-\ell}^{\ell} \frac{Q_{\ell m}}{2\ell+1} \frac{Y_{\ell m}(\theta, \varphi)}{r^{\ell+1}} \quad (5.8a)$$

with

$$Q_{\ell m} := \int d^3x' r'^{\ell} \rho(r', \theta', \varphi') Y_{\ell m}^*(\theta', \varphi') . \quad (5.8b)$$

The coefficients are defined simply comparing to the external solution, ϕ^{ext} . This way one obtains the exterior solution as an expansion in spherical harmonics where the coefficients are determined by the spherical harmonics decomposition of the source. It should be evident that the $\ell = 0$ multipole is the mass (or charge) of the source distribution; the $\ell = 1$ terms are the dipole moment; the $\ell = 2$ are the quadrupole moment, etc.

Cartesian coordinates and STF. Consider now the Taylor expansion of the Green function $G(x - y)$ in the variable $y = x'$

$$\frac{1}{|x - y|} = f(y) = f(0) + y^i \frac{\partial f}{\partial y^i} \Big|_0 + \frac{1}{2!} y^i y^j \frac{\partial^2 f}{\partial y^i \partial y^j} + \dots \quad (5.9a)$$

$$= \frac{1}{|x|} - y^i \frac{\partial}{\partial x^i} \frac{1}{|x|} + \frac{1}{2} y^i y^j \frac{\partial}{\partial x^i} \frac{\partial}{\partial x^j} \frac{1}{|x|} + \dots = \sum_{\ell=0} \frac{(-1)^\ell}{\ell!} y^{i_1} \dots y^{i_\ell} \frac{\partial}{\partial x^{i_1}} \dots \frac{\partial}{\partial x^{i_\ell}} \frac{1}{|x|} , \quad (5.9b)$$

where in the second line we used the results:

$$\frac{\partial f}{\partial y^i} \Big|_0 = \frac{\partial}{\partial y^i} \left(\frac{1}{|x - y|} \right) \Big|_{y=0} = - \frac{\partial}{\partial x^i} \left(\frac{1}{|x - y|} \right) \Big|_{y=0} = - \frac{\partial}{\partial x^i} \frac{1}{|x|} \quad (5.10a)$$

$$\frac{\partial^\ell f}{\partial y^{i_1} \dots \partial y^{i_\ell}} \Big|_0 = (-1)^\ell \frac{\partial^\ell}{\partial x^{i_1} \dots \partial x^{i_\ell}} \frac{1}{|x|} . \quad (5.10b)$$

The terms $\partial_{i_1} \dots \partial_{i_\ell} 1/|x|$ of the expansion in Eq. (5.4) have two important properties

- Symmetric in the indexes $i_1 \dots i_\ell$
- Traceless, in the following sense

$$\delta^{ij} \frac{\partial^2}{\partial x^i \partial x^j} \frac{1}{|x|} = \Delta \frac{1}{|x|} = 0 \quad (|x| \neq 0) , \quad (5.11a)$$

$$\delta^{ij} \frac{\partial^\ell}{\partial x^i \partial x^j \partial x^{i_2} \dots \partial x^{i_\ell}} \frac{1}{|x|} = 0 \quad (\text{for any } i, j = i_1, \dots, i_\ell) . \quad (5.11b)$$

We will call such expansion, an expansion in *Symmetric—Trace-Free (STF) Tensors*.

The general solution of the Poisson equation can be written as STF expansion. Plugging the STF expansion of the Green function into the general solution Eq. 5.3, one obtains immediately

$$\Delta^{-1}\rho = \int d^3y \frac{\rho}{|x - y|} = \int d^3y \rho \sum_{\ell=0} \frac{(-1)^\ell}{\ell!} y^{i_1} \dots y^{i_\ell} \frac{\partial}{\partial x^{i_1}} \dots \frac{\partial}{\partial x^{i_\ell}} \frac{1}{|x|} = \quad (5.12a)$$

$$= \sum_{\ell=0} \frac{(-1)^\ell}{\ell!} \int d^3y \rho(y) y^{i_1} \dots y^{i_\ell} \frac{\partial}{\partial x^{i_1}} \dots \frac{\partial}{\partial x^{i_\ell}} \frac{1}{|x|} = \sum_{\ell=0} \frac{(-1)^\ell}{\ell!} Q_{i_1 \dots i_\ell} \frac{\partial}{\partial x^{i_1}} \dots \frac{\partial}{\partial x^{i_\ell}} \frac{1}{|x|} \quad (5.12b)$$

where

$$Q_{i_1 \dots i_\ell} := \int d^3y \rho(y) y^{i_1} \dots y^{i_\ell} \quad (5.13)$$

is the STF decomposition of the source. The symbol $\langle \dots \rangle$ indicate the symmetric—trace-free part. The coefficients can be directly defined as the STF part because they are contracted with a STF combination of derivatives and their antisymmetric and trace part does not contribute in the sum.

Example 5.1.1. Consider the tensor $T^{ij} := y^i y^j$. The symmetrized tensor is $T^{(ij)} = y^{(i} y^{j)} = 1/2(y^i y^j + y^j y^i)$, with trace $T = y^{(i} y^{j)} \delta_{ij} = 2y^j y_j / 2 = y^2$. Then the STF is constructed by first taking the symmetric and then subtracting $\delta_{ij} T/3$:

$$T^{(ij)} = y^{(i} y^{j)} = \frac{1}{2}(y^i y^j + y^j y^i) - \frac{1}{3} y^2 \delta^{ij} . \quad (5.14)$$

Example 5.1.2. $\square$ [TODO: Calculate the first two terms (charge and dipole) of the solution in both approaches.]

Relation between the two expansions. There is close relation between Eq. (5.4) and Eq. (5.9a) and between Eq. (5.8a) and Eq. (5.12a). To explore this let $x^i = rn^i$ with $r = |x|$ and use

$$\partial_i r = \partial_i (x_j x^j)^{1/2} = \frac{x_i}{r} = n_i, \quad \partial_i n_j = \partial_j \left(\frac{x_j}{r} \right) = \frac{1}{r} \delta_{ij} - \frac{1}{r^2} x_j \partial_i r = \frac{1}{r} (\delta_{ij} - n_i n_j) \quad (5.15)$$

to calculate the derivatives in Eq. (5.9a):

$$\partial_i \frac{1}{r} = -\frac{n_i}{r^2} \quad (5.16a)$$

$$\partial_i \partial_j \frac{1}{r} = -\frac{3}{r^3} n^{(i} n^{j)} \quad (5.16b)$$

$$\partial_i \partial_j \partial_k \frac{1}{r} = -\frac{3 \cdot 5}{r^4} n^{(i} n^j n^{k)} \quad (5.16c)$$

$$\partial_{i_1} \dots \partial_{i_\ell} \frac{1}{r} = \frac{(-1)^\ell (2\ell - 1)!!}{r^{\ell+1}} n^{(i_1} \dots n^{i_\ell)} . \quad (5.16d)$$

Hence, the STF expansion can be expressed in terms of the STF combination of unit vectors (let $y^i = r' n'^i$)

$$\frac{1}{|x - y|} = \sum_{\ell=0} \frac{(2\ell - 1)!!}{\ell!} \frac{r'^\ell}{r^{\ell+1}} n'^{i_1} \dots n'^{i_\ell} n^{(i_1} \dots n^{i_\ell)} \quad (5.17a)$$

$$\Delta^{-1} \rho = \sum_{\ell=0} \frac{(2\ell - 1)!!}{\ell!} r^{-\ell-1} Q_{i_1 \dots i_\ell} n^{(i_1} \dots n^{i_\ell)} . \quad (5.17b)$$

The first of the above equation should be directly compared to Eq. (5.4). Observe now that a function of the angles (θ, φ) can be considered a function of the unit vector n^i that points in the direction identified by the angles,

$$n^i = (\sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta) . \quad (5.18)$$

The relation between Eq. (5.4) and Eq. (5.9a) is now clarified by the following

Theorem 5.1.1. *The functions $F_\ell(\hat{n}) := Q_{i_1 \dots i_\ell} n^{(i_1} \dots n^{i_\ell)}$ are eigenfunctions of the L^2 operator (Laplacian on the 2-sphere) with eigenvalues $-\ell(\ell + 1)$.*

Proof.

(i) Take $\tilde{F}_\ell = r^\ell F_\ell$ and show that $L^2 \tilde{F}_\ell \equiv 0 \ \forall \ \ell$. This is immediate for $\ell = 0$ (F_0 is a constant) and for $\ell = 1$ since $F_1(x^i)$ is a linear function and its second derivative is zero. The first nontrivial term is $\ell = 2$ that vanishes because the coefficient are tracefree:

$$\Delta \tilde{F}_2 = \Delta(Q_{ij} r^2 n^i n^j) = \Delta(Q_{ij} x^i x^j) = Q_{ij} \partial_k \partial_k (x^i x^j) = Q_{ij} \partial_k (\delta_k^i x^j + \delta_k^j x^i) = 2Q_{ij} \delta_k^i \delta_k^j = 2Q_{ij} \delta^{ij} = 0 . \quad (5.19)$$

Similarly, the STF property of the expansion makes all the $\ell > 2$ terms to vanish [exercise].

(ii) Using the result above:

$$0 = \Delta \tilde{F}_\ell = \left(\frac{1}{r^2} \partial_r (r^2 \partial_r) + \frac{L^2}{r^2} \right) \tilde{F}_\ell = \frac{F_\ell}{r^2} \frac{d}{dr} (r^2 \frac{d}{dr} r^\ell) + \frac{r^\ell}{r^2} L^2 F_\ell = r^{\ell-2} (\ell(\ell + 1) F_\ell + L^2 F_\ell) . \quad (5.20)$$

Example 5.1.3. An axisymmetric system is invariant under rotation about an axis, say $\hat{n} = \hat{z}$. Axisymmetric functions $F(\theta, \varphi) = F(\theta)$ are usually expanded in Legendre polynomials $P_\ell(\cos \theta)$, but they can be expanded also in STF

$$F(\theta) = c_0 + c_1 z^{(i} n_i + c_2 z^{(i} z^{j)} n_i n_j + \dots + c_\ell z^{(i_1} \dots z^{i_\ell)} n_{i_1} \dots n_{i_\ell} + \dots , \quad (5.21)$$

as a particular case of generic functions of (θ, φ) . Considering $z_i n^i = \cos \theta$, it is immediate to see the two expansions are equivalent:

$$1 = 1 \quad (5.22a)$$

$$z^{(i} n_i = z^i n_i = \cos \theta \quad (5.22b)$$

$$z^{(i} z^{j)} n_i n_j = (z_i z_j - \frac{1}{3} \delta_{ij}) n^i n^j = \cos^2 \theta - \frac{1}{3} \quad (5.22c)$$

...

The theorem implies something very general. Given the spherical harmonic $Y_{\ell m}(\theta, \varphi)$, there exists a SFT tensor $\mathcal{Y}_{i_1 \dots i_\ell}^{\ell m}$ independent on (θ, φ) such that

$$Y_{\ell m}(\theta, \varphi) = \mathcal{Y}_{i_1 \dots i_\ell}^{\ell m} n^{i_1} \dots n^{i_\ell} . \quad (5.23)$$

It is possible to show that

Theorem 5.1.2. *The $2\ell + 1$ STF tensors $\mathcal{Y}_{i_1 \dots i_\ell}^{\ell m}$ form an orthonormal basis for the rank- ℓ STF tensors. In other terms, a generic STF $T_{i_1 \dots i_\ell}$ can be expanded as*

$$T_{i_1 \dots i_\ell} = \sum_{m=-\ell}^{\ell} T_{\ell m} \mathcal{Y}_{i_1 \dots i_\ell}^{\ell m} \quad (5.24)$$

where the $T_{\ell m}$ are called the spherical components of T_{ij} .

A direct calculation for $\ell = 2$ should convince of the general validity of this theorem and it is left as [exercise].

Properties of spherical components. Tensor spherical components play a role not in GWs physics but also in quantum mechanics. Few important properties are discussed in the following.

(i) Multiplying Eq. (5.24) by $n_{i_1} \dots n_{i_\ell}$ and using the definition of $\mathcal{Y}_{i_1 \dots i_\ell}^{\ell m}$ one obtains the identity

$$T_{i_1 \dots i_\ell} n^{i_1} \dots n^{i_\ell} = \sum_{m=-\ell}^{\ell} T_{\ell m} Y^{\ell m}. \quad (5.25)$$

This express the fact that in the STF expansion only the spherical harmonics with index ℓ contribute to the term with indexes $i_1 \dots i_\ell$.

(ii) The spherical components can be obtained using the orhogonality of $Y_{\ell m}$:

$$T_{\ell m} = 4\pi \frac{\ell!}{(2\ell + 1)!!} T_{i_1 \dots i_\ell} (\mathcal{Y}_{i_1 \dots i_\ell}^{\ell m})^*. \quad (5.26)$$

The general calculation is a as follow. First, multiply the identity Eq. (5.25) by the complex conjugate of the spherical harmonics and integrate; then

$$T_{\ell m} = T_{i_1 \dots i_\ell} \int d\Omega Y_{\ell m}^* n_{i_1} \dots n_{i_\ell} = T_{i_1 \dots i_\ell} \mathcal{Y}_{j_1 \dots j_\ell}^{\ell m} \int d\Omega n_{j_1} \dots n_{j_\ell} n_{i_1} \dots n_{i_\ell} \quad (5.27a)$$

$$= 4\pi \frac{\ell!}{(2\ell + 1)!!} T_{i_1 \dots i_\ell} (\mathcal{Y}_{i_1 \dots i_\ell}^{\ell m})^*. \quad (5.27b)$$

The last line uses the following identity:

$$\frac{1}{4\pi} \int d\Omega n_{i_1} \dots n_{i_\ell} = \frac{1}{(2\ell + 1)!!} (\delta_{i_1 i_2} \delta_{i_3 i_4} \dots \delta_{i_{2\ell-1} i_{2\ell}} + \dots \text{all possible pairing of indexes}), \quad (5.27c)$$

and the fact that, since $\mathcal{Y}_{i_1 \dots i_\ell}^{\ell m}$ is symmetric in $j_1 \dots j_\ell$, all the $\ell!$ permutations give the same result. To show this, do first the full calculation for $\ell = 2$. For any vector v^i :

$$I_2 := \int_0^{2\pi} d\varphi \int_0^\pi d\theta \sin \theta v_i n^i v^j n_j = \int d\varphi \int d\theta \sin \theta (v_1 \sin \theta \cos \varphi + v_2 \sin \theta \sin \varphi + v_3 \cos \theta)^2 = \quad (5.27d)$$

$$= \int d\varphi \int d\theta \sin \theta (v_1^2 \sin^2 \theta \cos^2 \varphi + v_2^2 \sin^2 \theta \sin^2 \varphi + v_3^2 \cos^2 \theta) \quad (5.27e)$$

$$+ 2v_1 v_2 \sin^2 \theta \underbrace{\cos \varphi \sin \varphi}_{\int d\varphi \dots = 0} + 2v_1 v_3 \sin \theta \cos \theta \underbrace{\cos \varphi}_{\int d\varphi \dots = 0} + 2v_2 v_3 \sin \theta \cos \theta \underbrace{\sin \varphi}_{\int d\varphi \dots = 0} =$$

$$= v_1^2 \underbrace{\int_0^{2\pi} \cos^2 \varphi d\varphi}_{=\pi} \underbrace{\int_0^\pi \sin^3 \theta d\theta}_{=4/3} + v_2^2 \underbrace{\int_0^{2\pi} \sin^2 \varphi d\varphi}_{=\pi} \underbrace{\int_0^\pi \sin^3 \theta d\theta}_{=4/3} + v_3^2 2\pi \underbrace{\int_0^\pi \sin \theta \cos^2 \theta d\theta}_{=2/3} \quad (5.27f)$$

$$= \frac{4\pi}{3} (v_1^2 + v_2^2 + v_3^2) = \frac{4\pi}{3} v_i v_j \delta^{ij} \quad (5.27g)$$

Then observe that all the odd integrals I_{2k+1} are zero because under a parity transformation $\theta \mapsto \theta - \pi$ the integral is equal to minus itself $I_{2k+1} \mapsto -I_{2k+1}$: $\int_0^\pi d\theta \sin \theta = \int_{-1}^1 du \mapsto \int_1^{-1} du$. Finally, all the even integrals I_{2k} are equal to the symmetric combination of the product of the δ 's times a coefficient C_ℓ that depends on ℓ . The latter can be found by contracting all the indexes of the integrand. For example, for $\ell = 2$

$$\int d\Omega n^i n_i = C_2 \delta_i^i = 4\pi \frac{1}{3}. \quad (5.27h)$$

Patience calculation gives the result in Eq. (5.27c).

(iii) The properties of the tensor components under rotation are determined by the properties of the spherical harmonics. Since $T_{i_1 \dots i_\ell} n^{i_1} \dots n^{i_\ell}$ is a scalar, it does not change under rotations, and if one performs a rotation around the z -axis of an angle $\delta\varphi$ (azimuthal angle) the identity in Eq. (5.25) implies that

$$Y_{\ell m} \mapsto e^{im\delta\varphi} Y_{\ell m} \Rightarrow T_{\ell m} \mapsto e^{-im\delta\varphi} T_{\ell m}. \quad (5.28)$$

In general, $T_{\ell m}$ transforms like $Y_{\ell m}^*$.

Summary 5.1.1. A rank- ℓ STF tensor $T^{(i_1 \dots i_\ell)}$ has $2\ell + 1$ independent components and it is an irreducible representation of dimension $2\ell + 1$ of the rotation group $SO(3)$. The spherical harmonics $Y_{\ell m}$ with $-\ell \leq m \leq \ell$ are a particular equivalent representation.

Remark 5.1.1. There is a useful compact notation for these multi-index sum: the indexes $i_1 \dots i_\ell$ are often indicated with a capital letter L :

$$\sum_{\ell=0} \frac{(-1)^\ell}{\ell!} Q_L \partial_L F(x) := \sum_{\ell=0} \frac{(-1)^\ell}{\ell!} Q_{i_1 \dots i_\ell} \frac{\partial}{\partial x^{i_1}} \dots \frac{\partial}{\partial x^{i_\ell}} F(x) . \quad (5.29)$$

5.1.2 Extension to wave equations & tensors

In order to be applied to gravitational waves, the above method must be generalized to tensorial fields that satisfy wave equations. Schematically we are interested in the extension

$$\Delta \longrightarrow \square , \quad \phi(x) \longrightarrow h_{\mu\nu}(t, x) .$$

One possibility is to perform a Fourier transform in time such that $\square \mapsto \Delta + k^2$ and to generalize the method for static potentials utilizing *tensor spherical harmonics* (Jackson, 1975). Tensor spherical harmonics are eigenfunctions of L^2 and can be used to separate the wave equation for tensor fields, thus generalizing the ansatz for the scalar case:

$$\square \phi = 0 ; \quad \phi = \sum_{\ell=0} \sum_{m=-\ell}^{\ell} \frac{c_{\ell m}(t - r/c)}{r} Y_{\ell m}(\theta, \varphi) . \quad (5.30)$$

They are useful in various contexts including perturbations of spherically symmetric spacetimes and will be introduced in Sec. 5.3. However, the Fourier approach does not allow one to explicitly link the multipoles to the time evolution of the source. The STF approach extended to tensor fields is preferable to that purpose, and it is heavily employed within the post-Newtonian method.

5.2 STF expansion

Scalars. Consider a scalar field solution of the wave equation

$$\square \phi = -4\pi \rho , \quad (5.31)$$

where ρ is a localized source (compact support in $r < R$) and thus $\phi \rightarrow 0$ at infinity. The solution to the boundary value problem is given in terms of the retarded Green function (Jackson, 1975)

$$\phi = \square^{-1} \rho = \int d^3 x' \frac{\rho(t - |x - x'|/c, x')}{|x - x'|} . \quad (5.32)$$

The exterior solution for $r > R$ (homogeneous equation) can be obtained by observing that functions of the retarded time $u = t - r/c$ of type $F(u)/r$ are particular solutions

$$\square (F(t - r/c)/r) = (-\partial_{tt} + c^2 \Delta) (F(t - r/c)/r) = \left(-\partial_{tt} + \frac{c^2}{r^2} \partial_r (r^2 \partial_r) + \frac{L^2}{r^2} \right) (F(t - r/c)/r) = \quad (5.33a)$$

$$= (-\partial_{tt} + \frac{c^2}{r^2} \partial_r (r^2 \partial_r)) (F(t - r/c)/r) = -\frac{1}{r} f''(u) + \frac{c^2}{r^2} \partial_r \left(r^2 \frac{-1}{c} \frac{f'(u)}{r} - r^2 \frac{f(u)}{r^2} \right) \quad (5.33b)$$

$$= -\frac{f''(u)}{r} + \frac{c}{r^2} f'(u) - \frac{c}{r^2} \left(f'(u) - \frac{r}{c} f'' \right) \equiv 0 . \quad (5.33c)$$

Thus, the general solution is a linear combination of functions $F(u)/r$, and it can be expressed by the STF expansion ($L = i_1 \dots i_\ell$)

$$\phi^{\text{ext}} = \sum_{\ell=0} \frac{(-1)^\ell}{\ell!} \partial_L \frac{F_L(u)}{r} . \quad (5.34)$$

The explicit expression for the F_L has been found by Blanchet and Damour (1989) by matching the exterior solution to the general solution. It reads

$$F_L(u) = \int d^3 y y_L \int_{-1}^1 dz \delta_\ell(z) \rho(u + \frac{z|y|}{c}, y) , \quad (5.35)$$

where

$$\delta_\ell(z) := \frac{(2\ell + 1)!!}{2^{\ell+1} \ell!} (1 - z^2)^\ell , \quad (5.36)$$

such that the integral $\int_{-1}^1 dz \delta_\ell(z) = 1$ and $\delta_\ell(z) \rightarrow \delta(z)$ for $\ell \rightarrow \infty$. Analogous solutions exist for vectors and tensors, and they are summarized in the following without proof.

Vectors. The general exterior solution to

$$\square A_\mu = -\frac{4\pi}{c} J^\mu , \quad (5.37)$$

is given by

$$A_0(u) = \sum_{\ell=0} \frac{(-1)^\ell}{\ell!} \partial_L \frac{F_L(u)}{r} \quad (5.38a)$$

$$A_i(u) = \sum_{\ell=0} \frac{(-1)^\ell}{\ell!} \partial_L \frac{G_{Li}(u)}{r} , \quad (5.38b)$$

where the $F_L(u)$ is the same as Eq. (5.35), i.e. the 0-component behaves like a scalar potential, and

$$G_{Li}(u) := \int d^3y y_L \int_{-1}^1 dz \delta_\ell(z) J_i(u + \frac{z|y|}{c}, y) . \quad (5.39)$$

The tensor G_{Li} is STF with respect to the multi-index $L = i_1 \dots i_\ell$ but not in the index i . Hence, it is usually re-expressed as a combination of STF tensors with $L+1$ indexes (an irreducible representation) as

$$G_{Li} = U_{iL} + \frac{\ell}{\ell+1} \epsilon_{ki\langle i_\ell} C_{L-1\rangle} + \frac{2\ell-1}{2\ell+1} \delta_{i\langle i_\ell} D_{L-1\rangle} \quad (5.40)$$

with

$$U_{L+1} := G_{\langle L+1 \rangle} , \quad C_L := G_{kl\langle L-1 \epsilon_{i_\ell} \rangle kl} , \quad D_{L-1} := G_{kkL-1} . \quad (5.41)$$

Intuition for this formulas for the case $\ell=1$ can come by proving that a generic rank-2 tensor can be decomposed into a scalar, a vector rank-2 STF tensor, and by explicitly constructing the decomposition [exercise].

Damour and Iyer (1991) showed that a gauge transformation that preserves the Lorentz gauge,

$$A_\mu \rightarrow A_\mu + \partial_\mu \theta : \quad \square \theta = 0 , \quad (5.42)$$

allows one to set to zero the D_{L-1} term, but changes the expansion of the 0-component. The final result is

$$A_0 = \sum_{\ell=0} \frac{(-1)^\ell}{\ell!} \partial_L \frac{Q_L(u)}{r} \quad (5.43a)$$

$$A_i = -\frac{1}{c} \sum_{\ell=1} \frac{(-1)^\ell}{\ell!} \partial_{L-1} \left[\frac{Q_{iL-1}^{(1)}(u)}{r} + \frac{\ell}{\ell+1} \epsilon_{ikl} \partial_k \left(\frac{M_{iL-1}(u)}{r} \right) \right] , \quad (5.43b)$$

where one defines the *electric* and *magnetic moment* as

$$Q_L(u) := \int d^3y y_L \int_{-1}^1 dz \left[\delta_\ell(z) y_L \rho(u + \frac{z|y|}{c}, y) - \frac{1}{c^2} \frac{2\ell+1}{(\ell+1)(2\ell+3)} \delta_{\ell+1}(z) y_{iL} J^{(1)}(u + \frac{z|y|}{c}, y) \right] \quad (5.44a)$$

$$M_L(u) := \int d^3y y_L \int_{-1}^1 dz \left[\delta_\ell(z) y_{\langle L-1 \epsilon_{i_\ell} \rangle jk} y_j J_k(u + \frac{z|y|}{c}, y) \right] \quad (5.44b)$$

and superscripts indicate derivatives w.r.t. to retarded time,

$$Q_{iL-1}^{(1)} := \frac{d}{du} Q_{iL} , \quad J^{(1)} := \frac{d}{du} J_i . \quad (5.44c)$$

The electromagnetic field outside the source can be thus decomposed in STF time-dependent electric and magnetic multipoles.

Tensors. A similar results applies to tensors whose components satisfy the wave equation

$$\square \bar{h}_{\mu\nu} = \frac{16\pi G}{c^4} T_{\mu\nu} . \quad (5.45)$$

One starts from the expansions

$$\bar{h}_{00} \sim \sum_\ell \partial_L \frac{F_L}{r} , \quad \bar{h}_{0i} \sim \sum_\ell \partial_L \frac{G_{Li}}{r} , \quad \bar{h}_{ij} \sim \sum_\ell \partial_L \frac{H_{Lij}}{r} , \quad (5.46)$$

and expresses G_{Li} and H_{Lij} by means of full STF tensors. After a transformation that preserves the Hilbert (Lorentz) gauge one gets (Damour and Iyer, 1991)

$$\bar{h}_{00} = \frac{4G}{c^2} \sum_{\ell=0} \frac{(-1)^\ell}{\ell!} \partial_L \frac{M_L(u)}{r} \quad (5.47a)$$

$$\bar{h}_{0i} = \frac{4G}{c^2} \sum_{\ell=1} \frac{(-1)^\ell}{\ell!} \partial_{L-1} \left[\frac{M_{iL-1}^{(1)}(u)}{r} + \frac{\ell}{\ell+1} \epsilon_{ikl} \partial_k \left(\frac{S_{iL-1}(u)}{r} \right) \right] \quad (5.47b)$$

$$\bar{h}_{ij} = \frac{4G}{c^2} \sum_{\ell=2} \frac{(-1)^\ell}{\ell!} \partial_{L-2} \left[\frac{M_{ijL-2}^{(2)}(u)}{r} + \frac{2\ell}{\ell+1} \epsilon_{ikl} \partial_k \left(\frac{\epsilon_{kl(i} S_{j)L-1}^{(1)}(u)}{r} \right) \right] . \quad (5.47c)$$

The gravitational fields far away from the source is expressed as a STF expansion in two sets of multipolar moments $M_L(u)$ and $S_L(u)$ (and their derivatives) that are related to the *active mass density*, *active current density* and *active stress tensors*

$$\sigma := \frac{1}{c^2} (T^{00} + T^{ii}) , \quad \sigma_i := \frac{1}{c} T^{0i} , \quad \sigma_{ij} := T^{ij} , \quad (5.48)$$

by the integrals (Damour and Iyer, 1991; Maggiore, 2007)

$$M_L(u) := \int d^3y \int_{-1}^1 dz \left[\delta_\ell(z) y_L \sigma - \frac{4(2\ell+1)}{c^2(\ell+1)(2\ell+3)} \delta_{\ell+1}(z) y_{iL} \sigma_i^{(1)} + \frac{2(2\ell+1)}{c^4(\ell+1)(\ell+2)(2\ell+5)} \delta_{\ell+2}(z) y_{ijL} \sigma_{ij}^{(2)} \right] \quad (5.49a)$$

$$S_L(u) := \int d^3y \int_{-1}^1 dz \left[\delta_\ell(z) \epsilon_{\langle k l i_\ell} y_{L-1 \rangle k} \sigma_l - \frac{(2\ell+1)}{c^2(\ell+1)(2\ell+3)} \delta_{\ell+1}(z) \epsilon_{\langle k l i_\ell} y_{L-1 \rangle k m} \sigma_{lm}^{(1)} \right] . \quad (5.49b)$$

Example 5.2.1. $\square$ [TODO: Derive the quadrupole formula from the general solution.]

Remark 5.2.1. The result above holds for the linear GR theory. But the same structure carries over on general background and the formulas remain the same with an appropriate substitution of $T_{\mu\nu}$ with the effective stress-energy tensor $\tau_{\mu\nu}$ that include the nonlinearities of the gravitational field.

5.3 Tensor spherical harmonics

To introduce tensor spherical harmonics it is useful to take a “quantum mechanical” approach. Given a field of spin s , its total angular momentum is the sum of the orbital angular momentum and of the spin, $J^i = L^i + S^i$. The operators J^2, J_z, L^2 and S^2 commute and can be diagonalized simultaneously.

Definition 5.3.1. Tensor spherical harmonics $\mathbf{Y}$ of a field of spin s can be defined as the simultaneous eigenfunctions of the operators J^2, J_z, L^2 and S^2 :

$$J^2 \mathbf{Y} = j(j+1) \mathbf{Y} , \quad J_z \mathbf{Y} = j_z \mathbf{Y} , \quad L^2 \mathbf{Y} = l(l+1) \mathbf{Y} , \quad S^2 \mathbf{Y} = s(s+1) \mathbf{Y} . \quad (5.50)$$

The state of the system belongs to a space that is the tensor product of the space of dimension $(2l_z + 1)$ spanned by the states $|l_z\rangle$ with the space of dimension $(2s_z + 1)$ spanned by the states $|ss_z\rangle$:

$$|ll_z ss_z\rangle = |l_z\rangle \otimes |ss_z\rangle . \quad (5.51)$$

The total state is thus expanded as

$$|jj_z\rangle = \sum_{l_z=-l}^l \sum_{s_z=-s}^s |ll_z ss_z\rangle \langle ll_z ss_z | jj_z\rangle , \quad (5.52)$$

where $\langle ll_z ss_z | jj_z\rangle$ are the Clebsh-Gordan coefficients. The total angular momentum eigenvalues can take the values

$$|l-s| \leq j \leq l+s . \quad (5.53)$$

The tensor spherical harmonics can be explicitly constructed from the eigenfunctions of the orbital angular momentum, the (scalar) spherical harmonics and of the spin operator,

$$L^2 Y_{ll_z} = l(l+1) Y_{ll_z} , \quad S^2 \mathbf{X}_{ss_z} = s(s+1) \mathbf{X}_{ss_z} , \quad (5.54)$$

as

$$\mathbf{Y} = \sum_{l_z=-l}^l \sum_{s_z=-s}^s \langle ll_z ss_z | jj_z \rangle Y_{ll_z} \mathbf{X}_{ss_z} . \quad (5.55)$$

The tensor spherical harmonics carry 4 labels $\mathbf{Y}_{jj_z}^{ls}$ (omitted above) and describe the angular distribution and the polarization of the field of spin s . Since L^2 is part of the Laplacian, the tensor spherical harmonics can be used to separate the wave equation for tensor fields, thus generalizing the scalar case. Note the spin eigenfunctions $\mathbf{X}_{ss_z}$ are tensor with indexes that depend on the spin; these tensor indexes are understood in the bold notation. For $s = 1/2$ they carry a spinor index; for $s = 1$ a vector index, and for $s = 2$ a pair of indexes.

Vectors. For $s = 1$, $\mathbf{X}_{1s_z}$ with $s_z = 0, \pm 1$ can be constructed from the Cartesian unit vectors

$$\mathbf{X}_{1\pm 1} = \mp \frac{1}{\sqrt{2}}(\hat{x} \pm i\hat{y}) , \quad \mathbf{X}_{10} = \hat{z} . \quad (5.56)$$

The tensor spherical harmonics are thus vectors for $s = 1$. They can be used to solve the wave equation for the electromagnetic potential.

These $\mathbf{Y}_{jj_z}^{ls=1}$ vector spherical harmonics have no special properties with respect to the vector n^i , in particular they are neither purely transverse nor longitudinal. For this reason, the vector spherical harmonics expansion of the electromagnetic potential is usually expressed in terms of combinations of $\mathbf{Y}_{jj_z}^{l1}$ that highlights the transverse/longitudinal character of the terms, i.e. the relation to the unit vector n^i along the direction of propagation of the wave. To write such an expansion, one defines the *pure-spin vector spherical harmonics*

$$\mathbf{Y}_{jj_z}^R := \sqrt{2j+1} \left[j^{1/2} \mathbf{Y}_{jj_z}^{j-1} - (j+1)^{1/2} \mathbf{Y}_{jj_z}^{j+1} \right] \quad (5.57a)$$

$$\mathbf{Y}_{jj_z}^E := \sqrt{2j+1} \left[(j+1)^{1/2} \mathbf{Y}_{jj_z}^{j-1} + j^{1/2} \mathbf{Y}_{jj_z}^{j+1} \right] \quad (5.57b)$$

$$\mathbf{Y}_{jj_z}^B := i \mathbf{Y}_{jj_z}^j , \quad (5.57c)$$

where the $s = 1$ index in $\mathbf{Y}_{jj_z}^{ls} = \mathbf{Y}_{jj_z}^{l1} = \mathbf{Y}_{jj_z}^l$ is omitted and one has to remember that $j = l-1, l, l+1$ ($j = 1$) for $l \neq 0$ ($l = 0$). These basis functions are orthonormal in all the indexes and are the physically more appropriate to describe the polarization states of the vector field¹. The definition of the pure-spin vector spherical harmonics can be now expressed in terms of scalar spherical harmonics in order to make explicit their relation with $\hat{n}$:

$$\mathbf{Y}_{\ell m}^R = Y_{\ell m} \hat{n} , \quad \mathbf{Y}_{\ell m}^E = \sqrt{\ell(\ell+1)} r \nabla Y_{\ell m} , \quad \mathbf{Y}_{\ell m}^B = \sqrt{\ell(\ell+1)} i \mathbf{L} Y_{\ell m} = \sqrt{\ell(\ell+1)} r \hat{n} \times \nabla Y_{\ell m} = \hat{n} \times \mathbf{Y}_{\ell m}^E . \quad (5.58)$$

Note we have conventionally renamed the total angular momentum: $(j, j_z) \mapsto (\ell, m)$. The above formulas show the $\mathbf{Y}_{\ell m}^E$ and $\mathbf{Y}_{\ell m}^B$ are transverse w.r.t. $\hat{n}$ while $\mathbf{Y}_{\ell m}^R$ is longitudinal. A generic vector field satisfying the wave equation is finally expanded as

$$V^i(t, r, \theta, \varphi) = \sum_{\ell=0}^{\ell} \sum_{m=-\ell}^{\ell} R_{\ell m}(t, r) (Y_{\ell m}^R)^i(\theta, \varphi) + \sum_{\ell=1}^{\ell} \sum_{m=-\ell}^{\ell} [E_{\ell m}(t, r) (Y_{\ell m}^E)^i(\theta, \varphi) + B_{\ell m}(t, r) (Y_{\ell m}^B)^i(\theta, \varphi)] . \quad (5.59)$$

Under reflection, $\hat{n} \mapsto -\hat{n}$ ², the parity of $Y_{\ell m}^E$ and $Y_{\ell m}^B$ are respectively $(-1)^\ell$ and $(-1)^{\ell+1}$. Since these are the transformation properties of the electric and magnetic fields, the $Y_{\ell m}^E$ and $Y_{\ell m}^B$ are called respectively of *electric-type* and *magnetic-type*. Note the second sum in Eq. 5.59 starts with $\ell = 1$, since the basis functions are defined for $\ell \geq 1$.

Exercise 5.3.1. Compute the parity of the quantities in Eq. 5.58.

Exercise 5.3.2. The electromagnetic field is characterized by two polarization states orthogonal w.r.t. each other and transverse to the direction of propagation of the wave. This suggests that in electrodynamics all the longitudinal degrees of freedom encoded in $R_{\ell m}$ can be eliminated by a gauge transformation. The vector potential in Lorentz gauge admits an expansion as in Eq. (5.59) (because it satisfies a wave equation) in which only the coefficients $E_{\ell m}$ and $B_{\ell m}$ are present. Recall that the Lorentz gauge is satisfied for any transformation of the vector potential $A_\mu \mapsto A_\mu + \partial_\mu f$ with f an harmonic function ($\square f = 0$). Show that $R_{\ell m}$ can be set to zero by a gauge transformation.

eq:mueq:multi:YREBlti:YREB

Exercise 5.3.3. We have seen above that performing the scalar spherical harmonics decomposition of a function satisfying the scalar Poisson equation, one obtains a ODE in r for the expansion coefficients. How do Maxwell equations look like when expanded Eq. 5.58?

¹Note $\mathbf{Y}_{jj_z}^E$ and $\mathbf{Y}_{jj_z}^R$ are not eigenfunctions of L^2

² In spherical coordinates this is

$$r \mapsto r, \quad \theta \mapsto \pi - \theta, \quad \phi \mapsto \phi + \pi \quad (5.60)$$

Tensors. The logic to obtain an expansion of rank-2 tensors in tensor spherical harmonics follows closely the one presented above for vectors. Many steps are here omitted; a full account can be found in (Thorne, 1980). The $\mathbf{Y}_{jjz}^{l2}$ ($s = 2$) are STF tensors build similarly to above by taking as eigenfunctions $\mathbf{X}_{2s_z}$ the appropriate combination of two $s = 1$ eigenfunction $\mathbf{X}_{1s_z}$ combined with Clebsh-Gordan coefficients. *Pure-spin $s = 2$ spherical harmonics* are then introduced as linear combination of $\mathbf{Y}_{jjz}^{l2}$. The scheme below shows from which tensor spherical harmonics are obtained and what is their relations to the scalar spherical harmonics. Note again the total angular momentum indexes are renamed: $(j, j_z) \mapsto (\ell, m)$.

$$\mathbf{Y}_{jjz}^{j+2}, \mathbf{Y}_{jjz}^j, \mathbf{Y}_{jjz}^{j-2} \rightarrow (\mathbf{Y}_{\ell m}^{S0})_{ij} = (n_i n_j - 1/3 \delta_{ij}) Y_{\ell m} \quad (5.61a)$$

$$\mathbf{Y}_{jjz}^{j+2}, \mathbf{Y}_{jjz}^j, \mathbf{Y}_{jjz}^{j-2} \rightarrow (\mathbf{Y}_{\ell m}^{E1})_{ij} = \left(\frac{2}{\ell(\ell+1)}\right)^{1/2} (r/2) (n_i \partial_j + n_j \partial_i) Y_{\ell m} \quad (5.61b)$$

$$\mathbf{Y}_{jjz}^{j+2}, \mathbf{Y}_{jjz}^j, \mathbf{Y}_{jjz}^{j-2} \rightarrow (\mathbf{Y}_{\ell m}^{E2})_{ij} = \left(\frac{2}{\ell(\ell+1)}\right)^{1/2} (i/2) (n_i L_j + n_j L_i) Y_{\ell m} \quad (5.61c)$$

$$i \mathbf{Y}_{jjz}^{j+1}, i \mathbf{Y}_{jjz}^{j-1} \rightarrow (\mathbf{Y}_{\ell m}^{B1})_{ij} = \left(2 \frac{(\ell-2)!}{(\ell+2)!}\right)^{1/2} r^2 \Lambda_{ij, i' j'} \partial_{i'} \partial_{j'} Y_{\ell m} \quad (5.61d)$$

$$i \mathbf{Y}_{jjz}^{j+1}, i \mathbf{Y}_{jjz}^{j-1} \rightarrow (\mathbf{Y}_{\ell m}^{B2})_{ij} = \left(2 \frac{(\ell-2)!}{(\ell+2)!}\right)^{1/2} (i/2) (\partial_{i'} L_{j'} + \partial_{j'} L_{i'}) Y_{\ell m} . \quad (5.61e)$$

where the $s = 2$ index in $\mathbf{Y}_{jjz}^{ls}$ is omitted. The five function set above are STF tensors basis for the five possible polarization of a field of spin $s = 2$ ³. The two transverse polarizations are

$$n^i (\mathbf{Y}_{\ell m}^{E2})_{ij} = 0 = n^i (\mathbf{Y}_{\ell m}^{B2})_{ij} . \quad (5.62)$$

The gravitational field in TT gauge solution of the homogeneous wave equation of linearized GR is expanded in tensor spherical harmonics as follows,

$$h_{ij}^{\text{TT}}(t, r, \theta, \varphi) = \frac{G}{c^4} \frac{1}{r} \sum_{\ell=2} \sum_{m=-\ell}^{\ell} [u_{\ell m}(t, r) (\mathbf{Y}_{\ell m}^{E2})_{ij}(\theta, \varphi) + v_{\ell m}(t, r) (\mathbf{Y}_{\ell m}^{B2})_{ij}(\theta, \varphi)] . \quad (5.63)$$

Remark 5.3.1. In GR, all the other components of the generic tensor spherical harmonics expansion can be eliminated by a gauge choice because there exist only two polarizations and their are both transverse degree of freedom. In alternative theories of gravity where the metric encodes the gravitational field there exist up to six different polarizations exist. The measure of the GW polarization thus provides a powerful test for theories of gravity (Will, 2014). A GW detector, whose scale is small compared to the gravitational wavelength, measures the local components of a symmetric 3×3 tensor given by the components of the Riemann tensor $S_{ij} = -R_{0i0j}$ via the equation of geodesic deviation,

$$\ddot{x}_i = -R_{0i0j} x^j = S_{ij} x^j = \begin{pmatrix} A_S + A_+ & A_\times & A_1 \\ * & A_S - A_+ & A_2 \\ * & * & A_L \end{pmatrix}_{ij} x^j . \quad (5.64)$$

In general, S_{ij} has six independent components that can be expressed in terms of polarizations. The polarization modes A_L and $A_{1,2}$ are longitudinal, with A_L an axially symmetric stretching mode in the propagation direction, and $A_{1,2}$ are quadrupolar modes in each of the two orthogonal planes containing the propagation direction. The polarization modes $A_{+, \times}$ are the transverse modes of GR while A_S is a transverse breathing mode. The effect of these polarization on a ring of test masses is shown in Fig. 5.1.

In order to measure all these polarization an array of at least six suitably oriented GW antennas is necessary. Actually, there are eight unknowns in the problem: six polarizations and two direction cosines determinig the source sky position. The latter can be determined either via time-delays between different detectors or other observations (electromagnetic). With fewer detectors one can hope to measure at most particular combinations of the polarizations.

The expansion in tensor spherical harmonics of h_{ij}^{TT} can be now compared to the multipolar expansion in Eq. (3.52) of Chap. 3 in order to express the $u_{\ell m}, v_{\ell m}$ coefficients in terms of the momenta of the stress-energy tensor. The computation involves some nontrivial integrals of type $\sim \int d\Omega (\mathbf{Y}_{\ell m}^{E/B2})_{ij} n_{i_1} \dots n_{i_\alpha}$, and the use of the basis $\mathcal{Y}_{i_1 \dots i_\ell}^{\ell m}$ introduced in Sec. 5.1.

The starting point is the two expressions for the TT wave far away from the source

$$h_{ij}^{\text{TT}} = \frac{G}{c^4} \frac{1}{r} \Lambda_{ij, kl} \sum_{\alpha=0} \frac{1}{\alpha!} (\partial_t^\alpha S^{kl, i_1 \dots i_\alpha}) n_{i_1} \dots n_{i_\alpha} \quad (5.65a)$$

$$= \frac{G}{c^4} \frac{1}{r} \sum_{\ell=2} \sum_{m=-\ell}^{\ell} [u_{\ell m}(t, r) (\mathbf{Y}_{\ell m}^{E2})_{ij}(\theta, \varphi) + v_{\ell m}(t, r) (\mathbf{Y}_{\ell m}^{B2})_{ij}(\theta, \varphi)] . \quad (5.65b)$$

³The labels 0, 1, 2 in the names marks the helicity for massless fields.

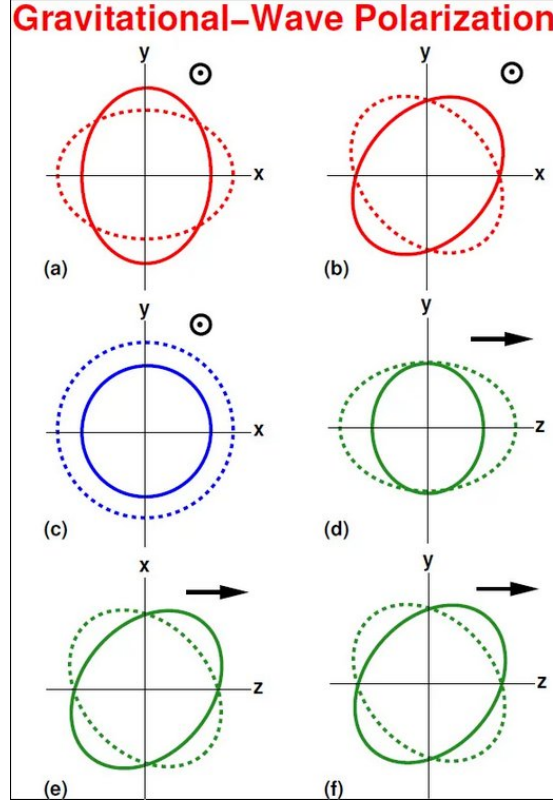


Figure 5.1: Effect on a ring of test masses on the $x - y$ plane at the passage of a GW in a generic metric theory of gravity. The wave propagates in the perpendicular direction in panel (a-c), while in the direction of the arrow in panels (d-f). Panels (a,b) are the transverse modes as in Fig. 3.1. Panel (c) is the transverse breathing mode A_s predicted for example by scalar-tensor theories. Panel (d) is the longitudinal mode A_L , predicted for example by massive scalar-tensor theories. Panels (e,f) are the $A_{1,2}$. The Einstein-Aether theory is an example that predicts all six modes. From Will (2014).

To express the coefficient $u_{\ell m}$ in terms of the source multipoles one equates both lines, multiplies by $(\mathbf{Y}_{\ell m}^{E2})_{ij}$ and integrates on the solid angle. The orthogonality of the tensors spherical harmonics isolates $u_{\ell m}$ on the l.h.s., and gives that the latter is equal to an integral that can be solved to leading order in v/c (Thorne, 1980; Maggiore, 2007):

$$u_{\ell m} = \sum_{\alpha=0} \frac{1}{\alpha!} (\partial_t^\alpha S^{kl, i_1 \dots i_\alpha}) \int d\Omega \underbrace{(\mathbf{Y}_{\ell m}^{E2})_{ij} \Lambda_{ij, kl}}_{=(\mathbf{Y}_{\ell m}^{E2})_{kl}} n_{i_1} \dots n_{i_\alpha} \sim \mathcal{Y}_{i_1 \dots i_\ell}^{\ell m *} (\partial_t^{\ell-2} S^{i_1 i_2, i_3 \dots i_\ell}) + \mathcal{O}((v/c)^2). \quad (5.66)$$

Using the conservation of the stress-energy tensor and the symmetries of the tensors, the contraction between the $\mathcal{Y}$ and the multipoles S can be written in terms of the contractions with the mass and current multipoles, e.g.

$$\mathcal{Y}_{i_1 \dots i_\ell}^{\ell m *} S^{i_1 i_2, i_3 \dots i_\ell} \sim \mathcal{Y}_{i_1 \dots i_\ell}^{\ell m *} \ddot{M}^{i_1 \dots i_\ell}. \quad (5.67)$$

The final results is

$$u_{\ell m} = \frac{d^\ell}{dt^\ell} I_{\ell m}, \quad \text{with } I_{\ell m} \sim \mathcal{Y}_{i_1 \dots i_\ell}^{\ell m *} M^{i_1 \dots i_\ell} \sim \int d^3 x r^\ell T^{00} \mathbf{Y}_{\ell m}^*. \quad (5.68)$$

A similar procedure can be followed for the $v_{\ell m}$ coefficients and leads to

$$v_{\ell m} = \frac{d^\ell}{dt^\ell} S_{\ell m}, \quad \text{with } S_{\ell m} \sim \mathcal{Y}_{i_1 \dots i_{\ell-1}}^{\ell m *} \epsilon_{ijk} P^{j, k i_1 \dots i_{\ell-1}} \sim \int d^3 x r^\ell T^{0i} (\mathbf{Y}_{\ell m}^B)_i. \quad (5.69)$$

6. Post-Newtonian formalism

Post-Newtonian formalism for self-gravitating sources. 1PN metric, basic ideas behind the formalism, relaxed EFE, key technical steps. Application to the two-body problem.

Suggested readings. *Chap. 4 of Maggiore (2007); Damour (1987); Damour and Deruelle (1986); Blanchet (2014); Schaefer and Jaranowski (2018).*

6.1 Self-gravitating sources

Gravitational waves in linearized gravity indicate that self-gravitating, compact and fast moving sources are the most powerful sources of GWs. In particular, for a source of mass M , extension R , velocity v at a distance D from the detectors the GWs we found the scaling relation

$$h \sim \left(\frac{R}{D}\right) \left(\frac{GM}{c^2 R}\right) \left(\frac{v}{c}\right)^2. \quad (6.1)$$

Linearized GR is not suitable to describe these sources since

- The background is flat and “fixed”;
- The source velocity is assumed small;
- The background metric and the source velocity are assumed independent.

For a self-gravitating source the velocity is related to the gravitational potential via the virial theorem

$$\left(\frac{v}{c}\right)^2 \sim \frac{2GM}{c^2 R} = \frac{R_S}{R}. \quad (6.2)$$

Hence the gravitational field and the source are not independent. The *post-Newtonian (PN) formalism* extends the linear theory to weakly self-gravitating system with small velocities (and stresses, “internal velocities”) by considering an expansion in $\epsilon = v/c$ (or equivalently R_S/R). The additional main assumption is a smooth stress-energy tensor with compact support in $r < R$. This formalism is the main building block to compute the GWs emitted by compact binaries, i.e. binary systems composed by black holes and neutron stars.

A naive approach to the problem would be to compute $\mathcal{O}(\epsilon^n)$ corrections to the equations of motion and then calculate the radiation via the multipolar expansion (that is an expansion in ϵ .) However, this is not possible because

- Gravitational-wave emission subtract energy from the motion;
- Gravitational waves at a given PN order are sources of GWs at higher order.

The PN formalism is significantly more involved than analogous perturbative expansion of linear field theories. Note that we here adopt the convention of using $\mathcal{O}(\epsilon^n) \sim \mathcal{O}(1/c^n)$ and call n PN order terms of order $\mathcal{O}(\epsilon^{2n})$. This is used in many modern references but not all of them.

Remark 6.1.1. *The PN formalism was started by Einstein, Landau & Lifshitz, Droste, de Sitter and others. Notable results are the Einstein-Infeld-Hoffmann 1PN Lagrangian for N bodies (particles) and the Peters and Mathews (1963) paper on the gravitational radiation from point masses (Chap. 4.) A systematic approach to the problem was however obtained only in the the 90s by Blanchet & Damour and Will & Wiseman. The formalism prescribes how to compute consistently arbitrary corrections to dynamics and radiation by matching the PN expansion in the near region to exterior (far zone) solutions. Note there is no guarantee the PN formalism can be pushed at arbitrary high-order in ϵ .*

6.2 1PN equations

Let us start discussing the PN formalism at 1PN and deriving the metric. Assumptions:

1. Small self-gravity, velocity and stresses $\epsilon \sim v/c \sim (R_S/R)^{1/2} \ll 1$, $|T^{0i}|/T^{00} = \mathcal{O}(\epsilon)$ and $|T^{ij}|/T^{00} \sim \mathcal{O}(\epsilon^2)$;
2. Neglect backreaction of the GWs.

The second assumption turns out to be correct at this order (see below). Combining the 1PN metric with the stress-energy tensor for N particles leads to *Einstein-Infeld-Hoffmann Lagrangian*, which is briefly presented as an example without derivation.

At Newtonian order the weak and static metric is Eq. (2.24), i.e. (with a slight change of notation)

$$g_{00} = -1 - 2\phi = -1 + \frac{2U}{c^2}, \quad g_{0i} = 0, \quad g_{ij} = \delta_{ij}. \quad (6.3)$$

where the static potential is the solution of the Poisson equation

$$U = \frac{G}{c^2} \int d^3x' \frac{T^{00}(x')}{|x - x'|}. \quad (6.4)$$

Given a PN order, different metric coefficients must be expanded up to different powers of ϵ in order for EFE to be fulfilled at that given order. This can be understood by the following argument. For the stress-energy tensor components one expects that

$$\frac{|T^{0i}|}{T^{00}} = \mathcal{O}(\epsilon), \quad \frac{|T^{ij}|}{T^{00}} = \mathcal{O}(\epsilon^2). \quad (6.5)$$

Consequently, using EFE $G_{\mu\nu} \sim \square g_{\mu\nu} \simeq GT_{\mu\nu}/c^4$, one sees that if $g_{00} = \mathcal{O}(\epsilon^n)$ then the contribution to the same order of the other metric components must be $g_{0i} = \mathcal{O}(\epsilon^{n-1})$ and $g_{ij} = \mathcal{O}(\epsilon^{n-2})$. Some symmetry consideration help to further specify the power counting. Without radiation reaction, there must be a symmetry under time reversal (Chandrasekhar and Esposito, 1970). For $t \mapsto -t$ ($v \mapsto -v$), g_{00}, g_{ij} are even while g_{0i} is odd. Hence, g_{00}, g_{ij} must contain even powers of v , while g_{0i} must contain odd powers. Indeed, it is found with complete calculations that if terms with the “wrong” parity under time reversal are included, then these terms satisfy homogenous wave equations and can be gauged away.

Table 6.1: Expansion in ϵ of metric and stress-energy tensor components. For a given PN order, different metric components enter with different powers of ϵ . $g_{\mu\nu}^{(n)}$ indicates the component $\mu\nu$ up to order ϵ^n . Compare to Table I of (Chandrasekhar and Esposito, 1970); note some conventions there are different from these notes.

Component	0PN	1PN	2PN	...	2nPN
g_{00}	$-1 + g_{00}^{(2)}$	$g_{00}^{(4)}$	$g_{00}^{(6)}$	...	$g_{00}^{(2n+2)}$
g_{0i}		$g_{0i}^{(3)}$	$g_{0i}^{(5)}$	...	$g_{0i}^{(2n+1)}$
g_{ij}	δ_{ij}	$g_{ij}^{(2)}$	$g_{ij}^{(4)}$	...	$g_{ij}^{(2n)}$

Near and far zones. Given the power counting above, one can proceed by equating EFE at each order PN and solve for the metric. In doing so, one must notice that

$$\partial_t = \mathcal{O}(v)\partial_i \Rightarrow -\frac{1}{c^2}\partial_{tt} + \Delta = [1 + \mathcal{O}(\epsilon)]\Delta, \quad (6.6)$$

and retardation effects are a small correction. The lowest order PN solution is given in terms of instantaneous potentials. In other terms, the PN expansion of a function is an expansion of type

$$F(u) = F(t - r/c) = F(t) - \frac{r}{c}\dot{F}(t) + \frac{r^2}{2c^2}\ddot{F}(t) + \dots, \quad (6.7)$$

where each time derivatives carries (in the Fourier space) a factor proportional to the characteristic frequency of the system $\omega = c/\lambda$,

$$\tilde{F}(t - r/c) = \tilde{F}\left(1 - \frac{r\omega}{c} + \frac{r^2\omega^2}{2c^2} + \dots\right) = \tilde{F}\left(1 - \frac{r}{\lambda} + \frac{r^2}{2\lambda^2} + \dots\right). \quad (6.8)$$

Hence, the PN expansion on the small velocity $\epsilon \sim 1/c \ll 1$ is an *near zone* expansion $r/\lambda \ll 1$, where λ is the reduced wavelength of the radiation.

Define:

- *near zone* $r < \mathcal{R}$ with $\mathcal{R} \gg R$ (here, $r \ll \lambda$);
- *overlap region* $R < r < \mathcal{R}$;
- *far/radiation zone* $R < r < \infty$ (here, $r \gg \lambda$).

We will discuss later that the PN expansion is not valid to study asymptotic gravitational-wave solutions (radiation zone), but it is only valid near the source. The solutions in the radiation zone are indeed better studied using an expansion in $R_S/R \ll 1$. These two expansion can be, in principle, matched if their domains of validity overlap in an intermediate region. These concepts of “zones” are necessary to formulate the systematic computations of high-order PN corrections.

1PN metric. Derive the 1PN metric using the Harmonic gauge

$$\partial_\mu(\sqrt{-g}g^{\mu\nu}) = 0 . \quad (6.9)$$

EFE at 0PN:

$$\Delta g_{00}^{(2)} = -\frac{8\pi G}{c^4} T_{(0)}^{00} \Rightarrow g_{00}^{(2)} = -2\phi = +\frac{U}{c^2} . \quad (6.10)$$

EFE at 1PN:

$$\Delta g_{ij}^{(2)} = -\frac{8\pi G}{c^4} \delta_{ij} T_{(0)}^{00} \Rightarrow g_{ij}^{(2)} = -2\phi \delta_{ij} \quad (6.11a)$$

$$\Delta g_{0i}^{(3)} = \frac{16\pi G}{c^4} T_{(1)}^{0i} \Rightarrow g_{0i}^{(3)} = -\frac{4G}{c^2} \int d^3x' \frac{T_{(1)}^{0i}}{|x-x'|} =: \xi_i \quad (6.11b)$$

$$\Delta g_{00}^{(4)} = \partial_{tt} g_{00}^{(2)} + g_{ij}^{(2)} \partial_i \partial_j g_{00}^{(2)} - \partial_i g_{00}^{(2)} \partial_j g_{00}^{(2)} - \frac{8\pi G}{c^4} (T_{(2)}^{00} + T_{(2)}^{ii} + 2g_{00}^{(2)} T_{(0)}^{00}) \quad (6.11c)$$

The equation for $g_{00}^{(4)}$ is solved by substituting the $g_{00}^{(2)}$ solution in the r.h.s., using the identity $\partial_i \phi \partial_i \phi = 1/2 \Delta \phi - \phi \Delta \phi$, setting $g_{00}^{(4)} = -2(\phi^2 + \psi)$, to obtain

$$\Delta \psi = \partial_t^2 \phi + \frac{4\pi}{c^4} (T_{(2)}^{00} + T_{(2)}^{ii}) \Rightarrow \psi = \int d^3x' \frac{d^3x'}{|x-x'|} \left[\partial_t^2 \phi + \frac{4\pi}{c^4} (T_{(2)}^{00} + T_{(2)}^{ii}) \right] , \quad (6.11d)$$

with the boundary condition $\psi|_{r \rightarrow \infty} = 0$.

The 1PN metric is thus expressed in terms of the 5 instantaneous potentials related to each other by a constraint equation given by the gauge conditions

$$(\phi, \xi_i, \psi) : 4\partial_t \phi + \partial_i \xi^i = 0 . \quad (6.12)$$

It is immediate to verify that the gauge condition is verified by conservation of the stress energy tensor at 1PN via the equations of motion above [exercise].

The 1PN metric can be re-expressed in terms of retarded potentials. Observe that

$$g_{00} = -1 - 2\phi - 2(\phi^2 + \psi) + \mathcal{O}(\epsilon^6) = -1 - 2(\phi + \psi) - 2\phi^2 + \mathcal{O}(\epsilon^6) = -1 - 2(\phi + \psi) - 2(\phi + \psi)^2 + \mathcal{O}(\epsilon^6) \quad (6.13a)$$

$$= -1 + 2\frac{V}{c^2} - \frac{V^2}{c^4} + \mathcal{O}(\epsilon^6) = -e^{-2V/c^2} + \mathcal{O}(\epsilon^6) , \quad (6.13b)$$

where in the first line the term $\psi^2 + 2\phi\psi$ can be introduced because it is higher order than ϕ^2 , and in the last line we introduced the potential $V := \phi + \psi$. The equations for ϕ and ψ can be now combined to obtain an equation for V . Observing that $\partial_i^2 \phi = \partial_i^2 (\phi + \psi)$ to the working order, one gets

$$\square_\eta V = -\frac{4\pi}{c^4} (T_{(0)}^{00} + T_{(2)}^{00} + T_{(2)}^{ij}) = -\frac{4\pi}{c^4} (T^{00} + T^{ii}) = -\frac{4\pi}{c^4} \sigma \Rightarrow V = -G \int d^3x' \frac{\sigma(t - |x-x'|/c, x')}{|x-x'|} , \quad (6.14)$$

where the total *active gravitational mass density* is introduced (see Chap. 5.) A similar calculation but starting from the ξ_i potential leads to the definition a new retarded potential V_i that is related to the *active mass-current density* σ_i [exercise]. The 1PN metric is then written

$$g_{00} = -1 + \frac{2V}{c^2} - \frac{2V^2}{c^4} + \mathcal{O}(\epsilon^6) , \quad g_{0i} = -\frac{4V_i}{c^3} + \mathcal{O}(\epsilon^5) , \quad g_{ij} = \delta_{ij} (1 + \frac{2V}{c^2}) + \mathcal{O}(\epsilon^4) . \quad (6.15)$$

Example 6.2.1. The Einstein-Infeld-Hoffman Lagrangian for a system of N particle is obtained from the 1PN metric. The first step is to take the stress-energy tensor for a system of N particles

$$T^{\mu\nu} = \frac{1}{\sqrt{-g}} \sum_{a=1}^N m_a \frac{d\tau_a}{dt} \frac{dx_a^\mu}{dt} \frac{dx_a^\nu}{dt} \delta^{(3)}(x^i - x_a^i) , \quad (6.16)$$

and compute the contributions $T_{(0)}^{00}, T_{(2)}^{00}, T_{(2)}^{ij}, T_{(1)}^{0i}$ summing up all the particles except one, say $a = b$. From the latter quantities one obtains the metric generated on particle b by all the other particles. The Lagrangian for particle b is then derived from the action associated to this metric,

$$S = -mc^2 \int dt \sqrt{-g_{00} - 2g_{0i} \frac{v^i}{c} - g_{ij} \frac{v^i v^j}{c^2}} . \quad (6.17)$$

The result is expressed as $L = L_0 + L_1/c^2$ where

$$L_0 = \sum_{a=1}^N \frac{1}{2} m_a v_a^2 + \sum_{a \neq b} \frac{G m_a m_b}{2 r_{ab}} \quad (6.18a)$$

$$L_1 = \sum_{a=1}^N \frac{1}{8} m_a v_a^4 - \sum_{a \neq b} \frac{G m_a m_b}{4 r_{ab}} (7 v_a^i v_b^i + \frac{(r_{ab}^j v_a^j)(r_{ab}^j v_b^j)}{r^2}) + \frac{3G}{2} \sum_a \sum_{b \neq a} \frac{m_a m_b v_a^2}{r_{ab}} - \frac{G^2}{2} \sum_a \sum_{b \neq a} \sum_{c \neq a} \frac{m_a m_b m_c}{r_{ab} r_{ac}}. \quad (6.18b)$$

Example 6.2.2. 1PN EOM and the PSR B1913+16 mass. *The 1PN EOM for a binary system were solved by Damour and Deruelle (1986) and were key to obtain the timing formulae for PSR B1913+16. Since the action is invariant under time translations and rotation, there exists two first integrals of the EOM representing the energy and the angular momentum. In the center of mass frame¹ the energy and angular momentum per reduced mass $\mu = m_1 m_2 / m$ are*

$$\frac{E}{\mu} = \frac{1}{2} v^2 - \frac{Gm}{r} + \frac{3}{8} (1 - 3\nu) \frac{v^4}{c} + \frac{Gm}{2c^2 r} [(3 + \nu) v^2 + \nu \frac{(r^i v_i)}{r} + \frac{Gm}{r}] \quad (6.19a)$$

$$\frac{J^i}{\mu} = [1 + \frac{1}{2} (1 - 3\nu) \frac{v^2}{c^2} + (3 + \nu) \frac{Gm}{c^2 r}] \epsilon_{ijk} r^j v^k, \quad (6.19b)$$

where $\nu = \mu/m$ is the symmetric mass ratio. The radial and azimuthal angle EOM can be recast to equations formally identical to the Newtonian equations by redefining parameters and coordinates. Thus, they can be solved analytically and the motion is defined by the usual Keplerian orbit parameters plus an additional set of post-Keplerian parameters. One finds that the periastron advance is a function of $1/r$; the derivative of its angular position ω averaged over one orbit is predicted to be

$$\langle \dot{\omega} \rangle = 2.11353 \left(\frac{m_1 + m_2}{M_\odot} \right)^{2/3} \text{ deg/yr}. \quad (6.20)$$

Because $\langle \dot{\omega} \rangle$ can be measured from the pulsar data, the comparison with the theory gives the binary mass.

6.3 Towards high-order PN computations

Pushing the calculation to higher PN order one encounters the following difficulties.

1. The Poisson equations for the static PN potentials become of type

$$\Delta g_{\mu\nu}^{(n)} = \underbrace{(\text{matter source})}_{\text{compact support}} + \underbrace{(\text{metric source } g_{\mu\nu}^{(n-1)})}_{\text{noncompact support}}. \quad (6.21)$$

They acquire terms from the metric at lower order that are not compact in space. The Poisson integrals are not convergent in this case because higher PN order means multipole expansions of $1/|x - x'|$ with $(x - x')^\ell$ for large ℓ and divergences at infinities. This was somehow anticipated by establishing that the PN expansion is a near zone expansion. Actually, the Poisson integrals are not the correct solution to the above equation in this case because the field does not vanish at infinity. The correct boundary condition would be to impose no incoming radiation from infinity (outgoing Sommerfeld condition) but this cannot be imposed within the PN expansion (see next point).

Example 6.3.1. *A simple example that shows that the Poisson integral is not always appropriate is the boundary value problem for equation*

$$\Delta u = \rho = \text{const}, \quad (6.22)$$

that clearly does not admit $u = 0$ at infinity as boundary condition. The Poisson integral is divergent but a solution is for example $u(r) = (-1/6)\rho r^2$.

2. The PN expansion of a function of the retarded time is a near zone expansion and blows up for $r \rightarrow \infty$, see Eq. (6.7). Instead, at the asymptotically flat spatial infinity the function $F(u)/r$ is expected to go to zero. This shows that the PN expansion is inadequate for large r and it is indeed an *asymptotic expansion* of type

$$F(\epsilon, r) = \sum_n c_n(r) \epsilon^n, \quad (6.23)$$

where the coefficients depend on a second scale (parameter) and the series is not uniformly convergent.

¹Actually it is a center of energy, and differs from the Newtonian definition.

3. Backreaction needs to be taken into account into the EOM at sufficiently high-order. This breaks invariance under time-reversal and the power counting done for the 1PN metric.

Using the result of linearized theory and invoking the energy balance between the energy radiated and the energy subtracted from the motion one finds at which PN order the backreaction enters the expansion.

$$E = K + V \simeq -\frac{1}{2}V + V = \frac{1}{2}V = -K = -\frac{1}{2}Mv^2 \quad \Rightarrow \quad \dot{E} = -Mv\dot{v} \quad (6.24a)$$

$$P = \dot{E}_{\text{GW}} \simeq \frac{GM^2}{c^5} \frac{v^6}{r} \quad (6.24b)$$

where the first line uses the virial theorem. Thus

$$\dot{E} = P = \dot{E}_{\text{GW}} \quad \Rightarrow \quad \dot{v} \simeq \frac{GM}{r^2} \left(\frac{v}{c}\right)^5 \quad (6.24c)$$

indicating the radiation backreaction enters at ϵ^5 , i.e. at 2.5PN. This estimate is correct, and the quadrupole formula does give the correct leading order result. However, in a nonlinear theory is not guaranteed that the energy balance holds.

Example 6.3.2. *Backreaction is present also in electromagnetism. The dynamics of a N charges produces electromagnetic waves propagating to infinity. The energy propagated with the waves is drained by the motion of the charges that must experience a radiation reaction force. Let us show that the work of the radiation reaction force corresponds to the radiated power by the ensemble of charges. The electromagnetic Lagrangian for charge q_a with mass m_a is*

$$L_a = -m_a c^2 \sqrt{1 - \frac{v_a^2}{c^2}} - q_a \phi + \frac{q_a}{c} v_a^i A_i, \quad (6.25)$$

where the potential $A_\mu = (\phi, A_i)$ is generated by the current $J_\mu = (\rho, J_i)$ produced by all the charges and calculated along the particle a trajectory. To solve for the motion one can proceed with a near-zone- v/c expansion. The potentials can be computed via a multipolar near-zone expansion

$$\phi = \square^{-1}[\rho] \simeq \frac{1}{r} \int d^3x' \rho(x') + \frac{n_i}{cr^2} \frac{d}{dt} \int d^3x' \rho(u, x') x'^i + \dots = \sum_a \frac{q_a}{r} + \dots \quad (6.26a)$$

$$A_i = \square^{-1}[J_i] \simeq \frac{1}{cr} \int d^3x' J_i(u, x') + \dots = -\frac{1}{cr} \sum_a q_a x_a^i + \dots \quad (6.26b)$$

The Lagrangian can be expanded in v/c and one finds that: (i) the term linear in v/c vanishes because charge conservation; (ii) the Lagrangian depends only on positions and velocities up to $(v/c)^2$ (is conservative), and (iii) at $(v/c)^3$ the dipole of the charge distribution $\dot{d}^i = \sum_a q_a x_a^i$ generates an electric field $E^i = 2/(3c^2) \ddot{d}^i$ that in turn exerts a Lorentz force $F^i = q_a E^i$ on the point mass a . The total work performed on all the charges is

$$W = \sum_a F_a^i v_{a i} = \frac{2}{3c^3} \sum_a q_a v_a^i \ddot{d}_i. \quad (6.27)$$

Taking the time-average power one verifies that the average work equals the total radiated power

$$\bar{W} = -\frac{2}{3c^3} \langle \ddot{d}^i \rangle. \quad (6.28)$$

□ [TODO: Expand on calculations (Maggiore P.275; LandauLifshitz 65-75)]

The modern PN formalism solves the *relaxed EFE* (Sec. 6.4) and tackle the issues presented above in the following way.

- (i) The solution to issue 2. is to employ a PN expansion in the near zone only and use instead a *post-Minkowskian* (PM) expansion (in R_S/r) to find exterior/asymptotic/vacuum solution. The PM expansion is then matched to the PN expansion in the overlap zone. The general method is called *matched asymptotic expansion*.
- (ii) The problem 1. of inverting the Poisson equations with the appropriate boundary condition is solved by a suitable regularization procedure and analytical continuation (Sec. 6.5).
- (iii) The full formalism predicts the validity of the quadrupole formula at leading order and the radiation enters at 2.5PN the EOM. Schematically, for compact binaries the energy balance has been verified to the following orders

Radiation at ∞	EOM/Metric in the near region
Quadrupole	2.5PN correction
1PN correction	3.5PN correction
1.5PN tail term	4PN correction

6.4 Relaxed EFE

Define the field

$$\mathbf{h}^{\mu\nu} := \sqrt{-g}g^{\mu\nu} - \eta^{\mu\nu} \quad (6.29)$$

to be considered in place of the metric in the full theory (no approximation; $\mathbf{h}^{\mu\nu}$ is not small in general). This corresponds to the “gothic metric” defined in the book of Landau and Lifschits (1975). The connection to the linear metric is obtained by linearizing the metric,

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} + \mathcal{O}(2) , \quad -g = 1 + h , \quad g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu} , \quad (6.30)$$

and in this limit the gothic metric reduces to minus the trace-reversed linear perturbation in the weak field regime

$$-\mathbf{h}^{\mu\nu} \simeq \eta^{\mu\nu} - (1 + h)^{1/2}(\eta^{\mu\nu} - h^{\mu\nu}) = h^{\mu\nu} - \frac{1}{2}\eta^{\mu\nu}h = \bar{h}^{\mu\nu} . \quad (6.31)$$

The harmonic gauge in terms of the gothic metric is

$$\partial_\mu \mathbf{h}^{\mu\nu} = 0 \quad (\text{Harmonic gauge}) \quad (6.32)$$

In this gauge and using the gothic metric the exact EFE read

$$\square \mathbf{h}^{\mu\nu} = \frac{16\pi G}{c^4} \tau^{\mu\nu} = \frac{16\pi G}{c^4} (-gT^{\mu\nu} + t_{LL}^{\mu\nu} + \partial_\alpha \partial_\beta \chi^{\alpha\beta\mu\nu}) \quad (\text{Relaxed EFE}) , \quad (6.33)$$

where

$$T^{\mu\nu} \quad (\text{stress energy tensor}) \quad (6.34a)$$

$$t_{LL}^{\mu\nu}(\mathbf{h}^{\mu\nu}, \partial \mathbf{h}^{\mu\nu}) \quad (\text{Landau-Lifshitz energy-momentum pseudotensor}) \quad (6.34b)$$

$$\chi^{\alpha\beta\mu\nu} := \frac{c^4}{16\pi G} (\mathbf{h}^{\alpha\mu} \mathbf{h}^{\beta\nu} - \mathbf{h}^{\mu\nu} \mathbf{h}^{\alpha\beta}) \quad (6.34c)$$

Eq. (6.33) and Eq. (6.32) are equivalent to EFE and are written in terms of partial derivatives instead of covariant. It is immediate to verify that the gauge condition implies the conservation of $\tau^{\mu\nu}$ in the form

$$\partial_\beta \tau^{\beta\mu} = 0 . \quad (6.35)$$

EFE imply the conservation of $T^{\mu\nu}$ and thus the matter EOM via the Bianchi identities. However, the Eq. (6.33) alone (without the gauge) does not imply Eq. (6.35). These equations are called *relaxed EFE* because their solution does not include the matter dynamics (and it is not a solution of EFE). Working with the REFE one considers separately Eq. (6.33) and Eq. (6.32) and (i) look for a solution Eq. (6.33) for a given dynamics by imposing the no-incoming-radiation conditions, and successively (ii) impose the gauge condition to guarantee the matter source obeys the correct EOM in the computed metric. The formal solution of Eq. (6.33) obeying the correct boundary conditions is given by the retarded Green function exactly as in the linearized theory:

$$\mathbf{h}^{\mu\nu} = -\frac{4G}{c^2} \int d^4x' \frac{\tau^{\mu\nu}(t', x')}{|x - x'|} \delta(t' - t + |x - x'|/c) = -\frac{4G}{c^2} \int d^3x' \frac{\tau^{\mu\nu}(t - |x - x'|/c, x')}{|x - x'|} . \quad (6.36)$$

However, differently from the linearized theory, now $\tau^{\mu\nu}$ contains the $\mathbf{h}^{\mu\nu}$ and the formula above cannot be employed to directly compute the solution. The latter is instead obtained by matching solutions given by suitable expansions in the near and radiation zones.

6.5 Main steps of the PN formalism

The basic logic and technical step of the Damour&Blanchet PN formalism are here summarize. More details are discussed in Maggiore (2007) book and in the Blanchet (2014) review.

General logic.

- (i) Find an approximate solution of REFE in the far region using an expansion in $R_S/r = GM/(c^2 r)$ or equivalently a *post-Minkowskian* (PM) expansion in G ,

$$\mathbf{h}^{\mu\nu} = \sum_n G^n \mathbf{h}_{(n)}^{\mu\nu} \quad (\text{PM expansion, radiation zone}) \quad r > R \quad (6.37)$$

The term $h_{(n)}^{\mu\nu}$ at a given order is given by a wave equation with no matter sources but with metric sources generated by lower order metric terms. From the discussion in Chap. 5, one can guess that the general exterior solution at each order can be expressed in terms of certain *source multipole moments*

$$h_{(n)}^{\mu\nu} \leftrightarrow \sum_L (I_L, J_L, W_L, X_L, Y_L, Z_L) . \quad (6.38)$$

These multipoles are initially unspecified (Cf. exterior solution of the scalar wave equation) and for this reason are also called algorithmic multipole moments. Some technical steps to compute $h_{(n)}^{\mu\nu}$ are discussed below.

- (ii) Find an approximate solution of REFE in the near region using the PN expansion in $\epsilon \sim 1/c$

$$h^{\mu\nu} = \sum_n \frac{1}{c^n} {}^{(n)}h^{\mu\nu} \quad (\text{PN expansion, near zone}) \quad r < \mathcal{R} \quad (6.39)$$

Each term ${}^{(n)}h^{\mu\nu}$ satisfies a Poisson equation in the form Eq. (6.21). To each PN order, the general solution at retarded time is composed of a particular finite solution of the inhomogeneous equation, that is found with a regularization procedure, plus the general solution of the homogeneous equation, that is expressed as a multipolar expansion of arbitrary STF tensors.

- (iii) Assume the validity of both expansions in the overlap zone, $R < r < \mathcal{R}$, and fix the source multipole moments of the MPM exterior solution in terms of the matter source by matching to the PN expansion. To perform the matching each term of the PM expansion must be re-expanded as a PN series in $1/c$. A crucial result due to Blanchet and Damour (1986) is that the PN expansion of a PM term of order n contains only a finite number of terms,

$$h_{(n)}^{00} = \mathcal{O}(\epsilon^{2n}) , \quad h_{(n)}^{0i} = \mathcal{O}(\epsilon^{2n+1}) , \quad h_{(n)}^{ij} = \mathcal{O}(\epsilon^{2n}) . \quad (6.40)$$

Equivalently, to work consistently at a given PN order one requires only a finite number of PM iterations. The match thus connects

$$(I_L, J_L, W_L, X_L, Y_L, Z_L) \leftrightarrow \int \bar{\tau}_{\mu\nu} [T^{\mu\nu}, h^{\mu\nu}] , \quad (6.41)$$

(where the bar indicate the PN contributions up to the working order) and fixes the arbitrary STF tensors in the solution of the homogeneous equations in the near zone. Note that to lowest order $I_L = M_L$ and $J_L = S_L$ as given in Chap. 5.

- (iv) The behaviour at scri+ of the PM solution $h_{(n)}^{\mu\nu}$ is

$$h_{(n)}^{\mu\nu}(u) \sim \sum_{k=1}^{n-1} \sum_{p=0} G_{L(k,p,n)}^{\mu\nu}(u) \frac{\hat{n}_L (\log r)^p}{r^k} , \quad (6.42)$$

where the $\log(r)$ terms are due to the use of harmonic coordinates. In order to compute the waveform in TT gauge, the gothic metric needs to be turned into radiative coordinates (denoted with capital letters, except for the retarded time) where one recovers the structure

$$H_{(n)}^{\mu\nu}(u) \sim \sum_{k=1} K_{L(k,n)}^{\mu\nu}(u) \frac{\hat{N}_L}{R^k} . \quad (6.43)$$

The transformation to radiative coordinates must be obtained order-by-order in G by explicit computation. The $1/R$ part of the expansion is then selected and projected using the Λ tensors. The final expression is

$$H_{ij}^{\text{TT}} = \frac{4G}{c^2 R} \Lambda_{ij,kl}(\hat{N}) \sum_{\ell=2} \frac{1}{c^\ell \ell!} \left[N_{L-2} U_{klL-2}(u) - \frac{2\ell}{(\ell+1)} N_{mL-2} \epsilon_{mn(k} V_{l)nL-2}(u) \right] , \quad (6.44)$$

where U_L and V_L are called the *radiative multipolar moments* and can be directly computed from the source multipole moments

$$(I_L, J_L, W_L, X_L, Y_L, Z_L) \rightarrow (U_L, V_L) . \quad (6.45)$$

To lowest order they are expressed as derivatives of the multipoles introduced in Eq. (5.49a),

$$U_L(u) = M_L^{(\ell)}(u) = \frac{d^\ell}{du^\ell} M_L(u) , \quad U_L(u) = S_L^{(\ell)}(u) = \frac{d^\ell}{du^\ell} S_L(u) . \quad (6.46)$$

Higher order computations reveal the appearance of *hereditary* terms that are given as integrals in the retarded time from ∞ to u of the multipole moments. These terms are a consequence of the propagation of waves on a curved background. These hereditary terms are in contrast to the instantaneous terms that one expects to propagate on a flat background. The hereditary terms are of two types. There are *tail terms* that quantify the contributions of the source from the past to u physically corresponding to the backscattering of the GW off the background curvature. The other type is composed of *memory terms*, physically interpreted as the re-radiation of the stress-energy of the propagating waves, and that contribute to a nonzero wave amplitude even after the generating source has switched off.

Technical steps.

- (i) *Multipolar post-Minkowskian (MPM) solution of REFE for $r > R$.* The structure induced by Eq. (6.33) for $r > R$ on the various terms of the PM solution is

$$\square h_{(1)}^{\mu\nu} = 0 \quad (6.47a)$$

$$\square h_{(2)}^{\mu\nu} = \Lambda_{(2)}^{\mu\nu}[h_{(1)}^{\mu\nu}] := M_{(2)}^{\mu\nu}[h_{(1)}^{\mu\nu}] \quad (6.47b)$$

$$\square h_{(3)}^{\mu\nu} = \Lambda_{(3)}^{\mu\nu}[h_{(2)}^{\mu\nu}, h_{(1)}^{\mu\nu}] := N_{(3)}^{\mu\nu}[h_{(1)}^{\mu\nu}, h_{(1)}^{\mu\nu}] + M_{(2)}^{\mu\nu}[h_{(2)}^{\mu\nu}, h_{(1)}^{\mu\nu}] + M_{(2)}^{\mu\nu}[h_{(1)}^{\mu\nu}, h_{(1)}^{\mu\nu}] \quad (6.47c)$$

$$\dots \quad (6.47d)$$

where $M_{(2)}^{\mu\nu}$ is a quadratic function, $N_{(3)}^{\mu\nu}$ a cubic function, etc, and where the gauge condition $\partial_\mu h_{(n)}^{\mu\nu} = 0$ has to be imposed at each order.

At the lowest order the general solution is given by an STF expansion

$$h_{(1)}^{\mu\nu} = \sum_{\ell=2} \frac{1}{r} \partial_L K_L^{\mu\nu}(t - r/c) \quad (6.48)$$

with the multipoles evaluated at the retarded time. As the gauge condition is imposed the 10 components are reduced to 6 independent components, and the general exterior solution can be written in terms of a symmetric tensor and a vector as

$$h_{(1)}^{\mu\nu} = k_1^{\mu\nu} + \partial^\mu \varphi_1^\nu + \partial^\nu \varphi_1^\mu - \eta^{\mu\nu} \partial_\alpha \varphi_1^\alpha. \quad (6.49)$$

In turn, the symmetric tensor and the vector can be expressed in terms of STF tensors (and their derivatives)

$$(k_1^{\mu\nu}, \varphi_1) \rightarrow (I_L, J_L, W_L, X_L, Y_L, Z_L). \quad (6.50)$$

Given the solution at lowest order, one can proceed to the higher order solutions. Here one should note that, differently from the homogenous equation for $n = 1$, the sources $\Lambda_{(n)}^{\mu\nu}$ extend to the whole space. Because all the multipoles in $h_{(1)}^{\mu\nu}$ diverge for $r \rightarrow 0$, the $h_{(2)}^{\mu\nu}$ diverges and also all the other corrections. In other words, the solutions valid for $r > R$ cannot be extended down to $r \rightarrow 0$. The retarded Green functions do not give the correct solution for the high-order PM terms.

The method to compute higher-order PM solution is as follows.

- (a) Observe that $h_{(n)}^{\mu\nu}$ is not needed as a full multipolar series but only as a truncated sum up to the order that is necessary to match later to the PN expansion. If one considers the truncated expansion, then the retarded integral defined as

$$I_{(n)}^{\mu\nu}(B) := \square_{\text{ret}}^{-1}(r^B \Lambda_{(n)}^{\mu\nu}) \quad (6.51)$$

is finite for a sufficiently large $B \in \mathbb{R}^+$ (One can always find a B such that ...)

- (b) The integral $I_{(n)}^{\mu\nu}(B)$ admits a unique analytical continuation to the B -complex plane except at some integer values and at the origin. In particular, for $B \rightarrow 0$ the integral admits a Laurent expansion, whose term $\propto B^0$ is a particular finite solution of the inhomogenous equation $\square h_{(n)} = \Lambda_{(n)}$. This solution is indicated as the *partie finie*

$$\text{FP}_{B=0} I_{(n)}^{\mu\nu}(B) = \text{FP}_{B=0} \left[\square_{\text{ret}}^{-1}(r^B \Lambda_{(n)}^{\mu\nu}) \right]. \quad (6.52)$$

The FP reduces to the retarded Green function integral if the source has compact support.

- (c) The general solution is given by summing up the particular solution with the FP with the general solution of the homogeneous equation that can be exactly determined.
- (ii) *Multipolar post-Newtonian (MPN) solution of REFE for $r < \mathcal{R}$.* The general structure of the PN Poisson equations is

$$\Delta^{(n)} h^{\mu\nu} = 16\pi G^{(n-1)} \tau^{\mu\nu} + \partial_t^2 h^{\mu\nu}. \quad (6.53)$$

Because the PN expansion is a near zone expansion, the Green function (Poisson) integrals of the above equation diverge for $r \rightarrow \infty$ for increasing large n . The correct way to solve the above equation in the near zone is to find a particular finite solution and add it to the general solution of the homogeneous equation.

The particular solution can be found by regularizing $r \rightarrow \infty$ in a similar way to what done above with the FP procedure. For a function satisfying the Poisson equations one defines

$$I(B) := \Delta^{-1}(r^B f) = -\frac{1}{4\pi} \int d^3x' \frac{|x'^B|}{|x - x'|} f(x'), \quad (6.54)$$

and, as above, $I(B)$ can be analytically continued and the coefficient $\propto B^0$ of the Laurent series in $B \rightarrow 0$ gives a particular solution. The way to apply this to the PN equation is to observe that taking the sum up of the PN series up to order n

$$\bar{h}^{\mu\nu} = \sum_{k=2}^n \epsilon^{k(k)} h^{\mu\nu}, \quad (6.55)$$

the equation above becomes a wave equation and the particular solution can be written

$$\bar{h}_p^{\mu\nu} = \frac{16\pi G}{c^4} \text{FP}_{B=0} [\Box^{-1}(r^B \bar{\tau}^{\mu\nu})] . \quad (6.56)$$

The general solution of the homogeneous equation is

$$\bar{h}_{\text{hom}}^{\mu\nu} = \frac{16\pi G}{c^4} \sum_{\ell=2} \frac{(-1)^\ell}{\ell!} \partial_L \left[\frac{\mathcal{R}_L^{\mu\nu}(t-r/c) - \mathcal{R}_L^{\mu\nu}(t+r/c)}{2r} \right] , \quad (6.57)$$

where the STF tensors $\mathcal{R}_L^{\mu\nu}$ are arbitrary functions of u and the antisymmetry follows from the regularity condition at $r=0$. These functions are determined in the matching of the solutions. It is immediate to see this term is odd under time reversal and thus contribute to the radiation reaction.

- (iii) An important result for the matching of the expansions is that the expression between the source multipole moments and the effective stress-energy tensor at a given PN order $\bar{\tau}^{\mu\nu}$ can be computed analytically and to any PN order (Blanchet, 2014). For the I_L and J_L the expression is formally identical to Eq. (5.49a), provided that the integral there are calculated with the FP procedure (if necessary) and where the active matter densities are taken from $\bar{\tau}^{\mu\nu}$. In particular, a remarkable fact holds: the quadrupole result from linearized GR is correct to 1PN.

6.6 Compact binaries

Applications of the PN formalism to the dynamics and radiation from compact binaries are further discussed in the Blanchet (2014) and the Schaefer and Jaranowski (2018) reviews. The starting point is the GR action with the stress-energy tensor of Eq. (6.16),

$$S = S_{\text{GR}} + S_{\text{pointmasses}} = \frac{c^3}{16\pi G} \int d^4x R \sqrt{|g|} - \sum_a m_a c^2 \int ds_a , \quad (6.58)$$

where $ds_a = (-g_{\mu\nu}(dx_a^\mu/dt)(dx_a^\nu/dt)/c^2)^{1/2} dt$ is calculated on the trajectory of body a . This action is also the starting point for the Einstein-Infeld-Hoffman 1PN approach. Differently from there high-order PN computations, require the regularization of the point-particles described by Dirac deltas. This can be done with field theory tools like dimensional regularization. We do not discuss here this technical aspects and refer instead the interested reader to the above reviews. In the following, we give an overview of the results for circular binaries and show how GW approximants are constructed from these results.

Consider a circular binary with objects with masses $m_1 \geq m_2$ and dimensionless spins $\chi_i = S_i/m^2$ $i=1,2$. We define the additional spin parameters

$$\tilde{a}_i := \frac{a_i}{m} = \frac{S_i}{m m_i} = \frac{\chi_i m_i}{m} , \quad \hat{a}_0 := \tilde{a}_1 + \tilde{a}_2 , \quad \tilde{a}_{12} := \tilde{a}_1 - \tilde{a}_2 . \quad (6.59)$$

For $m_1 \gg m_2$ ($\nu \rightarrow 0$) $\tilde{a}_1 \rightarrow S_1/(m_1)^2$ is the dimensionless spin of the large black hole and $\tilde{a}_{12} \rightarrow 0$. In place of the GW frequency, it is useful to introduce the parameter

$$x := \left(\pi \frac{Gm}{c^3} f \right)^{2/3} = \left(\frac{1}{2} \frac{Gm}{c^3} \omega \right)^{2/3} = \left(\frac{Gm}{c^3} \Omega \right)^{2/3} \sim \mathcal{O}((v/c)^2) , \quad (6.60)$$

where the scaling in ϵ simply follows from $(Gm/c^3 \Omega)^{2/3} \sim ((Gm/r(r\Omega)c^3)^{2/3} \sim (v^2 v/c^3)^{2/3}$. The energy of circular orbits can be written as

$$E(x) = E_{\text{orb}}(x) + E_{\text{SO}}(x) + E_{\text{SS}}(x) , \quad (6.61a)$$

where the different terms describe the orbital, spin-orbit and spin-spin interactions. The leading order contribution is $E_{\text{Newt}} = -\mu c^2 x/2$. The orbital energy is known at 4PN (Damour et al., 2014, 2016; Bernard et al., 2017)

$$\begin{aligned} \frac{E_{\text{orb}}(x)}{E_{\text{Newt}}} &= 1 - \left(\frac{3}{4} + \frac{\nu}{12} \right) x + \left(-\frac{27}{8} + \frac{19\nu}{8} - \frac{\nu^2}{24} \right) x^2 + \left(-\frac{675}{64} + \left(\frac{34445}{576} - \frac{205\pi^2}{96} \right) \nu - \frac{155\nu^2}{96} - \frac{35\nu^3}{5184} \right) x^3 \\ &\quad + \left[-\frac{3969}{128} + \left(\frac{9037\pi^2}{1536} - \frac{123671}{5760} + \frac{448}{15} (2\gamma_E + \ln(16x)) \right) \nu + \left(\frac{3157\pi^2}{576} - \frac{498449}{3456} \right) \nu^2 + \frac{301\nu^3}{1728} + \frac{77\nu^4}{31104} \right] x^4 \\ &\quad + \dots \end{aligned} \quad (6.61b)$$

with $\gamma_E = 0.577\dots$ is the Euler constant. Work towards 5PN computation is ongoing with different techniques Bern et al. (2019); Bini et al. (2019). The spin sector expansions start with

$$E_{\text{SO}} = -\frac{1}{6} (7\hat{a}_0 + \tilde{a}_{12} \sqrt{1-4\nu}) \nu x^{5/2} + \dots \quad ; \quad E_{\text{SS}} = \frac{1}{2} \hat{a}_0^2 \nu x^3 + \dots . \quad (6.61c)$$

and terms in the spin-orbit sector are known up to 4.5PN Levi et al. (2020); Antonelli et al. (2020).

The gravitational waveform for nonspinning binaries and generic masses is complete at 3.5PN order. Let $P_{\text{Newt}}(x) = 32c^5/(5G)\nu^2x^5$, the expression of the GW power (flux) is

$$\begin{aligned} \frac{P(x)}{P_{\text{Newt}}(x)} = & 1 + \left(-\frac{1247}{336} - \frac{35}{12}\nu\right)x + 4\pi x^{3/2} + \left(-\frac{44711}{9072} + \frac{9271}{504}\nu + \frac{65}{18}\nu^2\right)x^2 + \left(-\frac{8191}{672} - \frac{583}{24}\nu\right)\pi x^{5/2} \\ & + \left[\frac{6643739519}{69854400} + \frac{16}{3}\pi^2 - \frac{1712}{105}\gamma_E - \frac{856}{105}\log(16x) + \left(-\frac{134543}{7776} + \frac{41}{48}\pi^2\right)\nu - \frac{94403}{3024}\nu^2 - \frac{775}{324}\nu^3\right]x^3 \\ & + \left(-\frac{16285}{504} + \frac{214745}{1728}\nu + \frac{193385}{3024}\nu^2\right)\pi x^{7/2} + \dots \end{aligned} \quad (6.62)$$

Terms in the SO sector are known up to 4PN Marsat et al. (2014), 3PN at S^2 Bohé et al. (2015), and 3.5PN at S^3 Marsat (2015).

Taylor waveform approximants. In order to illustrate the application of the PN formalism to GW modeling from binary systems we go back to Example 4.1.1. The integration of Eq. (4.26) gives an equation for the GW frequency evolution in time, $\omega \sim (t_c - t)^{-3/8}$, where t_c is an integration constant called the *coalescence time*. The frequency formally diverges at $t = t_c$ but in those conditions the slow motion approximation is not valid anymore. The frequency evolution is then usually expressed in term of the parameter $x \sim \omega^{2/3}$ by introducing the dimensionless time

$$\tau := \nu \frac{c^3}{GM}(t_c - t) \Rightarrow x(t) = \frac{1}{4}\tau^{-1/4}. \quad (6.63)$$

The leading order GW phase can be then immediately calculated by integration,

$$\phi_{\text{Newt}}(t) = \int_0^{t_0} dt' \omega(t') = \phi_0 - \frac{1}{\nu} \tau^{5/8} = \phi_0 - \frac{x^{-5/2}}{32\nu}. \quad (6.64)$$

This procedure can be now extended to high PN order using the PN expanded expressions for the orbital energy and the flux and using the energy balance. The time-domain phase is an expansion in x

$$\frac{\phi(t)}{\phi_{\text{Newt}}(t)} = 1 + \mathcal{O}(x) + \mathcal{O}(x^{3/2}) + \mathcal{O}(x^2) + \mathcal{O}(x^{5/2}) + \mathcal{O}(x^3) + \dots \quad (6.65a)$$

$$= 1 + \left(\frac{3715}{1008} + \frac{55}{12}\nu\right)x - 10\pi x^{3/2} + \left(\frac{15293365}{1016064} + \frac{27145}{1008}\nu + \frac{3085}{144}\nu^2\right)x^2 + \dots \quad (6.65b)$$

The second line indicates the first terms for nonspinning binaries. The expansion for nonspinning binaries is fully known up to 3.5PN, i.e. $\mathcal{O}(x^{7/2})$, with several high-order terms known.

Waveform approximants build in this way are called *Taylor approximants*. The following example describe how to calculate analytically the Taylor approximants in frequency domain. The latter form is useful for gravitational-wave data analysis.

Remark 6.6.1. *The saddle point/stationary phase method approximates the definite integral (limits are not indicated)*

$$I(\lambda) = \int dt A(t) e^{\lambda \phi(t)} \quad (6.66)$$

where $A(t), \phi(t)$ are real functions, for $\lambda \ll 1$ ($\lambda > 0$), assuming the integration domain contains a stationary point $\phi'(t_0) = 0$. Around the stationary point, the Taylor expansion of the functions is

$$\lambda \phi(t) = \lambda \left(\phi(t_0) + \frac{1}{2} \phi''(t_0)(t - t_0)^2 + \mathcal{O}(2) \right) = \lambda \phi_0 + \frac{1}{2} \phi_0'' x^2 + \mathcal{O}(2) \quad (6.67a)$$

$$A(t) = A(t_0) + \mathcal{O}(1) = A_0 \left(1 + \frac{x}{\sqrt{\lambda}} \frac{A_0'}{A_0} + \mathcal{O}(2) \right) \quad (6.67b)$$

where functions evaluated at t_0 are indicated with a subscript $_0$ and where the second line changes to the (integration) variable to $x := \sqrt{\lambda}(t - t_0)$. The integral is thus

$$I(\lambda) = \frac{A_0}{\sqrt{\lambda}} e^{-\lambda \phi_0} \int dx e^{-\frac{1}{2} \phi_0'' x^2} [1 + \mathcal{O}(1/\sqrt{\lambda})] = \frac{A_0}{\sqrt{\lambda}} e^{-\lambda \phi_0} \sqrt{\frac{2\pi}{\phi_0''}} [1 + \mathcal{O}(1/\sqrt{\lambda})], \quad (6.68)$$

where one observe that for $\lambda \ll 1$ the contributions in brackets are progressively smaller, and that the first integral is dominated by the saddle point $x \simeq 0$ and that it can be approximated by a Gaussian integral on the real axis. The method can be extended to the complex functions. In this case, one must take a complex integral on a contour around the zero(s) $\phi'(z) = 0$ where $\Re(\phi)$ looks like a saddle. The result is the same as above; the detailed steps can be checked in analysis books.

Example 6.6.1. Stationary phase approximation (SPA) of GW. *The data analysis of GW data requires frequency-domain waveforms. Given a time-domain real waveform, $h_+(t) = A(t) \cos \phi(t)$, the complex Fourier transform is*

$$\tilde{h}(f) = \int dt A(t) \cos(t) e^{i2\pi f t} = \frac{1}{2} \int dt A(t) \left(e^{i\phi(t)} + e^{-i\phi(t)} \right) e^{i2\pi f t} \quad (6.69)$$

where we consider only $f > 0$ since $\tilde{h}(-f) = \tilde{h}^*(f)$. Looking for a SPA solution, the contribution $\propto e^{i\phi(t)+i2\pi f t}$ has no stationary points and is always oscillating and can be discarded. The stationary point of the other contribution is

$$2\pi f = \dot{\phi}(t_f) = \omega(t_f) , \quad (6.70)$$

and the SPA solution is

$$\tilde{h}(f) \simeq \frac{1}{2} \left(\frac{2\pi}{\ddot{\phi}(t_f)} \right)^{1/2} A(t_f) e^{i\Psi_+(f)} \quad (6.71)$$

where

$$\Psi_+(f) = 2\pi f t_f - \phi(t_f) - \frac{\pi}{4} . \quad (6.72)$$

The phase of the Fourier transform of the time-domain (quadrupolar) waveform is thus simply the Legendre transform of the quadrupolar time-domain phase $\phi(t)$. Using now Eq. (6.63) to substitute $t_f(\tau_f(f)) \sim f^{-8/3}$ into the Newtonian expressions for $A(t_f)$ and $\ddot{\phi}(t_f)$ one obtains that at leading order

$$\tilde{h}(f) = \tilde{A}(f) e^{-i\Psi(f)} \sim f^{-7/6} e^{-i\Psi(f)} . \quad (6.73)$$

Details and factors are left as [exercise]. Differentiating Eq. (6.72) twice with respect to f leads to

$$\frac{d^2 \Psi_{\text{SPA}}}{dw^2} = \frac{Q_\omega(w)}{w^2} , \quad (6.74)$$

where $w = 2\pi f$ ($\neq \omega$) is the Fourier domain circular frequency. Hence, the frequency-domain phase in the SPA can be calculated by integrating term-by-term the PN expansion of the Q_ω obtained from the energy balance,

$$Q_\omega = -\omega^2 \frac{\dot{E}}{P} = \frac{5}{48\nu} x^{-5/2} \left(1 + \frac{743 + 924\nu}{336} - 4\pi x^{3/2} + \mathcal{O}(x^2) \right) . \quad (6.75)$$

Remark 6.6.2. The Q_ω quantity also measures the number of GW cycles spent by the binary system within an octave of the GW frequency ω . It is therefore analogous to the “quality factor” of a damped oscillator. The accumulated time-domain phase between frequencies (ω_1, ω_2) is given by the following integral

$$Q_\omega = \frac{\omega^2}{\dot{\omega}} = \frac{d\phi(t)}{d \ln \omega} \Rightarrow \phi = \int_{\omega_1}^{\omega_2} Q_\omega d \ln \omega . \quad (6.76)$$

A change of Q_ω of the order ± 1 during a frequency octave $\ln \omega_2 / \omega_1 = 1$ corresponds to a local dephasing (around ω) of $\Delta\phi = \pm 1$ rad. Moreover, it can be shown that the squared signal-to-noise ratio (Chap. 9) can be written as (Damour et al., 2000),

$$\rho^2 = 4 \int d \ln \omega \frac{Q_\omega(\omega) A^2(\omega)}{\omega S_n(\omega)} , \quad (6.77)$$

where A is the amplitude of the time-domain waveform and S_n the one-sided noise power spectral density.

Finite size effects. The motion of bodies that have an internal structure and can be deformed (like neutron stars) is affected by the presence of tidal forces. We start with recalling how finite structure effect are described in Newtonian gravity.

Remark 6.6.3. Newtonian EOM for N perfect-fluid bodies. The Newton potential of N fluid bodies with density ρ and pressure P is

$$U = \sum_{b=1}^N \int_{V_b} d^3 x' \frac{\rho(t, x')}{|x - x'|} , \quad (6.78)$$

where V_b is the volume where the density of body b is nonzero. The potential for body a is thus given by an “external” contribution from the other bodies and a “self” contribution,

$$U_a = \int_{V_a} d^3 x' \frac{\rho(t, x')}{|x - x'|} + \sum_{b=1, b \neq a}^N \int_{V_b} d^3 x' \frac{\rho(t, x')}{|x - x'|} =: U_a^{(\text{self})} + U_a^{(\text{ext})} . \quad (6.79)$$

The force on body is also separated into a self-contribution and an external contribution; the self-contribution also contains the pressure of the fluid body (see next equation). Because the force is $F \sim Gm^2/r^2$, the external force

is $\sim GM/D^2$ where D is the characteristic distance between the bodies and M the total mass, while the self-force is $\sim GM^2/R_a^2$ where R_a is the characteristic size of body a ; one would estimate that

$$\frac{F_a^{(\text{self})}}{F^{(\text{ext})}} \sim \left(\frac{D}{R_a}\right)^2 \ll 1. \quad (6.80)$$

However, the self force is identically zero because in the integral

$$F_a^{(\text{self})\text{ i}} = \int_{V_a} d^3x' \left(-\partial_i P + \rho \partial_i U_a^{(\text{self})} \right) = 0 - \int d^3x \int d^3x' (x - x')^i \frac{\rho(t, x) \rho(t, x')}{|x' - x|^3} \equiv 0 \quad (6.81)$$

both terms are zero. The first term is zero because the pressure is zero at the surface; the second term vanishes because the integrand is odd under exchange $x' \leftrightarrow x''$, while the domain of integration is invariant under the same exchange. Note the densities ρ in the integrand are conceptually distinct: one is the source of the grav. field (the one in $U^{(\text{ext})}$), while the other is the “passive” density that measure the response to the external field. The external force acting on body a is

$$F_a^{(\text{ext})\text{ i}} = \int d^3x \rho \partial_i U_a^{(\text{ext})}(t, x) = \int d^3y \rho \partial_i U_a^{(\text{ext})}(t, z_a + y) \quad (6.82a)$$

$$= \int d^3y \rho \left[\partial_i U_a^{(\text{ext})}(t, z_a) + y^j \partial_i \partial_j U_a^{(\text{ext})}(t, z_a) + \frac{1}{2} y^j y^k \partial_i \partial_j \partial_k U_a^{(\text{ext})}(t, z_a) + \dots \right] \quad (6.82b)$$

where in the second passage one introduces the center of mass of coordinate $z_a(t)$ and a new coordinate y around it $y = x - z_a$; and in the last passage a multipolar expansion around the center of mass is performed. Hence, the EOM for body a is

$$m_a \frac{d^2 z_a^i}{dt^2} = m_a \partial_i U_a^{(\text{ext})}(t, z_a) + \frac{1}{2} Q_a^{jk} \partial_j \partial_k U_a^{(\text{ext})}(t, z_a) + \dots, \quad (6.83)$$

where Q_a^{jk} is the quadrupole of body a and the dipole term is zero in the center of mass frame. Considering that $\partial U^{(\text{ext})} \sim 1/D^2$ and $Q \sim \epsilon M R^2$ where $\epsilon \leq 1$ is the ellipticity of the body, it is immediate to estimate that $Q \partial^3 U^{(\text{ext})} \sim R^2/D^2$ and

$$\frac{Q_a^{jk} \partial_j \partial_k U_a^{(\text{ext})}}{\partial_i U_a^{(\text{ext})}} \sim \epsilon \frac{R_a^2}{D^2} \ll 1. \quad (6.84)$$

The above formula indicates the order of magnitude of the quadrupole correction to the EOM².

The above remark suggests that for a structured body in a binary system, the field of the companion induces, at leading order, a quadrupole moment that, in turn, modifies the orbit. A Newtonian argument indicate that the correction to the point-mass dynamics due to finite size enter at 5PN order. The Newtonian tidal force exerted by body a on body b of size R when they are separated by a distance D is of order

$$F_{a \rightarrow b}^{(\text{tidal})} \sim \frac{GM R_b}{D^3}. \quad (6.85)$$

Body b is distorted by this force and deforms; the first relevant correction for the dynamics is given by the quadrupole, $Q \sim \epsilon m R^2$. The ellipticity ϵ can be estimated as the ratio of the tidal force to the self-force of the body,

$$\epsilon_b \sim \frac{F_{a \rightarrow b}^{(\text{tidal})}}{F_b^{(\text{self})}} \sim \left(\frac{R_b}{D}\right)^3. \quad (6.86)$$

The quadrupole of body b induces a correction to the force $F_{ab}^k \sim m_b \partial_k U^{(\text{ext})}$ acting on body b . The induced force on body a is of order

$$F_{b \rightarrow a}^{(\text{quad})\text{ k}} \sim \frac{Q_b^{ij}}{m_b} \partial_i \partial_j F_{ab}^k \sim \epsilon \frac{R_b^2}{D^2} F_{ab}^k \quad (6.87)$$

from which

$$\frac{F_{b \rightarrow a}^{(\text{quad})\text{ k}}}{F_{ab}^k} \sim \epsilon \frac{R_b^2}{D^2} \sim \frac{R_b^5}{D^5} \sim \left(\frac{v}{c}\right)^{10}. \quad (6.88)$$

In the last passage one estimates the radius of a compact object as close to Schwarzschild one, and uses the virial theorem $r \sim GM/c^2 \sim v^2/c^2$. The estimate above is correct: tidal effects enter the action at order 5PN and several PN terms have been calculated for the dynamical and for the waveform up to 7.5PN (Damour, 1987; Damour et al., 2012).

The description of tidal interactions in the PN dynamics of self-gravitating and deformable bodies was formulated in a series of works by Damour, Soffel and Xu in the '90s (Damour, 1983; Damour et al., 1991, 1992, 1993, 1994). They

²The interested reader could, at this point, estimate the effect of the Sun's quadrupole moment on the Mercury perielion and compare to the GR correction.

developed a multi-chart approach whereby an *outer problem*, in which the bodies are “skeletonized” as worldlines with global properties, is matched to an *inner problem*, in which the effects of the other bodies in the worldtube around a given body are included. In the case of compact binaries, the inner problem corresponds to the description of the tidal response of a NS due to the external gravitational field of the companion. The matching with the outer problem allows one to include the effect of the tidal deformations on the orbital dynamics and the GW radiation. The presentation here follows closely that of Refs. (Damour and Nagar, 2009, 2010).

A fully relativistic treatment of the *inner problem* was developed in (Hinderer, 2008; Damour and Nagar, 2009; Binnington and Poisson, 2009). In the local frame of body A , the internally-generated mass M_L^A and spin S_L^A multipole moments, $L = i_1 i_2 \dots i_\ell$ being a multi-index, are related to the external gravitoelectric G_L^A and gravitomagnetic H_L^A tidal moments³ by the *tidal polarizability coefficients*,

$$M_L^A = \mu_\ell G_L^A, \quad S_L^A = \sigma_\ell H_L^A. \quad (6.89)$$

The gravitoelectric (gravitomagnetic) coefficient $G_{\mu\ell}$ has dimension $[\text{length}]^{2\ell+1}$ and measures the ℓ -th-order mass (spin) multipolar moment induced in the NS by the external ℓ -th-order gravitoelectric (gravitomagnetic) field. The dimensionless relativistic Love numbers are defined as

$$k_\ell = \frac{(2\ell-1)!!}{2} \frac{G_{\mu\ell}}{R^{2\ell+1}}, \quad j_\ell = \frac{(2\ell-1)!!}{2} \frac{G_{\sigma\ell}}{R^{2\ell+1}}, \quad (6.90)$$

with R being the NS radius. Note that the many works in the literature focus on the dominant quadrupole $\ell = 2$ gravitoelectric coefficient and drop the subscript, e.g. Hinderer (2008). For BH $\mu_\ell^{\text{BH}} = \sigma_\ell^{\text{BH}} = 0$ (Damour and Nagar, 2009; Binnington and Poisson, 2009; Grlebeck, 2015). Nevertheless BH can also be tidally polarized (Damour and Lecian, 2009). In practice, the calculation of the Love numbers reduces to the solution of stationary perturbations of spherical relativistic stars, because it is assumed that the external field varies sufficiently slowly (“adiabatic tides”). The tidal coefficients have a strong dependency on the NS compactness. Thus, Love numbers must be computed in GR, and not in the Newtonian limit. Love numbers depend on the EOS employed to construct the equilibrium NS. Hence, they carry the imprint of the EOS on the binary dynamics.

Finite-size effects are incorporated into the PN two-body dynamics by augmenting the effective action S with the nonminimal (worldline) couplings

$$S_{\text{nonminimal}} = \sum_a \frac{\mu_\ell^a}{2\ell!} \int (G_L^a)^2 ds_a + \frac{\ell \sigma_\ell^a}{\ell! 2(\ell+1)} \int (H_L^a)^2 ds_a. \quad (6.91)$$

The additional term alter the dynamics at 5PN in a way that is linear in the tidal deformations. The tidal contribution to the two-body Lagrangian at leading PN (Newtonian) order contains only the $\ell = 2$ gravitoelectric terms and reads

$$L_{\text{tidal}}^{\text{LO}} = k_2^A G M_B^2 \frac{R_A^5}{r^6} + (A \leftrightarrow B), \quad (6.92)$$

where r is the separation between the stars in the binary. Eq. (6.92) indicates that tidal corrections are attractive and short range. The effect of tides can be illustrated considering the modification to the Kepler law given by the quadrupolar gravitoelectric term

$$\Omega^2 r^3 = GM \left[1 + \dots + 12 \frac{M_A}{M_B} \frac{R_A^5}{r^5} k_2^A + (A \leftrightarrow B) \right]. \quad (6.93)$$

At a given radius the frequency is higher if the tidal interactions are present. In other words, the motion is accelerated by tidal effects and the system merges earlier and at a lower frequency. The contact GW frequency of the two NS can be estimated setting $r = R_A + R_B$ and finding $2GM\Omega \simeq 2(M_B/(MC_B) + M_B/(MC_B))^{-3/2}$ (Damour and Nagar, 2010). For equal masses the latter relation translates to

$$f_{\text{GW}}^{\text{contact}} \simeq 1.327 \left(\frac{C}{0.15} \right)^{3/2} \left(\frac{M}{2.8M_\odot} \right) \text{ kHz}. \quad (6.94)$$

Simulations show that the contact between the two NS happens approximately 2–4 GW cycles prior to merger at an even lower frequency $f_{\text{GW}}^{\text{NR,contact}} \simeq 700 (M/2.8M_\odot) \text{ Hz}$ (Bernuzzi et al., 2012).

³The tidal moments are defined as the symmetric-trace-free projection of the derivatives of the externally-generated parts of the local gravitoelectric $\bar{E}_a$ and gravitomagnetic fields $\bar{B}_a$, e.g. $G_L^A = \partial_{(L-1} \bar{E}_{a\ell)}^A|_{X^a \rightarrow 0}$, where X^a are local coordinates (Damour and Nagar, 2009).

7. Perturbations of spherical spacetimes

Basic ideas on the gauge-invariant perturbations framework, Regge-Wheeler-Zerilli equations, quasi-normal-modes solutions, stability of black holes.

Suggested readings. *Chap. 4 of Chandrasekhar (1985). Reviews: Kokkotas and Schmidt (1999); Nollert (1999); Nagar and Rezzolla (2005); Berti et al. (2009).*

7.1 Gauge invariant perturbation theory

Linear perturbations of spherical spacetimes describe

- Gravitational wave propagation on curved background;
- Stability of black holes solutions and spherical star equilibria;
- Deformation of spherical stars and relativistic Love numbers;
- Gravitational waves from black holes and neutron star out of equilibrium;
- Dynamics and GWs from small objects moving in black hole spacetime (in the test mass limit).

In linearized gravity around flat spacetime, one choses the Mikowski background metric in some global inertial coordinates and considers small perturbations in the sense that the components of the perturbed metric in those coordinates are smaller than those of the the background. Infinitesimal coordinate transformations are still possible as far as the coordinates' variation does not change the background metric and it is of the same order as the perturbation. This introduces the problem of gauge fixing the linearized theory.

For spherically symmetric spacetimes one can proceed similarly taking, for example, the Schwarzschild metric for the background and explicitly fixing the gauge. This was the first approach to the problem by Regge and Wheeler (1957); Zerilli (1970), where the *Regge-Wheeler (RW)* gauge was defined. Moncrief (1974a,b) and other authors afterwards have later proposed a gauge-invariant approach to the perturbation problem (Gerlach and Sengupta, 1979, 1980; Gundlach and Martin-Garcia, 2000; Martin-Garcia and Gundlach, 2001). Under an infinitesimal coordinate transformation

$$x^\mu \mapsto x^{\mu'} = x^\mu + \xi^\mu , \quad (7.1)$$

the perturbative tensor field changes as $\mathbf{h} \mapsto \mathbf{h}' = \mathbf{h} + \mathcal{L}_\xi \mathbf{h}$ and it is called gauge invariant iff

$$\mathcal{L}_\xi \mathbf{h} = 0 . \quad (7.2)$$

Starting from a general infinitesimal coordinate transformation, Gerlach and Sengupta (1979, 1980) cast all the possible linear perturbations into gauge-invariant quantities and write down the linearized Einstein equations for these gauge-invariant perturbation (See Sec. II of Gundlach and Martin-Garcia (2000) for a review of the formalism).

The main steps to compute spherical spacetime perturbations are:

1. *2+2 split.* Write the general spherically symmetric spacetime as a manifold $\mathcal{M} = M^2 \times S^2$, where M^2 is a Lorentzian 2D manifold with metric $g_{AB}(t, r)$ ($A, B, \dots = 0, 1$) and S^2 is the 2-sphere with metric $\gamma_{ab}(\theta, \varphi)$ ($a, b, \dots = 2, 3$ in this chapter) such that $g_{\mu\nu} = \text{diag}(g_{AB}, r^2 \gamma_{ab})$. Introduce covariant derivatives in M^2 and S^2 and a analogous decomposition of the stress-energy tensor. The spherically symmetric EFE are then re-written as equations for g_{AB} for the M^2 manifold.
2. *Even/odd and multipole decomposition.* Linear perturbation are decomposed into scalar, vector or tensor fields on M^2 times spherical harmonic scalar, vector or tensor fields on S^2 . Under a parity transformation

$$\theta \mapsto \pi - \theta , \quad \varphi \mapsto \varphi + \pi , \quad (7.3)$$

the perturbations pick either a factor $(-1)^\ell$ (even) or a factor $(-1)^{\ell+1}$ (odd). Because the background metric is invariant under a parity transformation, the two sectors do not mix. Thus, the generic perturbations decouple into *even* (or *polar*) and *odd* (or *axial*).

There are 3 odd and 7 even metric and matter perturbations. There are 1 odd and 3 even infinitesimal coordinate transformation. Thus, there are $3 - 1 = 2$ odd gauge-invariant perturbations, in the form of a vector field on M^2 , and $7 - 3 = 4$ even gauge-invariant metric perturbations, in the form of a symmetric tensor and a scalar.

The gauge-invariant formalism naturally specifies to the RW gauge, but has the advantage to exclude gauge-dependent contributions by construction.

The perturbations are not spherically symmetric (they are general) but their angular dependence is fully encoded in the tensor spherical harmonics in which they are expanded. Linearized EFE equations in M^2 apply for each multipole (ℓ, m) of these perturbative fields. Note however that gauge-invariant metric perturbations are possible only for $\ell \geq 2$.

3. *Gravitational waves.* The GW degrees of freedom in this formalism are encoded in the odd and even-parity Regge-Wheeler-Zerilli (RWZ) master functions $\Psi_{\ell m}^{(o/e)}$ satisfying wave equations on M^2 for each (ℓ, m) multipole with characteristic potentials determined by the background. The GW in the TT gauge far from the source can be expressed in terms of the RWZ master functions as

$$h_+ - ih_\times = \frac{G}{c^4 r} \sum_{\ell=2} \sum_{m=-\ell}^{\ell} \sqrt{\frac{(\ell+2)!}{(\ell-2)!}} {}^{-2}Y_{\ell m}(\theta, \varphi) \left(\Psi_{\ell m}^{(e)} + i\Psi_{\ell m}^{(o)} \right) + \mathcal{O}(1/r^2) \quad (7.4)$$

where ${}^{-2}Y_{\ell m}$ are the spin weighted $s = -2$ spherical harmonics. The result above is obtained by switching to Cartesian components for the perturbation and extracting the leading-order $1/r$ asymptotic behaviour.

2+2 split. □ [TODO: Write me.]

Even/odd perturbation of spherical spacetimes. Tensor perturbations of spherical spacetimes are naturally developed in tensor spherical harmonics and spherical coordinate taking advantage of the symmetries of the background. The tensor spherical harmonics needed here are a generalization of those discussed in Chap. 5. Maggiore (2018) (Sec. 12.2) discusses their explicit construction that consists in (i) an extension to four indexes of the $(\mathbf{Y}_{\ell m}^x)_{ij}$ with $x = S0, E1, E2, B1, B2$ ¹, (ii) their expression in terms of derivatives of the scalar $Y_{\ell m}$ (extension of Eq. (5.61)), and (iii) change of coordinate to spherical components. The computations are trivial but lengthy and are not reported here.

The tensor spherical harmonics form a complete basis on the 2-sphere to expand a tensor field of any rank. The spherical harmonics $Y_{\ell m}$ form a basis for scalar fields on S^2 . A vector basis can be constructed from them by taking the covariant derivative of the scalar spherical harmonics on S^2 (indicated as ${}_{|a}$),

$$Z_a^{\ell m} := Y_{|a}^{\ell m} ; \quad S_c^{\ell m} := \epsilon_{cd} \gamma^{de} Y_{|e}^{\ell m} , \quad (7.6)$$

where the first (second) expression should be used for even (odd) perturbation, see below. Above ϵ is the volume form in the 2-sphere, $\epsilon^{ac}\epsilon_{bc} = \gamma_{ab}$. A basis for even and odd symmetric tensors on S^2 are given by

$$\gamma_{ab} Y^{\ell m} , \quad Z_{ab}^{\ell m} := Y_{|ab} + \frac{\ell(\ell+1)}{2} \gamma_{ab} Y_{\ell m} ; \quad S_{ab}^{\ell m} := \frac{1}{2} (S_{a|b}^{\ell m} + S_{b|a}^{\ell m}) . \quad (7.7)$$

Note the even basis is formed by a pure trace and a traceless tensor. Explicit expressions in terms of the partial derivatives of $Y_{\ell m}(\theta, \varphi)$ can be found in e.g. Appendix A of Nagar and Rezzolla (2005).

The tensor spherical harmonics components in spherical coordinates for tensors on $\mathcal{M} = M \times S^2$ are then expressed in terms of the above quantities. Explicit expression can be found in Sec. 12.2.1 of Maggiore (2018). Advanced software for symbolic algebraic computation has been developed to this purpose² A generic tensor perturbation of spherical spacetime expanded in tensor spherical harmonics decomposes in even and odd

$$h_{\mu\nu} = h_{\mu\nu}^{(e)} + h_{\mu\nu}^{(o)} \quad (7.8)$$

where the even expansion contains contributions from $(\mathbf{Y}_{\ell m}^x)_{\mu\nu}$ with $x = tt, Et, Rt, L0, T0, E1, E2$ and the odd expansion with $x = Bt, B1, B2$. The metric perturbation is a special case. In terms of the basis functions for the 2-sphere discussed above, each term in the even and odd multipolar series is

$$(h_{\ell m}^{(e)})_{\mu\nu} = \left[\begin{array}{cc|c} H_0 Y_{\ell m} & H_1 Y_{\ell m} & h_A^{(e)} Z_a^{\ell m} \\ * & H_2 Y_{\ell m} & \\ \hline * & & r^2 K Y_{\ell m} \gamma_{ab} + r^2 G Z_{ab}^{\ell m} \end{array} \right] , \quad (h_{\ell m}^{(o)})_{\mu\nu} = \left[\begin{array}{c|c} 0 & h_A^{(o)} S_a^{\ell m} \\ * & h S_{ab}^{\ell m} \end{array} \right] \quad (7.9)$$

¹For the 00 component of a rank-2 tensor one uses the scalar $\delta_\mu^0 \delta_\nu^0 Y_{\ell m}$, while for the 0i component one uses the vector spherical harmonics. The basis function is still denoted as $\mathbf{Y}_{\ell m}^x{}_{ij}$ but with

$$x = tt, Et, Bt, Rt, L0, T0, E1, E2, B1, B2 . \quad (7.5)$$

The extension is here indicated as $(\mathbf{Y}_{\ell m}^x)_{\mu\nu}$ with an abuse of notation. These quantities correspond to the $(\mathbf{t}_{\ell m}^x)_{\mu\nu}$ of Sec. 12.2 in Maggiore (2018).

²The **Mathematica** package for computer algebra **xPert** is specifically design for metric perturbation theory. (See <http://www.xact.es/xPert>).

The even/odd perturbations are thus described by the functions (indexes ℓm are omitted for brevity)

$$(H_0, H_1, H_2, h_A^{(e)}, K, G), \quad (h_A^{(o)}, h), \quad (7.10)$$

Note the gauge invariant quantities are not these but suitable combinations. The relevant gauge-invariant combinations for the following are

$$k_A := h_A^{(o)} - h_{|A} + \frac{2h}{r} r_{|A}, \quad \kappa_1 := K + e^{-2b} \left(r \partial_r G - \frac{2}{r} h_1^{(e)} \right), \quad \kappa_2 := \frac{1}{2} [e^{-2b} H_2 - e^b \partial_r (r e^b K)], \quad (7.11)$$

where the background metric of M^2 is $\bar{g} = -e^{2a} dt^2 + e^{2b} dr^2$. A similar decomposition holds for the stress-energy tensor.

7.2 Regge-Wheeler-Zerilli (RWZ) equations

The Regge-Wheeler-Zerilli equations describe perturbation of Schwarzschild black hole. The background metric is taken with $e^{2a} = e^{-2b} = (1 - R_S/r) = (1 - 2GM/(c^2 r)) =: A(r)$ in Schwarzschild coordinates and we will make use of the tortoise coordinate

$$r_* = r + R_s \ln \left(\frac{r}{R_s} - 1 \right), \quad (7.12)$$

that maps the Schwarzschild exterior $[R_s, \infty)$ to $(-\infty, +\infty)$.

Let $\Lambda = \ell(\ell + 1)$, the odd (RW) and even (Zerilli) master functions for each multipoles are defined as the gauge-invariant quantities (ℓm are omitted for brevity)

$$\Psi^{(o)} := \frac{r}{\Lambda} \left[\partial_t h_1^{(o)} - r^2 \partial_r \left(\frac{h_0^{(o)}}{r^2} \right) \right], \quad \Psi^{(e)} := \frac{r^2 (\Lambda \kappa_1 + 4e^{-4b} \kappa_2)}{\Lambda [r(\Lambda - 2) + 6M]}. \quad (7.13)$$

These functions have dimensions of length. The linearized EFE lead, for both odd and even master functions, to wave equations of type

$$-\Psi_{tt}^{\ell m} + \Psi_{r_* r_*}^{\ell m} - V_\ell(r_*) \Psi^{\ell m} = S_{\ell m}, \quad (7.14a)$$

where the potentials are

$$V_\ell^{(o)} := A(r) \left(\frac{\Lambda}{r^2} - \frac{6M}{r^3} \right) \quad (7.14b)$$

$$V_\ell^{(e)} := A(r) \frac{\Lambda(\Lambda - 2)^2 r^3 + 6(\Lambda - 2)^2 M r^2 + 36(\Lambda - 2) M^2 r + 72 M^3}{r^3 ((\Lambda - 2)r + 6M)^2} \quad (7.14c)$$

These potentials are written in terms of the Schwarzschild radial coordinate not in terms of r_* . The potentials $V_\ell^{(e/o)}(r_*)$ decays exponentially near the horizon ($r_* \rightarrow \infty$) and as $1/r_*^2 \sim 1/r^2$ for $r_* \rightarrow \infty$; and they have maximum for $r_* \sim 2M$, Fig. 7.1. Chandrasekhar (1985) has shown that one can transform the odd equation to the corresponding one for even modes via a transformation involving differential operations. It can also be shown that both forms are connected to the Bardeen-Press perturbation equation derived via the Newman-Penrose formalism. The source term $S_{\ell m}$ comes from the perturbation stress-energy tensor. Here we mostly discuss the vacuum case $S_{\ell m} \equiv 0$, but a notable example of nonzero source is a particle perturbation. Note also that different conventions for the RWZ functions are used in the literature; we follow Nagar and Rezzolla (2005) where the different notations are also summarized.

Example 7.2.1. *An equation similar to the RWZ can be easily obtained for scalar perturbations of Schwarzschild by considering the box operator of the background metric,*

$$0 = \square_{\bar{g}} \phi = (-\bar{g})^{1/2} \partial_\mu ((-\bar{g})^{1/2} \bar{g}^{\mu\nu} \partial_\nu \phi). \quad (7.15)$$

Expanding the solution in spherical harmonics

$$\phi(t, r, \theta, \varphi) = \frac{1}{r} \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \phi_{\ell m}(t, r) Y_{\ell m}(\theta, \varphi), \quad (7.16)$$

one finds that the coefficients $\phi_{\ell m}(t, r)$ satisfy the same equation as for odd tensor perturbation with a very similar potential (and $S_{\ell m} \equiv 0$).

7.3 Asymptotic waves

Here we connect the odd and even-parity master function to the plus and cross polarization amplitudes of the GWs in the TT gauge in 4 steps.

1. Transform the metric to Cartesian components $h_{\hat{\mu}\hat{\nu}}$,

$$h_{\hat{\mu}\hat{\nu}} = e_{\hat{\mu}}^{\mu} e_{\hat{\nu}}^{\nu} h_{\mu\nu} , \quad \text{with } \mathbf{e} = \text{diag}(e^a, e^{-b}, r^{-1}, (r \sin^{-1} \theta)) ; \quad (7.17)$$

2. Require that the TT Cartesian components $h_{\hat{\theta}\hat{\theta}}, h_{\hat{\theta}\hat{\varphi}}, h_{\hat{\varphi}\hat{\varphi}}$ behave $\sim f(t-r)/r$ and satisfy the traceless condition $h_{\hat{\theta}\hat{\theta}} + h_{\hat{\varphi}\hat{\varphi}} = 0 + \mathcal{O}(1/r)$, while the others are $\mathcal{O}(1/r^2)$;
3. Constrain the leading-order dependency of the metric perturbation in Eq. (7.10) for large r ;
4. For both odd and even parity perturbations take the plus and cross components as

$$h_+^{(o/e)} := \frac{1}{2} \left(h_{\hat{\theta}\hat{\theta}}^{(o/e)} + h_{\hat{\varphi}\hat{\varphi}}^{(o/e)} \right) , \quad h_{\times}^{(o/e)} := h_{\hat{\theta}\hat{\varphi}}^{(o/e)} . \quad (7.18)$$

Comparing the expression of the Cartesian components in term of the metric perturbation to the assumed asymptotic expressions leads to

$$h \sim \mathcal{O}(r) , \quad h_A^{(o)} \sim \mathcal{O}(1/r) ; \quad H_2 \sim \mathcal{O}(1/r^2) , \quad h_1^{(e)} \sim \mathcal{O}(1/r) , \quad K \sim \frac{\Lambda}{2} G + \mathcal{O}(1/r^2) , \quad rG \sim \mathcal{O}(1) . \quad (7.19)$$

At this point one needs to connect the perturbations above to the master function making use of the gauge-invariant equations. Specifically, for the odd parity one can use the linearized Hamiltonian constraint while for the even parity it is sufficient to calculate the asymptotic expression of the $\kappa_{1,2}$ quantities. The result is

$$\partial_t h = e^a \partial_r (r \Psi^{(o)}) + h_0^{(o)} \Rightarrow \frac{h}{r} \sim -\Psi^{(o)} + \mathcal{O}(1/r) \quad (7.20a)$$

$$\Psi^{(e)} \sim \frac{r}{2} G + \mathcal{O}(1/r^2) \quad (7.20b)$$

The asymptotic expressions of the master functions are also found $\Psi^{(e/o)} \sim 1$ (although they have length dimension, they do not asymptotically depend on the radius). Putting together everything as per 4. one gets

$$h_+^{(o)} := \frac{1}{2} \left(h_{\hat{\theta}\hat{\theta}}^{(o)} + h_{\hat{\varphi}\hat{\varphi}}^{(o)} \right) = \frac{h}{r^2} \left(S_{\theta|\theta}^{\ell m} - \frac{1}{\sin^2 \theta} S_{\varphi|\varphi}^{\ell m} \right) + \mathcal{O}(1/r^2) \quad (7.21a)$$

$$h_{\times}^{(o)} := h_{\hat{\theta}\hat{\varphi}}^{(o)} = \frac{h}{r^2} \frac{1}{\sin \theta} S_{(\varphi|\theta)}^{\ell m} + \mathcal{O}(1/r^2) \quad (7.21b)$$

$$h_+^{(e)} := \frac{1}{2} \left(h_{\hat{\theta}\hat{\theta}}^{(e)} + h_{\hat{\varphi}\hat{\varphi}}^{(e)} \right) = \frac{G}{2} \left(Y_{|\theta\theta}^{\ell m} - \frac{1}{\sin^2 \theta} Y_{|\varphi\varphi}^{\ell m} \right) + \mathcal{O}(1/r^2) = \frac{G}{2} W^{\ell m} + \mathcal{O}(1/r^2) \quad (7.21c)$$

$$h_{\times}^{(e)} := h_{\hat{\theta}\hat{\varphi}}^{(e)} = G \frac{1}{\sin \theta} Y_{|\theta\varphi}^{\ell m} + \mathcal{O}(1/r^2) = \frac{G}{2} \frac{X^{\ell m}}{\sin \theta} + \mathcal{O}(1/r^2) . \quad (7.21d)$$

As a final step it is useful to define

$$W^{\ell m} := \frac{1}{\sin \theta} S_{(\varphi|\theta)}^{\ell m} = \partial_{\theta\theta} Y_{\ell m} - \cot \theta \partial_{\theta} Y_{\ell m} - \frac{1}{\sin^2 \theta} \partial_{\varphi\varphi} Y_{\ell m} \quad (7.22a)$$

$$X^{\ell m} := -\sin \theta \left(S_{\theta|\theta}^{\ell m} - \frac{1}{\sin^2 \theta} S_{\varphi|\varphi}^{\ell m} \right) \quad (7.22b)$$

and the *spin weighted spherical harmonics of spin $s = -2$*

$${}_2 Y_{\ell m}(\theta, \varphi) := \sqrt{\frac{(\ell-2)!}{(\ell+2)!}} \left(W_{\ell m}(\theta, \varphi) - i \frac{X_{\ell m}(\theta, \varphi)}{\sin \theta} \right) , \quad (7.23)$$

from which the famous result anticipated in Eq. (7.4) follows.

Remark 7.3.1. *In general the spin weighted spherical harmonics of spin s are defined as*

$${}_s Y_{\ell m}(\theta, \varphi) := e^{im\varphi} (-1)^2 \sqrt{\frac{2\ell+1}{4\pi}} d_{m,-s}^{\ell}(\theta) \quad (7.24)$$

where $d_{m,-s}^{\ell}$ are the Wigner small-d matrices. They generalize spherical harmonics for scalar functions on S^2 to functions on S^2 with spin weight s . The latter transforms under a rotation of an angle α about the poles $F \mapsto e^{is\alpha} F$. These functions represent fields with an additional $U(1)$ symmetry.

Remark 7.3.2. Post-Newtonian expression of the multipolar waveform. The multipoles $h_{\ell m}$ in Eq. (7.4) can be expressed as a PN series using the formalism of Chap. 5 and Chap. 6. The starting point are the expressions of the TT waveforms in terms of STF multipoles (Eq. (6.44)) and spherical tensor harmonics (Eq. (5.63)). Recalling the relations between STF tensors and spherical components of Chap. 5, the relations between the radiative multipole moments in Eq. (6.44) and the coefficients of the tensor spherical harmonics expansion in Eq. (5.63) are

$$u_{\ell m} \propto U_L \mathcal{Y}_{\ell m}^L, \quad v_{\ell m} \propto V_L \mathcal{Y}_{\ell m}^L, \quad (7.25)$$

where the constants and ℓ -dependent factors are omitted. It is now sufficient to recall the expression of the spherical harmonics in terms of the STF spherical harmonics for a given unit vector N^i , Eq. (5.23)

$$Y_{\ell m} = \mathcal{Y}_{\ell m}^L N_L, \quad (7.26)$$

and combine Eq. (7.4), Eq. (5.63), Eq. (5.23), and Eq. (5.63) to find that the multipoles $h_{\ell m}$ are a combination of the multipoles of the tensor spherical harmonics decomposition Kidder (2008)

$$h_{\ell m} = \frac{G}{\sqrt{2} R c^{\ell+2}} \left(u_{\ell m} - \frac{i}{c} v_{\ell m} \right). \quad (7.27)$$

To lowest order the $u_{\ell m}, v_{\ell m}$ are given by Eq. (5.68) and Eq. (5.69). In particular,

$$u_{2m} \propto \frac{d^2}{dt^2} \int d^3x r^2 T^{00} Y_{\ell m}^*, \quad (7.28)$$

thus indicating the $\ell = 2$ terms to lowest order reproduce the quadrupole formula.

7.4 Initial-boundary value problem (IBVP) for the RWZ

We now discuss the IBVP for the gravitational perturbations.

Since the RWZ potential goes to zero for $r_* \rightarrow \pm\infty$; the RWZ equation without sources reduces to the a one dimension “free” wave equation for large $|r_*|$. The most general asymptotic solutions at large radii $r_* \rightarrow \infty$ is thus described by the linear combination of ingoing and outgoing wave on Mikowski spacetime. For example, if the solution is composed of a single Fourier mode, $\Psi \sim e^{-\omega i(t \pm r_*)}$, then the outgoing to (ingoing from) infinity is solution the - (+) solution. The asymptotic general solution near the horizon ($r_* \rightarrow -\infty$) is also described by the superposition of an ingoing and an outgoing wave. Because no signals can come from the the horizon, the physical boundary condition to be imposed $r_* \rightarrow -\infty$ is that the solution is only composed of the ingoing wave³. The typical condition imposed at $r_* \rightarrow \infty$ is instead that the solution is only composed of the outgoing wave.

Further insights on the boundary conditions follow by introducing the Fourier transform in time $\tilde{\Psi}(\omega, r_*) = \mathcal{F}[\Psi(t, r_*)]$. Letting $x = r_*/c$, the RWZ equation becomes

$$\tilde{\Psi}_{xx} + (\omega^2 - V_\ell(x)) \tilde{\Psi} = 0. \quad (7.29)$$

The boundary conditions discussed above translate into the requirements

$$\tilde{\Psi} \sim e^{\pm i\omega x}, \quad x \rightarrow \pm\infty. \quad (7.30)$$

for each Fourier mode. Eq. (7.29) is similar to the Schroedinger equation for stationary states. The choice of “the others” boundary conditions, namely in/out-going for $x \rightarrow \pm\infty$, would imply $\tilde{\Psi} = 0$ at the boundaries, and would map into a Schroedinger problem for bound states. But because the RWZ potential is positive, such bound states cannot exist (for any ω^2).

The physical problem described by the RWZ with these boundary conditions is the following. At an initial time one has a nonzero profile (an external perturbation or a matter source term in case $S_{\ell m} \neq 0$) of $\Psi(t = 0, r_*)$ localized on a spatial interval of the r_* domain. The solution of the IBVP determines how the wave propagates on the Schwarzschild background (the V_ℓ potential) up to infinity and down to the horizon. The qualitative behaviour of the solution can be expected from a basic argument. Imagine the initial profile has compact spatial support on the right of the potential’s peak and sufficiently far away such that V_ℓ is small. If $\partial_t \Psi(t = 0, r_*) = 0$, the solution will initially propagate by separating in the in/outgoing waves (characteristics) on the approximately flat spacetime. The outgoing wave will reach infinity by keeping its amplitude approximately constant. A detector at very large radius would thus record a first burst of radiation. The ingoing wave will interact with a larger and larger potential as time advances; part of it will continue down to the horizon but part of it could be reflected back to infinity⁴. This behaviour can be also interpreted as a wave on flat background with an effective refraction index $\omega_{\text{eff}}^2 = \omega^2 - V_\ell$. The detector at very large radius would thus record a second burst of radiation different from the previous because modified by its interaction with the potential Fig. 7.1. We shall see below that this burst can carry information about the black hole (potential).

³This boundary condition also follows from requiring smoothness of the solution.

⁴One would be tempted to consider an analogy to the Schroedinger for scattering states. In those problems, however, one considers incoming waves from infinity that are not compatible with the outgoing boundary condition discussed above. A different boundary should be used for scattering.

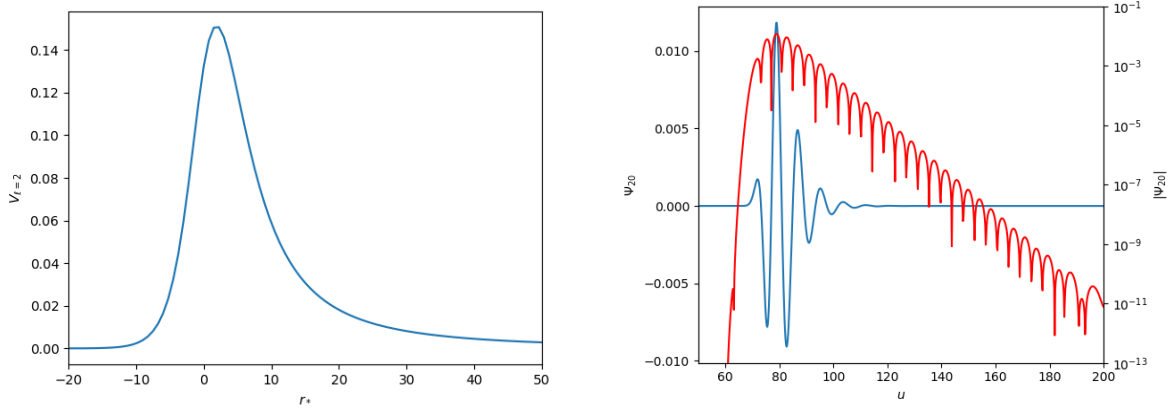


Figure 7.1: RW potential for $\ell = 2$ and $M = 1$ as function of the tortoise coordinate (left) and time domain solution for $\Psi_{20}^{(o)}$ in vacuum as a function of the retarded time $u = t - r_*$ given a time-symmetric pulse as initial data (right). The solution is composed by precursor-burst and the ringdown (power law tails are not visible in the plot).

7.5 Quasi-normal-modes (QNMs)

Quasi-normal modes (QNMs) are damped and oscillating (exponentials with complex frequencies) solutions of a large class of mathematical problems describing physical “open systems”. Examples of such systems are waves on an infinite string, resonances in a particles scattering experiment or gravitational emission from vibrating black holes. Note the spectrum of the differential operators describing there problems is is purely continuous. QNMs are reviewed by many authors (Kokkotas and Schmidt, 1999; Nollert, 1999; Berti et al., 2009).

In perturbations of black holes, QNMs complex frequencies carry information on the mass (and spin) of the background BH solutions. Since these frequencies characterize the GW, they are observable and are of primary importance to probe black-hole spacetimes. The QNMs of the RWZ master function are visible in Fig. 7.1 as the exponentially damped oscillations starting at $u \sim 80$. QNMs can be studied considering the Fourier transform of the the RWZ. In this approach, the boundary conditions discussed above are imposed on the general solution of the RWZ for $\tilde{\Psi}$ and they select these discrete frequencies in a similar way the regularity/junctions condition determine the quantum levels of a quantum mechanical system described by the Schroedinger equation for stationary states. Another approach, is to work with the Lapace transform, a general method that allows one to solve the initial-value problem via Green functions including the initial data. The QNM complex frequencies emerge here as complex poles of the Green function (see below).

Let us start from the Fourier approach for the RWZ (Leaver, 1985). Using the Schwarzschild radial coordinates rescaled to the black hole mass (equivalent to let $M = 1$) the frequency-domain RWZ equation writes

$$r(r-1)\tilde{\Psi}_{rr} + \tilde{\Psi}_r + \left[\frac{\omega^2 r^2}{r-1} - \Lambda + \frac{\epsilon}{r} \right] \tilde{\Psi} = 0, \quad (7.31)$$

where $\epsilon = -1, 0, 3$ for scalar, electromagnetic and gravitational perturbations. The equation above is known as the *generalized spheroidal wave equation*. The boundary conditions discussed above in the Schwarzschild radial coordinates are

$$\Psi \sim (r-1)e^{-i\omega}, \quad r \rightarrow 1; \quad \tilde{\Psi} \sim e^{i\omega(r+\ln r)}, \quad r \rightarrow +\infty. \quad (7.32)$$

The problem admits a general solution given by the series

$$\tilde{\Psi} = (r-1)^s r^{-2s} e^{-s(r-1)} \sum_{n=0}^{\infty} a_n \left(\frac{r-1}{r} \right)^n, \quad (7.33a)$$

where $s := -i\omega$ and the series' prefactor incorporates the boundary conditions that are satisfied if the series converges. The a_n coefficients satisfy the recursion relations

$$a_0 = 1, \quad \alpha_0 a_1 + \beta_0 a_0 = 0, \quad \alpha_n a_{n+1} + \beta_n a_n + \gamma_n a_{n-1} = 0, \quad (7.33b)$$

with

$$\alpha_n = n^2 + (2s+2)n + 2s+1 \quad (7.33c)$$

$$\beta_n = -(2n^2 + (8s+2)n + 8s^2 + 4s + \Lambda - \epsilon) \quad (7.33d)$$

$$\gamma_n = n^2 + 4sn + 4s^2 - \epsilon - 1. \quad (7.33e)$$

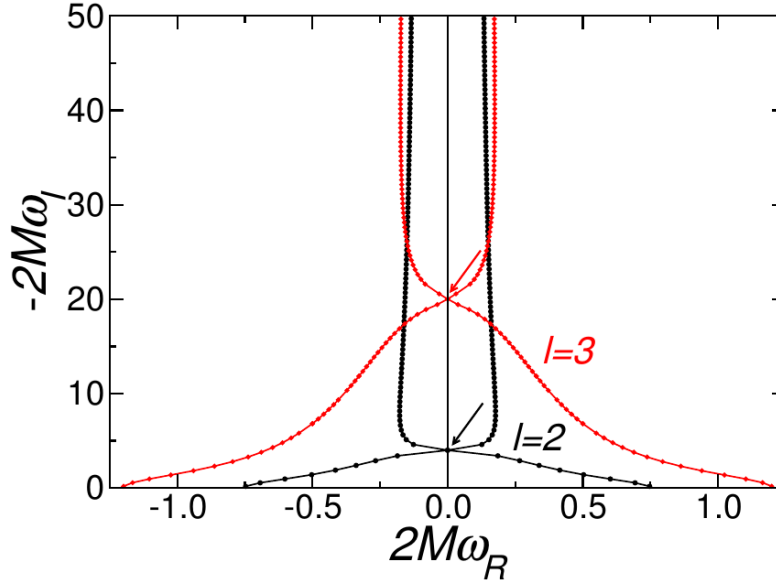


Figure 7.2: The spectrum of QNM for a Schwarzschild black-hole, for $\ell = 2$ (diamonds) and $\ell = 3$ (crosses). The arrow indicate the purely imaginary frequency. From Berti et al. (2009).

A further condition to impose is that the series converges and it is finite. The behaviour of the series at infinity

$$\frac{a_{n+1}}{a_n} \sim 1 \pm \sqrt{\frac{2s}{n}} + \frac{2s - 3/4}{n} + \dots \quad n \rightarrow \infty, \quad (7.34)$$

indicates that the uniform convergence is obtained only with the minus sign. This condition selects particular values of s in the following way. For a given frequency s , the three-term recurrence relation for a_n possesses two linearly independent solutions. The convergence condition of the series translates into the convergence of the continued fraction,

$$\frac{a_{n+1}}{a_n} = \frac{-\gamma_{n+1}}{\beta_{n+1} - \frac{\alpha_{n+1}\gamma_{n+2}}{\beta_{n+2} - \frac{\alpha_{n+2}\gamma_{n+3}}{\beta_{n+3} + \dots}}} \quad (7.35)$$

and, in particular, a_n is the minimal solution, i.e. a solution that converges to zero for $n \rightarrow \infty$, for the values of s that satisfies the $n = 0$ continued fraction equation. The QNM are then defined by the infinite roots s of the (infinite) continued fraction equation

$$0 = -\frac{\beta_0}{\alpha_0} = \frac{a_1}{a_0} = \frac{-\gamma_1}{\beta_1 - \frac{\alpha_1\gamma_2}{\beta_2 - \frac{\alpha_2\gamma_3}{\beta_3 + \dots}}} \quad (7.36)$$

Their explicit determination require to truncate the recursion and a numerical method to find the root.

Figure 7.2 shows some of the modes of the Schwarzschild black hole. Tables and scripts to compute them are available on e.g. Berti's website (Berti et al., 2009). The number of modes (overtones) for each harmonic index ℓ is infinite Bachelot and Motet-Bachelot (1993). The imaginary part of the frequency grows very quickly (this is also true for the higher ℓ modes): this means that the higher modes do not contribute significantly in the emitted gravitational wave signal. Note the real part is not monotonically decreasing with the overtones and it is largest for the fundamental tone. This is in contrast to the picture of a vibrating and damped string for which the fundamental mode has the lowest frequency and longest damping time and the overtone have a monotonic dependence. An interpretation of QNMs in terms of forced harmonic oscillator modes is discussed in Maggiore (2008). There is also a special purely imaginary QNM frequency whose value increases with ℓ ⁵.

Example 7.5.1. A toy model reproducing the key feature of QNMs is the RWZ master equation with the Pöschl-Teller potential

$$V(x) = \frac{V_0}{\cosh^2(\alpha x - x_0)} \quad (7.37)$$

where $V_0 = V(x_0)$ is the maximum of the RWZ potential and $\alpha^2 := -V''(x_0)/(2V_0)$. This potential is rapidly decaying for $x \rightarrow \pm\infty$ and allows one to find the QNM analytically. Introduce now a new independent variable and a rescale the master function as

$$\xi := (1 + e^{-2\alpha(x-x_0)})^{-1}, \quad y := \Psi/(\xi(1-\xi))^{i\omega/2\alpha}. \quad (7.38)$$

⁵The curious reader can find more about this starting from the referenced reviews.

Near infinity ($\xi = 1$) $1 - \xi \sim e^{-2\alpha(x-x_0)}$ and near the horizon ($\xi = 0$) $\xi \sim e^{2\alpha(x-x_0)}$. Eq. (7.29) reduces to a hypergeometric equation

$$\xi(1-\xi)\partial_{\xi\xi}y + [c - (a_+ + a_- + 1)\xi]\partial_{\xi}y - a_+a_-y = 0, \quad (7.39)$$

where $a_{\pm} = [\alpha \pm \sqrt{\alpha^2 - 4V_0 - 2i\omega}]/(2\alpha)$, $c = 1 - i\omega/\alpha$. The solution is therefore expressed via the hypergeometric series $F(a, b, c, z)$ as

$$\Psi = A\xi^{i\omega/2\alpha}(1-\xi)^{-i\omega/2\alpha}F(a_+ - c + 1, a_- - c + 1, 2 - c, \xi) + B(\xi(1-\xi))^{-i\omega/2\alpha}F(a_+, a_-, c, \xi), \quad (7.40)$$

where A, B are two constants determined by the properties of the hypergeometric function and by the boundary conditions. Because $F(a, b, c, 0) = 1$ and $\xi e^{i\omega/2\alpha} \sim e^{i\omega x}$, the boundary condition at the horizon implies that $A \equiv 0$. To impose the boundary condition at infinity one observes that $\Psi \sim B e^{i\omega x} F$, and thus the latter translates in to a sort of “quantization rule” for the parameter $q_{\pm}$. By considering the transformation of $F(a, b, c, \xi)$ under $\xi \rightarrow 1 - \xi$, (Abramowitz and Stegun, 1964),

$$F(a, b, c, z) = (1-z)^{c-a-b} \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)} F(c-a, c-b, c-a-b+a, 1-z) + \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} F(a, b, -c+a+b+1, 1-z), \quad (7.41)$$

the “ B ” term of the solution is consistent with the boundary condition if it is a constant. Since $c - a_+ - a_- = i\omega/\alpha$ and $(1-\xi)^{c-a_+-a_-} \sim e^{-i\omega x}$, one must demand that $1/\Gamma(a_{\pm}) = 0$. The QNM of the Pöschl-Teller potential follows from this condition,

$$\omega = \pm \sqrt{V_0 - \alpha^2/4 - i\alpha(2n+1)/2}, \quad n = 0, 1, 2, \dots \quad (7.42)$$

The first overtone approximate rather well the RWZ value $M\omega = 0.3737 - i0.0890$.

Remark 7.5.1. An approximate analytic expression for the QNM can be found by applying the Wentzel-Kramers-Brillouin (WKB) method of quantum mechanics. The physical scenario is the following. Eq. (7.29) is analogous to a 1D Schrödinger equation for a mass m in a potential barrier $V(x)$

$$\tilde{\Psi}_{xx} + Q(x)\tilde{\Psi} = 0, \quad \text{with } Q(x) := \frac{2m[E - V(x)]}{\hbar}. \quad (7.43)$$

The potential (and $Q(x)$) tends to constant values for $x \pm \infty$. One considers an energy below the peak of the potential, a wave package comes-in from $x \sim +\infty$ and gets in part reflected and in part transmitted. The reflected amplitude is generally comparable to the incident amplitude, while the transmitted amplitude is much smaller. In the WKB approach one formally multiplies the Ψ_{xx} by a small parameter and looks for a series solution of the small transmitted wave whose leading order term is given by the solution of the Eikonal equation,

$$(\tilde{\Psi}_x^{(0)})^2 = -Q \Rightarrow \tilde{\Psi}^{(0)} = \int dx \sqrt{-Q(x)}. \quad (7.44)$$

The transmitted amplitude is obtained by integrating $(-Q)^{1/2}$ between the turning points of the potential barrier $V(x)$. The gravitational problem is different in the sense that there is no incoming wave from infinity (the boundary condition there is outgoing) but some gravitational perturbation is initially prescribed. Thus, in general, one does not expect the transmitted wave to be smaller than the reflected but of approximately the same amplitude. However, there exists a quantum mechanical case similar to this: the case $\max[Q] \sim 0$, i.e. for the energies in the quantum mechanics (squared frequencies for the BH case) comparable to the peak of the potential. Following this idea, Schutz and Will (1985) consider $\max[Q] \gtrsim 0$ and propose to match the two WKB solutions, valid for $x < x_1$ and $x > x_2$ such that $Q(x_{1,2}) = 0$, to an approximate solution in the region $x_1 < x < x_2$ around $x = x_0$, the point of $\max[Q]$. This gives QNM frequencies approximated by the value of the potential and its second derivative at the peak,

$$(M\sigma)^2 \approx V_{\ell}(r_0) - i(n + \frac{1}{2})(-2V_{\ell}''(r_0))^{1/2}, \quad (7.45)$$

where n is the quantum number labelling the overtones. This method works well for estimating the fundamental mode and can be applied also to other than gravitational perturbations.

Laplace transform approach to solve the Cauchy problem. We discuss the Laplace method for the general solution of the initial value (Cauchy) problem and then specialize it to the case of the RW equation.

Let us consider the Cauchy problem for the wave equation for $\Psi(t, x)$ with $x \in \mathbb{R}$ and $t \in \mathbb{R}^+$ with a potential $V(x)$

$$\Psi_{tt} - \Psi_{xx} + V(x)\Psi = 0 \quad (7.46a)$$

$$\Psi(0, x) = \psi(x) \quad (7.46b)$$

$$\Psi_t(0, x) = \psi_t(x) \quad (7.46c)$$

If the initial data have compact support or are sufficiently localized, then the solution ψ is bound and admits a Laplace transform. The Laplace transform of the solution $\psi(t, x)$ is

$$\phi(s, x) = \int_0^\infty e^{-st} \psi(t, x) dt \quad (7.47)$$

and it is defined for positive, real s and can be continued into $\Re(s) > 0$. By integrating the equation for Ψ one finds that the function $\phi(s, t)$ satisfies the equation

$$\phi_{xx} - (s^2 + V(x)) \phi = F(s, x) , \quad (7.48)$$

with F defined by the problem's initial data

$$F(s, x) := -s\psi(x) - \psi_t(x) . \quad (7.49)$$

Given two independent solutions, $f^\pm(s, x)$, of the homogeneous equation ($F = 0$), the solution of Eq. (7.48) is

$$\phi(s, x) = \int_{-\infty}^{+\infty} G(s; x, x') F(s, x') dx' , \quad (7.50)$$

where the Green function generated from the homogeneous independent solutions is defined as

$$G(s; x, x') := \frac{1}{W(s)} [\theta(x - x') f^-(s, x') f^+(s, x) + \theta(x' - x) f^-(s, x) f^+(s, x')] = \frac{f^-(s, x_-) f^+(s, x_+)}{W(s)} , \quad (7.51)$$

with $x_\mp = \frac{\min}{\max}(x, x')$ and the *Wronskian* is defined as

$$W(s) := f^-(s, x) f_x^+(s, x) - f_x^-(s, x) f^+(s, x) . \quad (7.52)$$

Note $W(s)$ does not depend on x because $W_x = 0$, which follows by the fact that $f^\pm$ are solutions of the homogeneous equation:

$$W_x = \cancel{f_x f_x^+} + f^- f_{xx}^+ - \cancel{f_x^+ f_x^-} - \cancel{f_x^- f_{xx}^-} = (V + s^2)(f^+ f^- - f^+ f^-) \equiv 0 . \quad (7.53)$$

It is straightforward to verify that the Green function satisfy Eq. (7.48) with source function the Dirac's δ , $F(x) = \delta(x)$ [exercise]. The formal solution of the Cauchy problem is then obtained from the inverse Laplace transform of ϕ

$$\psi(t, x) = \frac{1}{2\pi i} \lim_{R \rightarrow \infty} \int_{\epsilon - iR}^{\epsilon + iR} e^{st} \phi(s, x) ds , \quad (7.54)$$

where $\epsilon = \Re(s)$ is greater than the real part of all singularities of $\phi(s)$. In practice, the integral is evaluated by choosing a closed path around the singularities and using Cauchy's residue theorem.

Note that the solution of the Cauchy problem through its Laplace transform already contains the initial data and also the requirement to be bound at spatial infinity. The latter condition is fulfilled just choosing the two independent functions $f^\pm(s, x)$ as bound functions.

Example 7.5.2. A simple application of the Laplace method is the wave equation for a string with fixed ends at $x = 0, \pi$. Here one take the wave equation with zero potential, arbitrary initial data and boundary conditions at all times $\Psi(\cdot, x = 0) = 0 = \Psi(\cdot, x = \pi)$. The general solution to this problem is given by a superposition of normal modes ⁶

$$\Psi(t, x) = \sum_{n=1}^{\infty} \sin(nx) (A_n \cos(nt) + B_n \sin(nt)) \quad (7.55)$$

where the constants A_n, B_n are determined by the initial data as

$$A_n = \frac{2}{\pi} \int_0^\pi \Psi(t = 0, x) \sin(nx) dx , \quad B_n = \frac{2}{n\pi} \int_0^\pi \partial_t \Psi(t = 0, x) \sin(nx) dx . \quad (7.56)$$

The result above can be directly verified and then proven using the separation of variable technique [exercise].

Apply now the the Laplace method to obtain the same result. The solutions of the homogeneous equation are simply two independent solution of the harmonic oscillator equation,

$$f^+ = \sin(\omega x) , \quad f^- = \sin(\omega(x - \pi)) . \quad (7.57)$$

The Wronskian is $W = -\omega \sin(\pi\omega)$ and has zeros at integer values of the frequencies $\omega_n = n$. The Taylor expansion near the zeros is $W \sim -[n\pi \cos(n\pi)](\omega - n)$, so the Green function has simple poles at ω_n . The ω_n are the normal

⁶Normal modes differently from QNM form a complete set for the solution.

frequencies of the string. By imposing $\omega = \omega_n = n$ in the homogeneous solution, one finds the latter are no longer independent but coincide with the normal modes of the solution. The general solution is given by

$$\Psi(t, x) = \frac{1}{2\pi} \int dx' \int ds e^{-st} [\omega \Psi(0, x') - \partial_t \Psi(0, x')] G(x, x') , \quad (7.58)$$

where $s \in \mathbb{C}$ and the complex integral in s requires the choice of a contour. For the latter one takes a closed path composed of a segment in the top half complex plane $R + iI$ with $I > 0$ closed by a semicircle γ_R connecting the segment's extrema in the bottom half complex plane. The closed integral is equal to the sum of the residuals, i.e. the poles of $G(x, x')$; while the limit of the integral on the segment is the solution we search for,

$$\begin{aligned} \frac{1}{2\pi} \int dx' \int ds e^{-st} [\omega \Psi(0, x-) - \partial_t \Psi(0, x')] G(x, x') &= \sum_n \frac{-i}{n\pi} \left(\int dx' \sin(nx') [\omega \Psi(0, x-) - \partial_t \Psi(0, x')] \right) e^{-int} \sin(nx) \\ &+ \lim_{R \rightarrow \infty} \int_{\gamma_R} (...) . \end{aligned} \quad (7.59)$$

It is left as [exercise] to verify that the above expression returns the expected result.

QNMs in the Laplace approach. Let us see how QNMs emerge in the solution of the Cauchy problem obtained with the Laplace method. Assume to compute the integral Eq. (7.54) considering a contour made of a segment $[\epsilon - iR, \epsilon + iR]$ parallel and on the right of the imaginary s -axis and close it with a half circle Γ_R , taking then the limit $R \rightarrow \infty$. Assume for simplicity that

1. There are no essential singularities;
2. $f^\pm(s, x)$ are analytic;
3. The integral on Γ_R vanishes for $R \rightarrow \infty$;
4. The initial data has compact support $[x_L, x_R]$.

Then one computes the integral in Eq. (7.54) from

$$\sum_k \text{res}(e^{st} \phi, s_k) = \oint ds e^{st} \phi = \int_{\Gamma_r} e^{st} \phi + \int_{\epsilon - iR}^{\epsilon + iR} e^{st} \phi = 0 + \int_{\epsilon - iR}^{\epsilon + iR} e^{st} \phi , \quad (7.60)$$

where in the last passage we used hypothesis 3. and the residuals are computed on the poles s_k of the Wronskian $W(s)$. The solution of the IBVP is thus

$$\psi(t, x) = \frac{1}{2\pi i} \int_{\epsilon - i\infty}^{\epsilon + i\infty} e^{st} \phi(s, x) ds \quad (7.61a)$$

$$= \frac{1}{2\pi i} \int_{\epsilon - i\infty}^{\epsilon + i\infty} ds \frac{e^{st}}{W(s)} \left[f^+(s, x) \int_{-\infty}^x dx' f^-(s, x') F(s, x') + f^-(s, x) \int_x^{+\infty} dx' f^+(s, x') F(s, x') \right] \quad (7.61b)$$

$$= \sum_k \frac{e^{s_k t}}{W'(s_k)} \left[f^+(s_k, x) \underbrace{\int_{x_L}^{x_R} dx' f^-(s_k, x') F(s_k, x')}_{=: c_-(s_k)} + f^-(s_k, x) \underbrace{\int_{x_L}^{x_R} dx' f^+(s_k, x') F(s_k, x')}_{=: c_+(s_k)} \right] \quad (7.61c)$$

In the second passage we have substituted the expression of $\phi(s, x)$ in terms of the Green function (the Heaviside functions split the integral). In the third passage we used hypothesis 4. and the residual theorem with the pole given by the zeroes of the Wronskian. The latter behaves near the poles as

$$W(s) = W'(s_k)(s - s_k) + \mathcal{O}((s - s_k)^2) . \quad (7.62)$$

The condition $W = 0$ implies that the homogeneous solution $f^\pm(s, x)$ calculated at $s = s_k$ are no longer independent; from Eq. (7.52) it immediately follows that

$$f^+(s_k, x) \propto f^-(s_k, x) , \quad (7.63)$$

where the proportionality coefficient is a function of s_k .

The next step is to inspect the asymptotic behaviour of the homogenous solutions $f^\pm$ of Eq. (7.48) and select bounded solutions. The general solutions of the homogenous Eq. (7.48) at large $|x|$ are

$$f_{xx}(s, x) - s^2 \phi(s, x) \simeq 0 \Rightarrow f \sim e^{sx} \Rightarrow f_{\text{gen}} = \begin{cases} a_1(s) e^{sx} + a_2(s) e^{-sx} & x \rightarrow +\infty \\ b_1(s) e^{sx} + b_2(s) e^{-sx} & x \rightarrow -\infty \end{cases} . \quad (7.64)$$

For $\Re(s) > 0$, the solutions are finite with the choice $a_1(s) = 0 = b_2(s)$. Inspecting now Eq. (7.61c), it is clear that $\phi(s_k, x)$ (or ψ) is finite bounded for $x \rightarrow +\infty$ ($x \rightarrow -\infty$) iff $f^+(s_k, x)$ ($f^-(s_k, x)$) is bounded in the same limit.

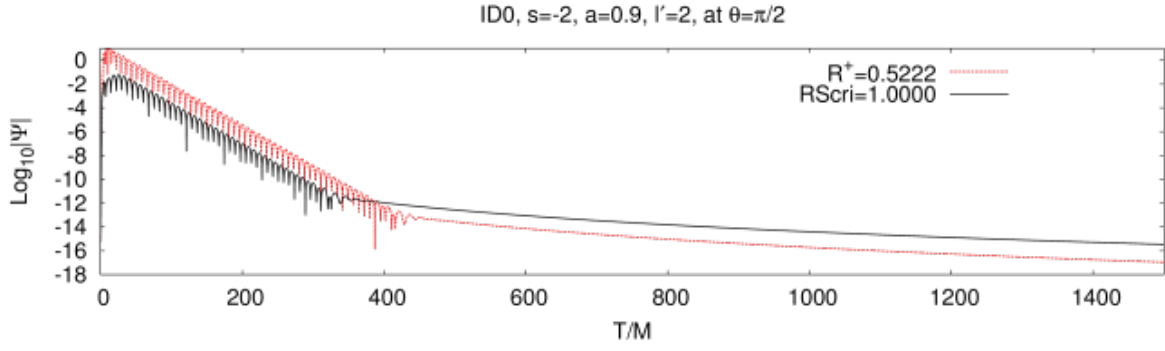


Figure 7.3: Perturbations of a Kerr BH with dimensionless spin $a = 0.9$. The perturbative field is in this case the Weyl scalar Ψ_4 decomposed into azimuthal m -modes and evolved according to the Teukolsky equation. The field is extracted at null infinity and at a large distance. One can appreciate the same structure as for Schwarzschild perturbations and that the power-law tails have different exponents depending on whether they are extracted at null infinity or finite distance. (From (Harms et al., 2013).)

Thus, f^+ must be the particular solution with $a_1(s_k) = 0$ for large positive x and f^+ must be the particular solution with $b_2(s_k) = 0$ for large negative x . Therefore, in Eq. (7.61c) the particular homogeneous solutions are those that asymptotically

$$f^+(s_k, x) \propto f^-(s_k, x) \propto e^{-s_k|x|}, \quad x \rightarrow \pm\infty. \quad (7.65)$$

In other terms, the choice $s = s_k$ is such that the bounded solution $f^- \propto e^{sx}$ for $x \rightarrow -\infty$ matches the bounded solution $f^- \propto e^{-sx}$ at $x \rightarrow +\infty$.

The above equation highlights the connection with the definition of normal modes via the Fourier transform. The case considered above shows that the solution of the initial value problem can be represented by a superposition of QNMs. Indeed Eq. (7.65) is the same boundary condition as Eq. (7.30) by setting $s_k = i\omega_k$, and for $x > x_R$ Eq. (7.61c) the solution is in the form of a sum of complex modes

$$\psi(t, x) = \sum_k \underbrace{\frac{c_-(s_k)}{W(s)}}_{=:c_k} \underbrace{f^+(s, x)e^{s_k t}}_{=:u_k(t, x)} \sim \sum_k c_k u_k(t, x). \quad (7.66)$$

This happens in cases when the potential falls off faster than exponentially or has compact support. In the general case, however, QNMs do not form a complete set in the space of solutions. Physically, this translates to the fact that it is not possible to associate a well defined energy to each QNM.

QNMs of Schwarzschild BH. In the case of Schwarzschild perturbations, the RW $V(x)$ decays exponentially for $x \rightarrow -\infty$, it reaches a maximum and then decays as $1/x^2$ for $x \rightarrow +\infty$. Bachelot and Motet-Bachelot (1993) proved that

1. $f^-(s, x)$ is analytic;
2. $f^+(s, x)$ has an essential singularity in $s = 0$, a branch cut in the negative real axis, $\Re(s) < 0$ and $\Im(s) = 0$, and poles in the plane $\Re(s) < 0$.

The contour for the integral calculation must be chosen made of a segment $[\epsilon - iR, \epsilon + iR]$ parallel and on the right of the imaginary s -axis, a small circle γ_r around the origin and two segments $\delta_r^\pm$ above and below the negative real axis. The contour is closed with two quarters of a circle $\Gamma_R^\pm$ in the IV and III quadrants of the complex plane. The contributions to the solution are schematically given by

$$\psi \sim \int_{\epsilon - iR}^{\epsilon + iR} (...) = \underbrace{\int_{\Gamma_R^+} (...) + \int_{\Gamma_R^-} (...)}_{\text{early times}} - \underbrace{\sum_k \text{res}(..., s_k)}_{\text{QNMs}} + \underbrace{\int_{\delta_r^+} (...) + \int_{\delta_r^-} (...) + \int_{\gamma_r} (...)}_{\text{late times}}. \quad (7.67)$$

The different integrals contribute to the solution at different times. The general solution has the structure *precursor*, *burst*, *ringdown* and *tail*, Fig. 7.1. The precursor and the burst are early time contributions, dependent on the initial data. QNMs follow the peak of the radiation. The late time solution (not shown in the figure) is determined by the potential at large radii and it is composed of power-law tails $\psi \sim t^{-p}$ with p dependent on the multipolar index ℓ , on the field spin (for a generic perturbation of spin s), on the class of initial data, and also on the spatial position at which the time solution is evaluated, Fig. 7.3. For example, tail decays are different for scalar and gravitational perturbations, for the solutions at scri and a finite radius and are not present for compact support potentials ⁷.

⁷The Poschl-Teller potential mimics the peak of the Regge-Wheeler potential but has exponential decay for $x \rightarrow \infty$; no tails are present.

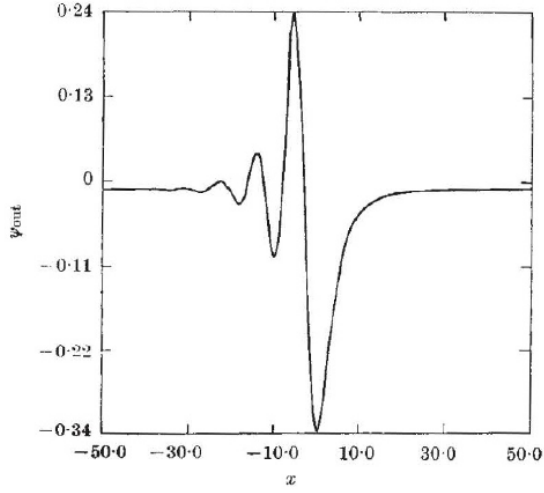


Fig. 3. The outgoing wave packet $\psi_{\text{out}}(x)$ at spatial infinity corresponding to the incident Gaussian wave packet $\psi_{\text{in}}(x) = e^{-ax^2}$ with $a=1$.

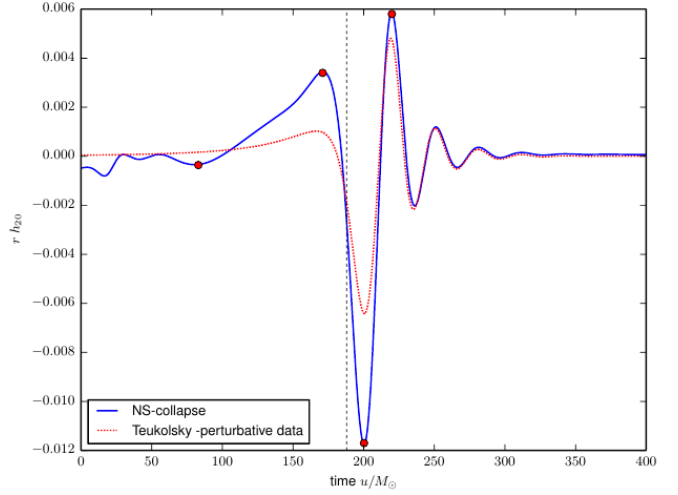


Figure 7.4: Left: The original plot of Vishveshwara (1970a). Note the x -axis is not not the asymptotic (null) time but the space coordinate, thus the “opposite” sign ($u = t - x$). Right: Ψ_{20} waveform from a 3+1 simulation of gravitational collapse to BH in numerical relativity and a waveform from a particle’s radial plunge on a Kerr black hole with the same spin (From (Dietrich and Bernuzzi, 2015)).

7.6 Effectiveness of perturbation theory

Perturbation theory describes several key astrophysical processes.

- ‘Scattering’ of GWs. Vishveshwara (1970a) first pointed that the response of a black hole to a perturbation is of the type summarized Fig. 7.1, and that the ringdown waveform carries in the QNM the “signature” of the BH.
- Plunges of small objects (particles) into a BH. Davis et al. (1999) studied the waveform produced by the radial plunge of a particle, Fig. 7.5. The waveform has the same structure precursor-burst-ringdown-tail highlighted by Vishveshwara.
- In a series of works Ipser and Thorne (1968); Thorne (1969) studied perturbations of spherical neutron stars opening the way for GW asteroseismology (Andersson and Kokkotas, 1998).
- Deformation (polarization coefficients) of neutron stars (Hinderer, 2008; Damour and Nagar, 2009; Binnington and Poisson, 2009);
- Nonspherical Gravitational collapse to BH, e.g. perturbations of Oppenheimer-Snyder solution (Oppenheimer and Snyder, 1939); Cunningham et al. (1978, 1979); Seidel (1990) have computed with perturbation theory the main feature of the waveform generated in the gravitational collapse (see below).

The above results are also relevant to understand the main features of processes in which perturbation theory does not apply.

- The black hole’s ringdown is a generic late-time feature in any binary black hole collisions.
- Numerical relativity simulations of rotational collapse to black holes shown that the waveform morphology is the same predicted by perturbation theory, see Fig. 7.5.
- GW asteroseismology is relevant to understand the properties and the signatures of radiating isolated neutron stars but also the proto- neutron star resulting from the core collapse supernovae remnants. The ultimate applications are related to constraining the nuclear equation of state and the explosion mechanism.

7.7 Stability of black hole spacetimes

A fundamental question about exact stationary solutions of EFE like Minkowski, Schwarzschild or Kerr is their stability under small perturbations. Assessing stability is key for (astro)physical realism of these solutions. Rigorous proofs are very nontrivial and usually build on several results. A very rough scheme is the following:

- Linear mode stability: within linear perturbation theory one should prove that the time evolution of each mode $\Psi_{\ell m}$ is bounded (in some norm) for a suitable class of initial data (say, with compact support).
- Linear stability: mode stability does not, in general, guarantee that a solution composed of an infinite sum of modes remains bound. One should prove that all solutions to the linearised EFE remain bounded for all time by a suitable norm of their initial data. Note that mode stability is a necessary condition to linear stability.
- Nonlinear stability: here one considers the more general Cauchy problem in GR with initial data given by Minkowski/Schwarzschild or Kerr plus small deviation. For example, nonlinear stability of Minkowski for asymptotically flat vacuum initial data has been proven by Christodoulou and Klainerman (1993).

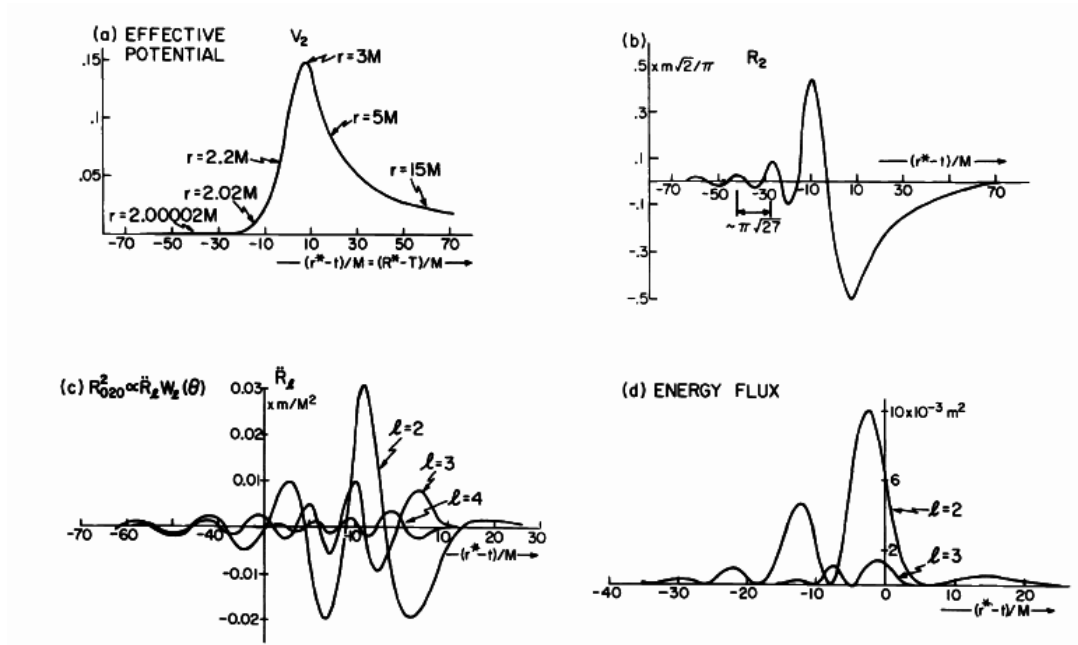


FIG. 1. Asymptotic behavior of the outgoing burst of gravitational radiation compared with the effective potential, as a function of the retarded time $(t - r^*)/M = (T - R^*)/M$. (a) Effective potential for $l=2$ in units of M^2 as a function of the retarded time $(t - r^*)/M = (T - R^*)/M$. For selected points the value of the Schwarzschild coordinate r is also given. (b) Radial dependence of the outgoing field $R_l(r, t)$ as a function of the retarded time for $l=2$. (c) $\ddot{R}_l(r^*, t)$ factors of the Riemann tensor components (see text) given as a function of the retarded time for $l=2, 3, 4$. (d) Energy flux integrated over angles for $l=2, 3$; the contributions of higher l are negligible.

Figure 7.5: Davis et al. (1999) results for the waveform from particle's radial plunge into Schwarzschild.

The question of stability for black hole spacetimes is still open, although many positive results are available. An incomplete list of results follows.

- Regge and Wheeler (1957); Vishveshwara (1970b) proposed the first arguments for mode stability. The solutions of the RWZ equation composed of QNM and power law tails discussed above guarantee that the perturbations radiate away and decay in time.
- Linear stability of scalar perturbation to Schwarzschild has been proven by Kay and Wald (1987).
- Linear stability of Schwarzschild was proven by Dafermos et al. (2019).
- Nonlinear stability of Schwarzschild was proven by Klainerman and Szeftel (2017) for a class of nontrivial perturbations.
- Mode stability of Kerr was proven by Whiting (1989).
- Linear stability of scalar perturbation of nonextremal Kerr BH was proven by Dafermos et al. (2014).
- All extremal Kerr BH are unstable to gravitational perturbation along their event horizon (Aretakis, 2015).

8. Effective-one-body framework

The effective-one-body (EOB) is an Hamiltonian formalism that allows one to map the general-relativistic two-body problem into the dynamics of an effective particles in an effective metric. It is a powerful formalism that incorporates results from PN, black hoe perturbation theory (including Gravitational Self-Force, GSF) and numerical relativity to provide gravitational wave templates.

Suggested readings. *Damour (2012) review; Papers: (Buonanno and Damour, 1999, 2000).*

8.1 General idea

The key components of the EOB formalism are

1. the effective Hamiltonian for the conservative dynamics;
2. the radiation reaction force;
3. the waveform.

In prescribing each of the three component, one starts from the PN knowldge but tries to introduce suitable resummations of the PN Taylor series in x .

Starting points of the method and basic building blocks:

1. PN Hamiltonian in ADM coordinate and related results (Damour and Schäfer, 1988; Damour et al., 2015) (50+ years of efforts starting from the works of Arnowitt et al. (1960); Kimura (1961) and summarized by Schaefer and Jaranowski (2018))

$$\hat{H} = \frac{H}{\mu} = H_N + H_{1PN} + H_{2PN} + \dots \quad (8.1a)$$

$$H_N = \frac{p^2}{2} - \frac{1}{q} \quad (8.1b)$$

$$H_{1PN} = \frac{1}{8}(3\nu - 1)(p^2)^2 - \frac{1}{2}[(3 + \nu)p^2 + \nu(n \cdot p)^2] + \frac{1}{2q^2} \quad (8.1c)$$

...

2. Hamitonian of a test-mass on Schwzrschild;
3. Pade' resummation of the circular GW flux (Damour et al., 1998, 2000, 2001);
4. The smooth transition plunge-ringodwn studied in Davis et al. (1999) and following works.

Here, the presentation of the formalism focuses on nonspinning circular binaries, although the method has been extended to spinning binaries and noncircular motion. Before starting we review some classical mechanics (and a bit of quantum) with the following

Remark 8.1.1. *In classical mechanics, periodic systems with a time-independent Hamiltonian can be described using Hamilton-Jacobi (HJ) equations and the action-angle variables. Let us briefly remind these concepts and their connection to the Bohr-Sommerfeld quantization rule.*

For a time-independent Hamiltonian H the action can be written in terms of Hamilton's characteristic function W , that satisfies the HJ equation (time-independent eigenvalue problem)

$$S(p, q, t) = W(q, p) - Et ; \quad W : H(q, \frac{\partial W}{\partial q}) = E . \quad (8.2)$$

Solutions of the HJ can be searched for using separation of variables. If one of the q coordinates is cyclic, then it can be always be separated. If additionally the motion is periodic ¹ one introduces (for each q) the action variable

$$J := \frac{1}{2\pi} \oint q dq = \oint \frac{\partial W}{\partial q} dq = const , \quad (8.3)$$

¹Periodic motion refer to phase space periodicity. This includes closed orbits (oscillations, q and p are period functions with same frequency) but also rotations (p period but q unbounded).

where the integral is taken over one period in the phase space. The conjugate variable $\theta := \partial W / \partial J$ evolves linearly in time at the characteristic frequency of the motion $\theta \propto \Omega t$ with $\Omega = \partial H / \partial J$. Note that J has dimension of an angular momentum and θ has no dimension. The Bohr-Sommerfeld quantization rule is

$$J = \hbar n . \quad (8.4)$$

Let us apply the formalism to the Coulomb potential $V(r) = -\alpha/r$, specifying for simplicity to in-plane $\theta = \pi/2$ motion from the beginning. The Hamiltonian and the HJ equations are

$$H = \frac{m}{2}(p_r^2 + \frac{p_\varphi^2}{r^2}) + V(r) ; \quad (\frac{\partial W_r}{\partial r})^2 + \frac{1}{r^2}(\frac{\partial W_\varphi}{\partial \varphi})^2 = 2m(E - V(r)) . \quad (8.5)$$

The variable φ is cyclic, so $p_\varphi = \text{const}$. For $E < 0$ the motion is periodic. The action variables are

$$J_\varphi := \frac{1}{2\pi} \oint p_\varphi d\varphi = p_\varphi , \quad J_r := \frac{1}{2\pi} \oint \sqrt{2m(E - V(r) - \frac{p_\varphi^2}{r^2})} dr = -J_\varphi + (\alpha/2)\sqrt{2m/E} . \quad (8.6)$$

The second integral gives ²

$$E = -\frac{\alpha^2 m}{2(J_r + J_\varphi)^2} . \quad (8.7)$$

The above equation the correct orbital frequency in a bound Kepler orbit, $\Omega_r = \partial E / \partial J_r = \alpha^2 (J_r + J_\varphi)^{-3} = 1/(\alpha)\sqrt{-2E^3/m}$. Note that $\Omega_\varphi = \Omega_r$. The application of the Bohr-Sommerfeld quantization rule to Eq. (8.7)

$$E_n = -\frac{\alpha^2 m}{2\hbar^2(n_r + \ell)^2} = -\frac{\alpha^2 m}{2\hbar^2 n^2} \quad (8.8)$$

where $n := n_r + \ell$ is the principal quantum number (Coulomb degeneracy).

8.2 EOB Hamiltonian

The EOB is a relativistic generalization of the well-known Newtonian property that the relative motion is equivalent to the motion of a particle of mass $\mu = \nu M$ in an effective potential. The GR dynamics can, in fact, be mapped into the motion of an effective particle μ into an effective metric. Here we see how to construct such a dictionary between the real Hamiltonian for the relative motion and the effective motion,

$$\text{Real Hamiltonian (relative motion)} \leftrightarrow \text{Effective EOB Hamiltonian} . \quad (8.9)$$

The EOB Hamiltonian is built in three conceptual steps.

1. *Delaunay PN ADM Hamiltonian.* The ADM PN Hamiltonian for the relative motion is written in terms of the action variables. It is convenient for notation purposes (but not only, see Rem. 8.2.1) to indicate the action variables using their quantum numbers n, ℓ . Let $\alpha = GM\mu/\hbar$, the expression to 2PN is (Damour et al., 1998)

$$E_{\text{real}} = -\frac{\mu}{2} \frac{\alpha^2}{n^2} \left[1 + \frac{\alpha^2}{c^2} \left(\frac{c_{11}}{n\ell} + \frac{c_{20}}{n^2} \right) + \frac{\alpha^4}{c^4} \left(\frac{c_{13}}{n\ell^3} + \frac{c_{22}}{n^2\ell^2} + \frac{c_{31}}{n^3\ell} + \frac{c_{40}}{n^4} \right) + \dots \right] \quad (8.10)$$

with $c_{n\ell} = c_{n\ell}(\nu)$ known PN coefficients.

2. *Effective metric and Delaunay Hamiltonian.* Consider the effective metric

$$g_{\text{eff}} = -A(R; \nu) dT^2 + B(R; \nu) dR^2 + R^2 d\Omega^2 \quad (8.11a)$$

$$A(R; \nu) = 1 + \tilde{a}_1(\nu) \frac{GM}{c^2 R} + \tilde{a}_2(\nu) \left(\frac{GM}{c^2 R} \right)^2 + \dots = 1 + \tilde{a}_1(\nu) u + \tilde{a}_2(\nu) u^2 + \dots \quad (8.11b)$$

$$B(R; \nu) = 1 + \tilde{b}_1(\nu) \frac{GM}{c^2 R} + \tilde{b}_2(\nu) \left(\frac{GM}{c^2 R} \right)^2 + \dots = 1 + \tilde{b}_1(\nu) u + \tilde{b}_2(\nu) u^2 + \dots , \quad (8.11c)$$

with unknown metric potentials A, B given by a PN series in $u := \frac{GM}{c^2 R}$. The Newton limit for $u \rightarrow 0$ fixes $\tilde{a}_2 = 2$; while the Schwarzschild limit $\nu = 0$ fixes $A(R, \nu = 0) = 1 - 2GM/R = B^{-1}(R, \nu = 0)$. The Delaunay Hamiltonian of an (effective) particle of mass μ on this metric is (quantum number N, L)

$$\mathcal{E}_{\text{eff}} = \mu c^2 - \frac{1}{2} \mu \frac{\alpha^2}{N} \left[1 + \frac{\alpha^2}{c^2} \left(\frac{C_{11}}{NL} + \frac{C_{20}}{N^2} \right) + \frac{\alpha^4}{c^4} \left(\frac{C_{13}}{NL^3} + \frac{C_{22}}{N^2 L^2} + \frac{C_{31}}{N^3 L} + \frac{C_{40}}{N^4} \right) + \dots \right] , \quad (8.12)$$

where the coefficients are combinations of the tilde coeffs, $C_{NL} = C_{NL}(\tilde{a}_i, \tilde{b}_i)$.

²The integral should be familiar from mechanics classes. Because the motion is confined in $r \in [r_0, r_1]$ it can be calculated using

$$\oint \sqrt{\dots} = \int_{r_0}^{r_1} \sqrt{(1 - r_0/r)(r_1/r - 1)} = \pi/2(r_1 - r_0) - \pi\sqrt{r_0 r_1} .$$

3. *Dictionary between energy levels.* The EOB Hamiltonian is defined by the map that connect the real (known as a PN expansion) and the effective problem. The map is constructed by first demanding that

$$n = N, \quad \ell = L, \quad (8.13)$$

and then searching for a relation $E_{\text{real}} \leftrightarrow \mathcal{E}_{\text{eff}}$. The Hamiltonians (energy levels) of the two problems are schematically given by

$$E = Mc^2 + E_{\text{real}} = Mc^2 + \text{Newt} + 1PN + 2PN + \dots \quad (8.14a)$$

$$\mathcal{E}_{\text{eff}} = \mu c^2 + \text{Newt} + 1PN + 2PN + \dots \quad (8.14b)$$

Buonanno and Damour (1999) postulate

$$\frac{\mathcal{E}_{\text{eff}}}{\mu c^2} = 1 + f\left(\frac{E_{\text{real}}}{\mu c^2}\right) = 1 + \frac{E_{\text{real}}}{\mu c^2} \left(1 + \alpha_1 \frac{E_{\text{real}}}{\mu c^2} + \alpha_2 \left(\frac{E_{\text{real}}}{\mu c^2}\right)^2 + \dots\right) \quad (8.15)$$

with unknown coefficients $\alpha_i(\nu)$. The above equation translates into a relation between the unknown coefficients at each PN order,

$$C_{NL}(\tilde{a}_i, \tilde{b}_i) \leftrightarrow \alpha_i(\nu). \quad (8.16)$$

For example, at 2PN there are six algebraic equations connecting the set of six coefficients

$$(\tilde{a}_2, \tilde{a}_3, \tilde{b}_1, \tilde{b}_2, \alpha_1, \alpha_2). \quad (8.17)$$

Assuming $\tilde{b}_1 = +2$, implies that (i) the effective metric coincides with Schwarzschild at this order and that (ii) there exists a unique solution for the 2PN coefficients, given by

$$\tilde{b}_1 = +2 \Rightarrow \tilde{a}_1 = 0, \quad \tilde{a}_3 = 2\nu, \quad \tilde{b}_2 = 2 - 6\nu, \quad \alpha_1 = \nu/2, \quad \alpha_2 = 0. \quad (8.18)$$

The 2PN A-potential is

$$A_{2\text{PN}} = 1 - 2u + 3\nu u^3, \quad (8.19)$$

while the 2PN energy map is

$$\frac{\mathcal{E}_{\text{eff}}}{\mu c^2} = 1 + \frac{E_{\text{real}}}{\mu c^2} \left(1 + \frac{\nu}{2} \frac{E_{\text{real}}}{\mu c^2}\right) = \frac{s - m_1^2 c^4 - m_2^2 c^4}{2m_1 m_2 c^4}; \quad s := (Mc^2 + E_{\text{real}})^2. \quad (8.20)$$

A similar derivation can be performed also at 3PN and leads to the same map for the energy. After verifying this order, one assumes the maps holds also at any order.

The EOB Hamiltonian is found by solving Eq. (8.20) for E_{real} and then by solving the effective Hamilton-Jacobi equation to get $\mathcal{E}_{\text{eff}}$ in terms of the effective phase space coordinates. Defining the coordinates

$$t := \frac{T}{GM}, \quad r := \frac{R}{GM}, \quad R_* := \int dR \frac{B}{A}, \quad p_{r_*} := \frac{P_{R_*}}{\mu}, \quad p_\varphi := \frac{P_\varphi}{\mu GM}, \quad (8.21)$$

The Hamiltonian reads

$$H_{\text{EOB}} = Mc^2 \sqrt{1 + 2\nu(\hat{H}_{\text{eff}} - 1)}, \quad (8.22a)$$

where

$$\hat{H}_{\text{eff}} = \frac{H_{\text{eff}}}{\mu} = \sqrt{A(u; \nu)(1 + p_\phi^2 u^2 + 2\nu(4 - 3\nu)u^2 p_{r_*}^4) + p_{r_*}^2}. \quad (8.22b)$$

One can immediately verify that for $\nu \rightarrow 0$, $H_{\text{EOB}} \rightarrow H_{\text{eff}}$ and the effective Hamiltonian H_{eff} reduces to the Hamiltonian of a particle in Schwarzschild spacetime, where $A(u; 0) = 1 - 2u$. Also, for small u

$$H_{\text{EOB}} \simeq Mc^2 + \frac{\mu}{2} P^2 + \frac{\mu}{2} (A - 1) = Mc^2 + \frac{\mu}{2} P^2 - \frac{GM\mu}{c^2 R}. \quad (8.23)$$

The metric potential $A(u)$ is thus interpreted as the inter-binary energy.

Remark 8.2.1. *Following from Rem. 8.1.1, it should be clear that the EOB map can be interpreted from a quantum-mechanical point of view. The construction presented above is done in such a way the gauge invariant energy levels of the two conservative dynamics match each other, Fig. 8.1.*

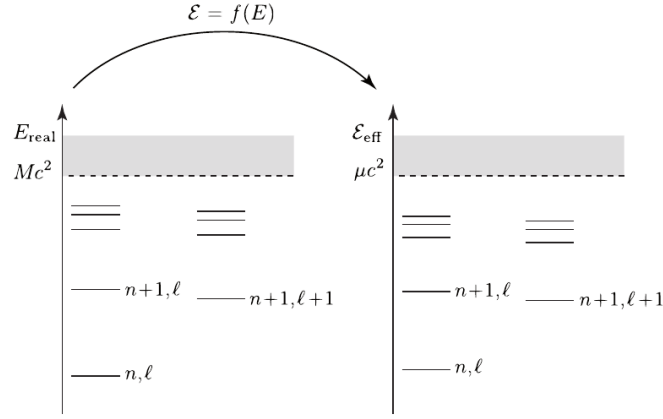


Figure 8.1: Quantum interpretation of the EOB dictionary for the conservative dynamics. From Damour (2012).

Metric potentials and resummation. At 3PN the metric potentials calculated from the EOB dictionary are (Damour, 2012)

$$A_{3\text{PN}} = 1 - 2u + 3\nu u^3 + a_4(\nu)u^4, \quad a_4(\nu) = \left(\frac{94}{3} - \frac{41\pi}{32}\right)\nu. \quad (8.24)$$

and

$$D_{3\text{PN}} := \frac{A_{3\text{PN}}}{B_{3\text{PN}}} = 1 - 6\nu u^2 + 2(3\nu - 26)\nu u^3. \quad (8.25)$$

It is remarkable that the 3PN Hamiltonian within the EOB formalism can be written in such a compact form (Cf. the starting ADM Hamiltonian given e.g. in (Damour, 2012)). The double-square-root structure of Eq. (8.22) is a first example of resummation in the EOB formalism. The EOB Hamiltonian contains by construction all the PN but in a more compact form. Moreover, since the square roots contains all the infinite terms of their Taylor expansion, one can hope that more information than the PN series is present in the EOB Hamiltonian³.

The convergence properties of the 3PN EOB Hamiltonian can be improved introducing a further resummation. One can observe that the coefficient $a_4(\nu)$ becomes rather large for $\nu = 1/4$, and wonder whether this is a real feature of the (unknown) exact effective metric or an unwanted feature introduced by the PN series. How to assess this? There is a simple way: continuity. We expect that the dynamics is smooth in the parameter ν . In fact, this should be required, as there is no fundamental reason to expect a different behaviour. The Schwarzschild's test-mass dynamics then provides an exact limit for $\nu \rightarrow 0$. Thus, we expect that for $\nu \neq 0$ the structure of the $A(u; \nu)$ and the Hamiltonian must be similar to Schwarzschild and must be a continuous-in- ν deformation of the Schwarzschild expressions. For example, the Schwarzschild spacetime has an horizon at $R_S = 2GM/c^2$ ($u = 0.5$) and a last stable orbit (LSO) at $R = 6M/c^2$. The effective spacetime should also possess these features albeit at different coordinate locations (corrected/deformed by functions of the mass ratio, ν). Inspection of the 3PN potential shows that $A_{3\text{PN}}(u, \nu)$ diverges for $\nu \rightarrow 1/4$ and does not admit any zero, Fig. 8.1. By contrast the 2PN potential follows qualitatively the behaviour of the Schwarzschild potential and admits a horizon (zero). A possible way to improve the model is, for example, to apply a Padé' function to the 3PN potential,

$$A_{3\text{PN}}(u) \mapsto P_3^1[A_{3\text{PN}}(u)] = \frac{1 + n_1 u}{1 + d_1 u + d_2 u^2 + d_3 u^3}, \quad (8.26)$$

where the n_1, d_1, d_2, d_3 can be algebraically computed from the coefficients of $A_{3\text{PN}}(u)$ by requiring that the first four terms of the Taylor expansion of $P_3^1[A(u)]$ in powers of u coincide with $A_{3\text{PN}}$ ⁴. As shown in Fig. 8.2 this potential is much closer to the 1PN and 2PN potential and retains similar properties.

For finite mass ratio the function $A(u; \nu)$ is completely known analytically up to 4PN (Damour et al., 2015) with few terms known at 5PN (and even beyond in the test-mass limit). The expression at 4PN is remarkably simple

$$A_{4\text{PN}} = 1 - 2u + \nu(2u^3 + a_4 u^4 + a_5(\nu, \ln u)u^5), \quad (8.27)$$

where $a_5(\nu, \ln u)$ is a linear function of ν and $\ln u$. Unknown terms at 5PN are currently parametrized by one or few free coefficients that can be fixed by comparing gauge invariant quantities like the waveform or the energetics to numerical relativity simulations. Moreover, the metric potentials in the EOB Hamiltonian are always employed in

³We have started from PN series and obtained a result that (potentially) contains more information than the starting one. The accuracy/faithfulness of the resummation is essentially a guess and must be verified against all the available exact results. But there is no trick and, on the contrary, this is a well known/defined approach for asymptotic series. A relevant example is given by Padé' approximants. The latter are rational polynomials known to give a better approximation of functions than the respective truncated Taylor series. In fact these approximants can work where the Taylor series does not converge. Padé' approximants are a well known technique used extensively also in computer calculations.

⁴A fast way to obtain these coefficients is to use computer algebra, e.g. with *Mathematica* <https://mathworld.wolfram.com/PadeApproximant.html>.

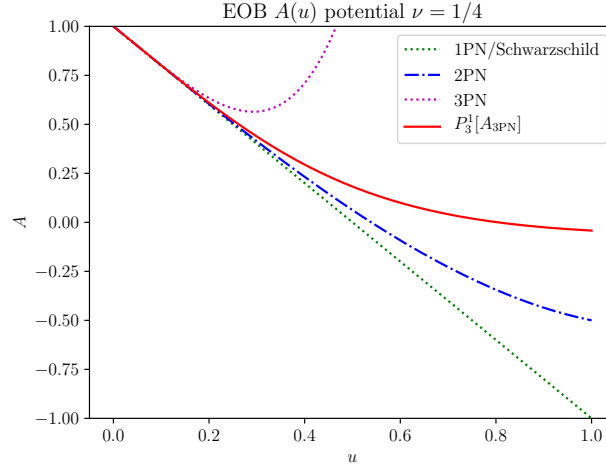


Figure 8.2: EOB potential at various PN order and the resummation of the 3PN potential.

some resummed form. For example, the $A(u)$ potential can be resummed with P_3^1 at 3PN, with P_4^1 at 4PN and with P_5^1 at 5PN, and the resulting “resummed PN series” shows a smooth behaviour in the PN order and in ν . The 3PN D potential

$$D_{3\text{PN}}(u) \mapsto P_3^0[D_{3\text{PN}}(u)] = \frac{1}{1 + 6\nu u^2 - 2\nu(3\nu - 26)u^3} . \quad (8.28)$$

The above expression guarantees that the D function remains positive when $R \rightarrow 0$ as demanded by the fact that $D(\nu \rightarrow 0) \rightarrow 1$.

Example 8.2.1. Circular orbits are given by setting $p_{r_*} \equiv 0$ in the Hamiltonian and looking for the stationary point,

$$\hat{H}_{\text{eff}} = \sqrt{A(u)(1 + p_\varphi^2 u^2)} , \quad \frac{d}{du} (A(u)(1 + p_\varphi^2 u^2)) = 0 . \quad (8.29)$$

The angular momentum of the circular orbits is thus given by ($d/du = ')$)

$$p_\varphi^2 = j_0^2 = -\frac{A'(u)}{(u^2 A(u))'} . \quad (8.30)$$

One can now recall that the effective potential for Schwarzschild geodesics is given by

$$V_j^{(s)} = A(R) \left(\frac{j^2}{R^2} + s \right) \quad (8.31)$$

where $s = 1(0)$ for timelike(null) geodesics and j is the angular momentum. Within EOB the concepts of LSO and light-ring can be generalized by simply considering $A(u) \mapsto A(u; \nu)$ and using the same formulas of the Schwarzschild spacetime. It is left as [exercise] to study the LSO and the LR for the potentials given above.

8.3 Radiation reaction force & Factorized circular waveform

The Hamiltonian EOM from the EOB Hamiltonian are

$$\dot{r} = \left(\frac{A}{B} \right)^{1/2} \frac{\partial \hat{H}_{\text{EOB}}}{\partial p_{r_*}} \quad (8.32a)$$

$$\dot{\varphi} = \frac{\partial \hat{H}_{\text{EOB}}}{\partial p_\varphi} := \Omega \quad (8.32b)$$

$$\dot{p}_{r_*} = - \left(\frac{A}{B} \right)^{1/2} \frac{\partial \hat{H}_{\text{EOB}}}{\partial r_*} \quad (8.32c)$$

$$\dot{p}_\varphi = \hat{\mathcal{F}}_\varphi = -\frac{1}{\Omega} F^{\ell_{\text{max}}} , \quad (8.32d)$$

The last equation contains a new ingredient: the *radiation reaction force* expressed in terms of the circular GW flux truncated at some multipole ℓ_{max} . The latter is given in terms of the modulus of the multipolar waveform discussed in Rem. 7.3.2,

$$F^{\ell_{\text{max}}} = \frac{2}{16\pi G} \sum_{\ell=2}^{\ell_{\text{max}}} \sum_{m=1}^{\ell} (m\Omega)^2 |Rh_{\ell m}|^2 . \quad (8.33)$$

The mutlipolar waveform introduced in Rem. 7.3.2 specified for circular orbits is written in general as Kidder (2008)

$$h_{\ell m}(x) = h_{\ell m}^N(x) \hat{h}_{\ell m}(x) ; \quad h_{\ell m}^N = \frac{GM\nu}{c^2 R} n_{\ell m} c_{\ell+\epsilon} x^{(\ell+\epsilon)} Y_{\ell-\epsilon, -m}(\frac{\pi}{2}, \phi) , \quad (8.34)$$

where x is the PN frequency parameter; $\epsilon = \pi(\ell + m)$ denotes the parity of $\ell + m$, i.e. $\epsilon = 0$ for even-parity (mass-generated) and $\epsilon = 1$ for odd-parity (current-generated) ones multipoles; $n_{\ell m}$ and $c_{\ell+\epsilon}(\nu)$ are numerical coefficients given in e.g. (Damour and Nagar, 2011); and $Y_{\ell m}$ the spherical harmonics. The subleading order term is given by a PN expression in the Taylor form

$$\hat{h}_{\ell m}^{\text{Taylor}} = 1 + c_1^{\ell m} x + c_{3/2}^{\ell m} x^{3/2} + c_2^{\ell m} x^2 + \dots \quad (8.35)$$

Damour et al. (2009) proposed a way to resum the above Taylor series and thus improve its convergence properties in the high-velocities (high-frequencies) regime. This waveform is known as *factorized waveform* because is it the product of several factors. Specifically it writes

$$\hat{h}_{\ell m}^{\text{DIN}}(\epsilon) = \hat{S}_{\text{eff}}^{(\epsilon)} T_{\ell m} e^{i\delta_{\ell m}} f_{\ell m} . \quad (8.36)$$

In the above expression the first two terms are known and are specific (factorized) contributions to the PN waveform ⁵, while the last terms are calculated by dividing the Taylor form with the known factors. The different terms in the factorized waveform are

- *Effective source factor*. Derived from leading order contributions to the circular waveform in the test-mass limit. For even parity it is chosen as the effective energy since the leading order source of gravitational radiation is given by the energy density. For odd parity it is taken as the circular angular momentum.

$$\hat{S}_{\text{eff}}^{(0)}(x) = \hat{H}_{\text{eff}} , \quad \hat{S}_{\text{eff}}^{(1)}(x) = \sqrt{x} j(x) . \quad (8.37)$$

- *Tail factor*. A resummed expression for an infinite number of “leading logarithms” entering the transfer function between the near-zone multipolar wave and the farzone one, due to tail effects linked to its propagation in a Schwarzschild background of mass $M_{\text{ADM}} = H_{\text{EOB}}$ (circular orbits):

$$T_{\ell m} = \frac{\Gamma(\ell + 1 - 2i\hat{k})}{\Gamma(\ell + 1)} e^{\pi\hat{k}} e^{2i\hat{k} \ln(2kr_0)} \quad (8.38)$$

where $k := m\Omega$, $\hat{k} := GM_{\text{ADM}}k$, and $r_0 = 2GM/\sqrt{e}$.

- *Phase factor*. Computed as $\delta_{\ell m} = \arg[\hat{h}_{\ell m}^{\text{Taylor}}/(\hat{S}_{\text{eff}} T_{\ell m})]$;
- *Amplitudes*. Computed as the ration of the moduli, $f_{\ell m}(x) = |\hat{h}_{\ell m}^{\text{Taylor}}/(\hat{S}_{\text{eff}} T_{\ell m})|$, they have a PN form $f_{\ell m}(x) = 1 + f_1^{\ell m} x + f_2^{\ell m} x^2 + \dots$ that is further resummed with various techniques (Damour, 2012).

There are several design choices in the factorized waveform that are currently subject of research and being explored/improved. A crucial aspect is the verification of the convergence of the analytical flux using exact result at high frequencies. These results are mostly available in the test-mass limit $\nu \rightarrow 0$ from the solution of the RWZ (or Teukolsky) equations. An early study in this direction is shown in Fig. 8.3.

8.4 Binary black hole (BBH) waveforms

The EOB formalism successfully predicted the waveform emitted by binary black holes and it is nowadays the most sophisticated approach to compute high-precision templates. Let us discuss the the main steps to compute EOB waveform for BBHs.

- Define the *moment of merger* t_m as the instant at which the orbital frequency $\Omega(t)$ reaches its maximum. This happens in the vicinity of the EOB (ν -deformed) adiabatic (no rad.reac. force) lightning. This can be seen by explicetly writing the EOM equation for Ω

$$\Omega = \frac{Au^2 p_\varphi}{\nu \hat{H}_{\text{EOB}} \hat{H}_{\text{eff}}} , \quad (8.39)$$

and observing that the maximum in $A(u)u^2$ on a circular orbit corresponds to a maximum of Ω . The maximum of Ω (or a neighbour) is a natural point where to stop the EOB dynamics. The circular waveform can be thus evaluated on the EOB dynamics obtaining $\hat{h}^{\text{insp}}$.

- Close to the moment of merger the amplitude of the EOB (2,2) waveform also has an absolute maximum that marks the end of the chirp signal. Note both the max of Ω and the max of the waveform cannot be predicted by the PN waveform, but they are observed in the GW computed by numerical relativity simulations. Comparing to the latter, the EOB circular waveform describing the circular inspiral motion $\hat{h}^{\text{insp}}$ is the least

⁵This expression has no formal derivation, but was proposed based on physical intuition. Some of the motivations are technical and are mentioned in Damour et al. (2009). The factors essentially absorb terms that would otherwise make the PN coefficients to diverge faster.

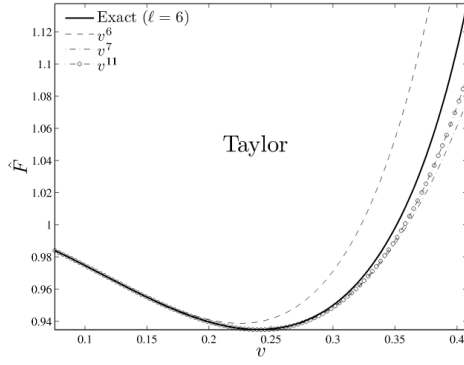


Fig. 3 The extreme-mass-ratio limit ($\nu \rightarrow 0$): the Newton-normalized energy flux emitted by a particle on circular orbits. The figure illustrates the scattering of the standard Taylor expansion of the flux around the “exact” numerical result (computed up to $\ell = 6$) obtained via perturbation theory.

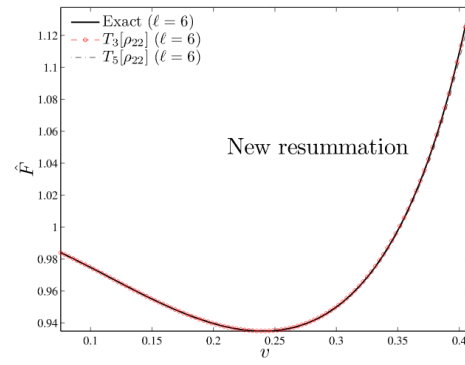


Fig. 6 Performance of the new resummation procedure described in Ref. [50]. The total GW flux $\hat{F}$ (up to $\ell_{\max} = 6$) computed from inserting in Eq. (62) the factorized waveform (41) with the Taylor-expanded $\rho_{\ell m}$'s (with either 3PN or 5PN accuracy for ρ_{22}) is compared with the “exact” numerical data.

Figure 8.3: Factorized circular flux compare to exact data in the test-mass limit ($\nu \rightarrow 0$, Left: PN waveform truncated at different PN order. Note the divergence from the exact result and the oscillatory character of the PN series. Right: Factorized waveform. From Damour and Nagar (2011).

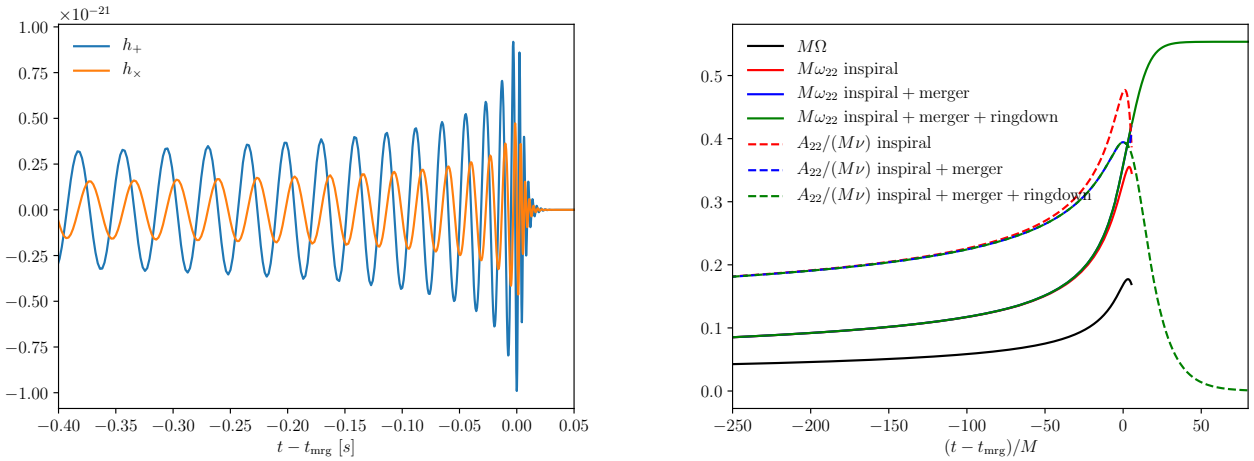


Figure 8.4: EOB BBH waveforms. Left: EOB waveform compatible with GW150914. Right: EOB (2, 2) waveform around at merger. The different contributions to the waveform are highlighted.

accurate towards the end of the inspiral. Beside the fact that the dynamics is reaching the highest velocity regime, another source of inaccuracy is that the motion is not circular. A way to improve the waveform is to introduce multiplicative corrections to the circular waveform called *Next-to-Quasi-Circular (NQC)* corrections. This defines the inspiral+plunge waveform, that for the (2, 2) looks

$$\hat{h}^{\text{insplunge}} = \hat{h}^{\text{insp}} \hat{h}^{\text{NQC}} = \hat{h}^{\text{insp}} \left(1 + a_1 \frac{p_{r*}^2}{(r\Omega)^2} + a_2 \frac{\ddot{r}}{r\Omega^2} \right) e^{i(b_1 p_{r*}/(r\Omega) + b_2 p_{r*} r\Omega)}, \quad (8.40)$$

where the coefficients $a_{1,2}, b_{1,2}$ (4 for each multipoles and dependent on ν) can be fixed by comparing to selected numerical data.

- (iii) The $\hat{h}^{\text{insplunge}}$ waveform is then smoothly connected to the *ringdown* waveform,

$$h^{\text{ringdown}}(M_f, a_f) = \sum_n C_n e^{\sigma_n(t-t_m)}. \quad (8.41)$$

The matching point (or interval) is chosen around t_m (maximum of Ω) as suggested by Davis et al. (1999). Here one needs a prediction for the final Kerr BH mass M_f and spin a_f , from which the Kerr QNM frequencies can be calculated, and a prescription for determining the excitation amplitudes C_n . Quantitative models of the ringdown make use of numerical relativity data.

An example of BBH waveform is shown in Fig. 8.4.

8.5 Tidal interactions

Damour and Nagar (2010) incorporated the treatment of tidal effect into the formalism, thus extending the model's

applicability to binaries with neutron stars. Tidal interactions are included by augmenting the EOB potential

$$A \mapsto A + A_{\text{tidal}} , \quad (8.42)$$

with an expression inferred from the tidal Lagrangian Eq. (6.92), (Bini et al., 2012).

The tidal potential has the form

$$A_{\text{tidal}} = \sum_{\ell \geq 2} \left[\kappa_{\ell}^{A+} u^{2\ell+2} (1 + \alpha_1^{(\ell+)} u + \alpha_2^{(\ell+)} u^2 + \dots) + \kappa_{\ell}^{A-} u^{2\ell+3} (1 + \alpha_1^{(\ell-)} u + \dots) + (A \leftrightarrow B) \right] \quad (8.43)$$

where $\alpha_i^{(\ell)}(\nu)$ are coefficients and

$$\kappa_{\ell}^{A+} := 2k_{\ell}^A \left(\frac{M_A}{MC_A} \right)^{2\ell+1} \frac{M_B}{M_A} , \quad \kappa_{\ell}^{A-} := 2j_{\ell}^A \left(\frac{M_A}{MC_A} \right)^{2\ell+1} \frac{M_B}{M_A} , \quad (8.44)$$

are the *multipolar tidal polarizability coupling constants*. The current analytical knowledge comprise gravitoelectric terms $\ell = 2, 3$ up to the next to next leading order (NNLO; coefficients $\alpha_{1,2}^{(2+,3+)}$) and gravitomagnetic terms up to NLO (coefficient $\alpha_1^{(2-)}$) (Vines and Flanagan, 2010; Damour et al., 2012; Bini et al., 2012; Bini and Damour, 2014).

Taking the Newtonian limit illustrates the meaning of the above coefficients,

$$H_{\text{EOB}} \simeq Mc^2 + \frac{\mu}{2} P^2 + \frac{\mu}{2} (A - 1) = Mc^2 + \frac{\mu}{2} P^2 + \frac{\mu}{2} \left(-\frac{2GM}{c^2 R^2} + \dots - \frac{\kappa_2^T}{R^5} \right) . \quad (8.45)$$

The constant $\kappa_2^T = \kappa_2^A + \kappa_2^B$ encodes the effect of tidal interactions at leading order. For a large span of EOS, masses in $[1, 2]M_{\odot}$ and mass ratios in $q \in [1, 2]$ its values are $\kappa_2^T \sim [50, 500]$. A common alternative notation uses the quantities $\Lambda_2^i \equiv 2/3 k_2^i (c^2 R_i / GM_i)^5$ with $i \in \{A, B\}$, in place of the κ_2^A and defines

$$\tilde{\Lambda} = \frac{16}{13} \frac{(M_A + 12M_B)M_A^4}{M^5} \Lambda_A + (A \leftrightarrow B) . \quad (8.46)$$

Note that using the κ quantities leads to more compact formulas.

Tidal corrections are introduced also in the waveform (Flanagan and Hinderer, 2008; Vines et al., 2011; Damour et al., 2012; Banihashemi and Vines, 2018) At leading order the stationary phase approximation of the waveform reads

$$\tilde{h}(f) = \tilde{A} f^{-7/6} e^{-i(\Psi_0(x) + \Psi_{\text{tidal}}(x))} = \tilde{A} f^{-7/6} e^{-i(\Psi_0(x) - 39/4 \kappa_2^T x^{5/2})} , \quad (8.47)$$

where $x(f) = (\pi G M f / c^3)^{2/3}$ and $\Psi_0(x)$ is the point-mass phase. Note that the tidal contribution at leading order is again fully determined by κ_2^T . For this reason, the latter (or equivalently $\tilde{\Lambda}$) is the quantity that is best measured from GW observations.

The EOB formalism is currently the only method to obtain an accurate GW measurement of tidal parameters and thus constraints on the neutron star matter. The reason is that the measurement of κ_2^T is more sensitive at high frequencies where post-Newtonian approximants but most of the signals are detected at frequencies that are too low for performing relativity simulations.

9. Experiments and data analysis

Discussion on the GW detectors and on characterization of noise curves. Data analysis application based on match-filtering techniques.

Suggested readings. *Chap. 7,8,9 of Maggiore (2007); Abbott et al. (2019a).*

9.1 GW detectors

Experimental GW physics began in the 1960s with J. Weber, who proposed resonant bars detectors. Bar detectors have resonance frequencies that can be excited by the passage of a GWs. The frequency range of sensitivity of these instruments is centered in a narrow interval around the resonance at kiloHertz frequencies. A bar detector with a mass of 2 tons can measure a relative displacement as small as $\Delta L/L \sim 10^{-18} - 10^{-19}$. No GW events were observed with these apparatus. Detections of GW passing through Earth have been possible with *ground-based laser interferometers*. These experiments are kilometer-size Michelson-Morley interferometers with Fabry-Perot cavities started in the 1980s and operated by the LIGO and Virgo collaborations. More recently, the KAGRA collaboration has joined the experimental effort. Future experiments will include the space-based LISA mission and the ground-based next-generation (the third, hence 3G) detectors such as the Einstein Telescope (ET).

In the following, we discuss key aspects of ground-based laser interferometers, the main sources of noise, current GW antennas and the two most important GW events observed so far. Next-generation detectors are discussed briefly at the end.

Ground-based laser interferometers. A simple scheme of a ground-based interferometer used for GW detections is shown in Fig. 9.1. GW interferometers are instruments for measurements of light time-intervals.

The basic concept is the following: a monochromatic laser beam starts from its source and reaches a *beam-splitter*, which separates the light in a beam travelling along one arm and another beam traveling in a second orthogonal arm. At the end of each arm, a highly reflective mirror (test-mass) is located at a precise distance from the beam-splitter. After travelling forth and back, the two beams recombine at the beam-splitter and the resulting light is collected in a photodiode, which measures its intensity. If the two beams are reconstructed with a coherent phase, there is constructive interference. If the phases of the laser beams are not aligned, there is destructive interference.

Key components of the interferometers are:

- **Mode cleaners:** the beam is cleaned in input and in output in order to ensure a monochromatic light beam;
- **Recycling mirrors:** these instruments are located before and after the beam-splitter and they are useful to reintroduce the light beam into the interferometer in order to minimize the losses of power;
- **Fabry-Prot cavities:** the arms of the interferometers have Fabry-Prot cavities constituted by the input test-masses and the end test-masses. Here the light resonates many times in order to increase the effective length of its path and to boost the intensity of the laser (which reaches ~ 50 kW);
- **Suspensions:** based on the concept of inverse pendulum, they are used to keep the test-masses in a fixed position, avoiding motions due to external unwanted contributions (e.g. seismic motions).

What does an interferometer measure? Let $\omega_{\text{lsr}}, \mathbf{k}_{\text{lsr}}$ the frequency and the wavenumber of the laser beam (different from the frequency of the GW we want to detect). The electric (and magnetic) field of the laser is (complex notation)

$$E(t, \mathbf{x}) = E_0 e^{-i\omega_{\text{lsr}}t + i\mathbf{k}_{\text{lsr}} \cdot \mathbf{x}}. \quad (9.1)$$

We denote L_x and L_y the lengths of the two arms and we assume that the beam-splitter divides the light in two equal beams. A photon that reaches the beam-splitter at time t_0 , travels through an arm and comes back, arrives again at the beam-splitter at a time $t' = t_0 + 2L_x/c$ (or $t' = t_0 + 2L_y/c$, depending on the direction). Then, considering the reflection and transmission factors ¹, we can write the electric fields of the two beams at the position of the

¹The incoming field is divided in two at the beam-splitter. Here the first beam is reflected ($r = 1/\sqrt{2}$), then it goes through the interferometer arm, it reaches the perfectly reflecting end-mirror ($r = -1$) and it goes back to the beam-splitter where it is transmitted ($t = 1/\sqrt{2}$). The other beam is immediately transmitted through the beam-splitter ($t = 1/\sqrt{2}$), then it also reaches the perfectly reflecting end-mirror ($r = -1$) and goes back to the beam-splitter, but here it is reflected from the other side of the mirror ($r = -1/\sqrt{2}$). The overall is $-1/2$ for the field in the x -arm and $+1/2$ for the field in the y -arm.

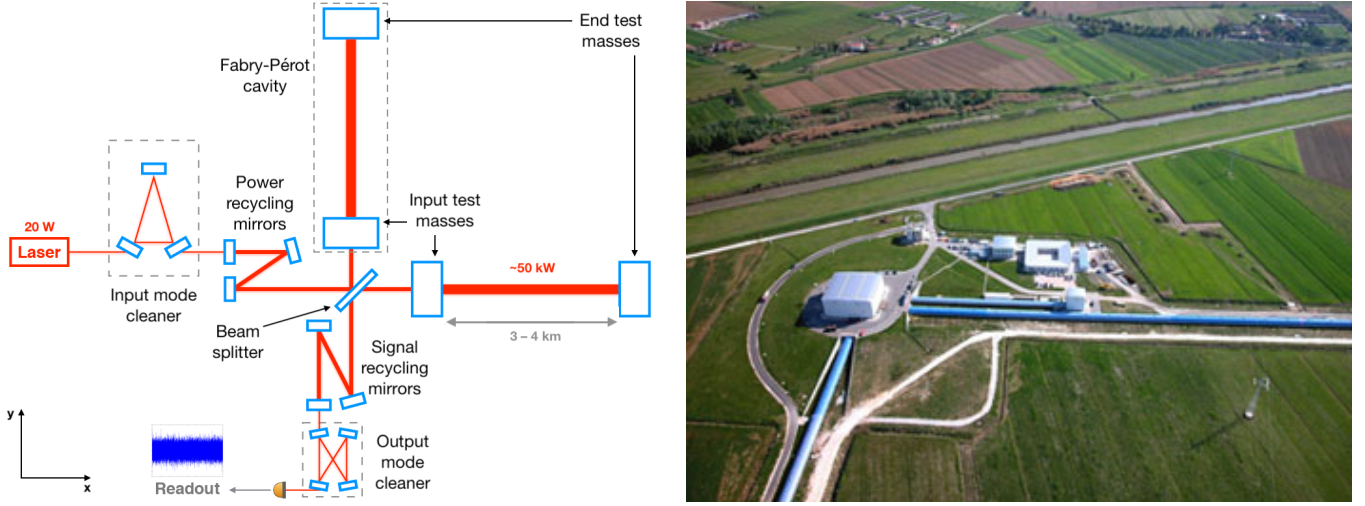


Figure 9.1: Characterization of GWs detectors. Left: Schematic example of ground-based laser interferometer, as LIGO or Virgo. The figure highlights the most important components of the instrument. Right: Picture of the European Gravitational Observatory (EGO) center and the Virgo interferometer, located near Pisa, Italy.

beam-splitter before the beams' recombination as

$$E_x(t) = -\frac{1}{2}E_0 e^{-i\omega_{\text{lsr}}t + 2ik_{\text{lsr}}L_x}, \quad E_y(t) = \frac{1}{2}E_0 e^{-i\omega_{\text{lsr}}t + 2ik_{\text{lsr}}L_y}. \quad (9.2)$$

The recombined field is $E_{\text{out}}(t) = E_x(t) + E_y(t)$. Writing $2L_x = (L_x + L_y) + (L_x - L_y)$ and the same for $2L_y$, we see that

$$E_{\text{out}} = -iE_0 \sin[k_{\text{lsr}}(L_y - L_x)] e^{-i\omega_{\text{lsr}}t + ik_{\text{lsr}}(L_x + L_y)}, \quad (9.3)$$

and the power measured at the photodiode is proportional to

$$|E_{\text{out}}|^2 = E_0^2 \sin^2[k_{\text{lsr}} \underbrace{(L_y - L_x)}_{=:\Delta L}]. \quad (9.4)$$

Therefore any variation in the flight-time of the beams (ΔL) results in a corresponding variation of the power at the photodetector.

During the scientific observation cycles (technically speaking when the detectors are *locked*), the interferometers work in the dark fringe: the phases of the light beams are taken to be anti-aligned in order to have destructive interference at the photodiode. This means that, in absence of GW signal in the detector, the laser power gathered at the photodiode is zero, on average. The dark fringe is used because it is easier to detect deviations from the null value. Indeed, what we want to measure is the fractional displacement of the arms $\Delta L/L \propto h(t)$, which is encoded in the phase of the recombined beam. Since this deviation is expected to be small, by expanding Eq. (9.4), at the lowest order we get $|E_{\text{out}}|^2 \propto \Delta L^2/L^2$. At this point is clear that if we want to measure a small deviation which affect the detector at quadratic order, we need to set the interferometer in the most sensitive condition, and then the dark fringe is preferred.

Sources of noise. Noise can be roughly defined as the output of the detectors that is not a GW signal. Full characterization of the noise sources is essential to detect GW with amplitudes of the order of $\sim 10^{-20}$. There are many sources of noise in ground-based detectors, the main one are discussed in the following. The noise at each frequency is described by its power spectral density (PSD) $S_n(f)$ (which we will define in Sec. 9.2).

- **Seismic noise:** this contribution has a spectrum in all three dimensions close to $10^{-7} f^{-2} \text{ m}/\sqrt{\text{Hz}}$. Isolation can be provided in a relatively simple way by making use of the fact that, for a simple pendulum system, the transfer function to the pendulum mass of the horizontal motion of the suspension point falls off as f^{-2} above the pendulum resonance. In a similar way, isolation can be achieved in the vertical direction by suspending a mass on a spring. In the case of the Virgo detector, operations are possible below 10 Hz. The system design is made of a seven-stage horizontal pendulum arrangement with six of the upper stages being suspended with cantilever springs to provide vertical isolation.
- **Gravity gradients:** this effect is caused by direct gravitational coupling of mass density fluctuations to the suspended mirrors. The dominant source of gravity gradients arise from seismic surface waves, where density fluctuations of the Earth's surface are produced near the location of the individual interferometer test masses. The seismic contribution to the PSD of the rms motion $\tilde{x}(f)$ of the test-masses can be shown to be

$$S_n^{\text{seism}}(f) \propto \left(\frac{G\rho}{\pi f^2} \right)^2, \quad (9.5)$$

where ϱ is the Earth's density near the test-masses. A monitor and subtraction method can be used to control this source of noise. An array of seismometers can be distributed strategically around each test mass to monitor the relevant ground motion (and ground compression) that would be expected to couple through local gravity. A subtraction signal can be developed from knowing how the observed density fluctuations couple to the motion of each test-mass, and can potentially allow a significant reduction in gravity-gradient noise.

- **Thermal noise:** The biggest contributions are associated with the mirror masses and the last stage of their suspensions, especially in the low frequency range. Current detectors are limited by thermal noise at their most sensitive frequency band. In fact, above the operating range there are the internal resonances of the test masses. The thermal noise in the operating range comes from the tails of these resonant modes. In order to estimate the internal thermal noise of a test mass, each resonant mode of the mass can be regarded as a harmonic oscillator. Using a direct application of the fluctuation-dissipation theorem to the optically-sensed position of the mirror surface, one finds that the PSD of the test-masses position can be described by

$$S_n^{\text{therm}}(f) = \frac{2 k_B T}{\pi^2 f^2} \frac{W_{\text{diss}}}{F_0}, \quad (9.6)$$

where F_0 is the peak amplitude of the oscillatory force and W_{diss} is the power dissipated in the mirror. This expression highlights where mechanical dissipation is located with respect to the sensing laser beam. In particular, the thermal noise associated with the multi-layer dielectric mirror coatings, required for high reflectivity, will in fact limit the sensitivity of second-generation gravitational-wave detectors at their most sensitive frequency band. In order to keep thermal noise as low as possible the mechanical loss factors of the masses and pendulum resonances should be as low as possible. Furthermore, the test-masses must have a shape such that the frequencies of the internal resonances are kept as high as possible. They must be large enough to accommodate the laser beam spot without excess diffraction losses, and they must be massive enough to keep the fluctuations due to radiation pressure at an acceptable level.

- **Quantum noise:** There are two sources of quantum noise: the uncertainty of the number of detected photons in the photodiode, called shot-noise, and the perturbations due to the radiation pressure of the laser beam. An obvious point to choose is halfway up a fringe since the change in photon number produced by a given differential change in arm length is greatest at this point, and consequently we minimize the uncertainty in the dark fringe. It is possible to obtain the explicit expression for the PSD for the shot-noise assuming that the number of photon collected at the photodiode is a stochastic variable with Poissonian distribution. The result is

$$S_n^{\text{sh}} = \frac{\hbar c^2}{I_0 \omega_0}, \quad (9.7)$$

where I_0 and ω_0 are respectively the intensity and the frequency of the light beam inside of the detector. We can see that this term is not frequency-dependent. For the radiation pressure contribution, one can use the classical relation for the light pressure $F_{\text{rp}} = 2I_0/c$ and we get,

$$S_n^{\text{rp}} = \frac{2\hbar I_0 \omega_0}{m^2 c^2 (2\pi f)^4}, \quad (9.8)$$

where m is the mass of the mirror. From Eq. (9.7) and Eq. (9.8), it is possible to define an single quantum PSD as

$$S_n^{\text{quant}} = \left(\kappa + \frac{1}{\kappa} \right) \frac{h_{\text{SQL}}^2}{2} \geq h_{\text{SQL}}^2, \quad (9.9)$$

where

$$\kappa = \frac{2I_0 \omega_0}{m(\pi c f)^2}, \quad h_{\text{SQL}} = \sqrt{\frac{2\hbar}{m(\pi L f)^2}}.$$

The quantity h_{SQL} determines the standard quantum limit, which represents a fundamental threshold for the observations. However, it is possible to exceed this limit using a particular optical technique, called *squeezing*. This method acts on the coherent states of the laser beam selecting the quadrature of interest and reducing the quantum fluctuations around this value.

Current detectors. Currently, the international network for GW observation is constituted of four operative ground-based interferometers. In U.S.A., there are the two LIGO (Laser Interferometer Gravitational-Wave Observatory) detectors: these machines have arms of 4 km and the laboratories are located in Hanford, Washington, and Livingston, Louisiana. The other two detectors are in Europe. One is GEO600 in Hannover, Germany, with arms of 600 m and founded by the Max Planck Society and the Science and Technology Facilities Council (STFC) Collaboration. This detector is smaller than the others and used to develop technology and “astrowatch” mode. The other one is Virgo with arms of 3 km and located near Pisa, Italy. It was funded by INFN (Italy) and CNRS (France) organizations and run by the EGO (European Gravitational Observatory).

The initial LIGO and Virgo operations started in 2000s, but they did not detect any GWs. The interferometers were subsequently upgraded to their “advanced” configurations. This first observation period O1 of the Advanced

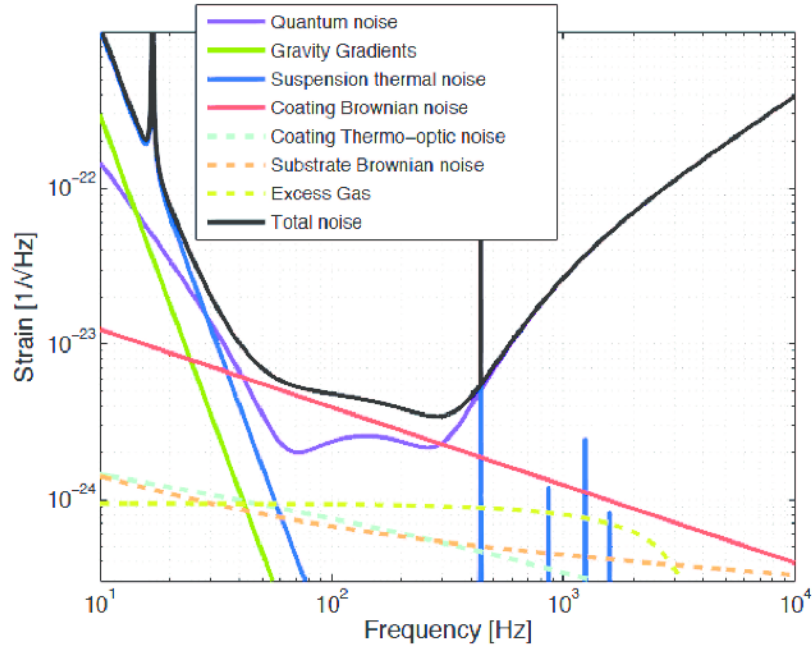


Figure 9.2: Amplitude spectral density of the design configuration planned and Advanced Virgo. The figure shows the different contributions of the several sources of noise present in the interferometers. At low frequency, the sensitivity is limited by the seismic motions, the several thermal contributions and the radiation pressure noise of the laser, while in the high frequency limit, the instruments are strongly limited by the quantum shot-noise. However, the realistic picture is more complex since many other external effects contribute in the noise and the ASDs show several high amplitude noise features. These peaks are due to several practical complications, e.g. the AC power grid (harmonics of 60 Hz in the U.S. and 50 Hz in Europe), mechanical resonances, injected calibration lines, contributions due to control systems and many others.

LIGO configurations started in 2015 and lasted for 3 months. Advanced Virgo started commissioning in 2016, joining the two LIGO detectors in May 2017 during the second observation period O2. In these two periods of observations (O1 and O2), the LIGO-Virgo collaboration (LVC) observed 11 gravitational transients coming from binary mergers of compact objects, such as black holes and neutron stars. These events are collected in a catalog and described in Abbott et al. (2019b). O3 runs started in 2019 and continued to spring 2020.

In 2020, another ground-based interferometer joined the network: KAGRA (Kamioka Gravitational Wave Detector) that is located in the Kamioka Observatory, Japan, near the neutrino physics experiments. KAGRA is 3 km long and it implemented two new technologies: it is built underground to minimize the effect of seismic motions and it uses cryogenic temperatures in order to reduce the Brownian noise. Moreover, another proposal comes from the LIGO collaboration for a detector in India.

Binary-black-hole merger observation: GW150914. The first direct observation of a gravitational transient was made on 14 September 2015 and was announced by the LIGO and Virgo collaborations on 11 February 2016. Formally, this event was observed by the LIGO detector before O1, during an engineering run (i.e. period in which the performances of the detector are tested). The whitened strains observed by the LIGO interferometers are shown in Fig. 9.6 with the respective spectrograms. This event corresponds to a coalescence of two black holes with masses respectively of $\sim 35 M_{\odot}$ and $\sim 30 M_{\odot}$ located at ~ 400 Mpc from the Earth. The spectrograms in Fig. 9.6 show that frequency increases with time coherently with the chirp-like evolution described in (Chap. 3) followed by a ringdown (Chap. 7). At the merger, the two objects reached a relative velocity of $\sim 0.55 c$ emitting $\sim 3.6 \times 10^{49}$ W. After the merger, the collision generated a single remnant black hole with mass $\sim 62 M_{\odot}$ and during the dynamics the system emitted in total $\sim 3 M_{\odot} c^2$ through GWs.

This first direct observation was reported around the world as a remarkable accomplishment for many reasons and now it represents a milestone of modern physics. First of all, this is a second proof of the existence of gravitational radiation after the pulsars observations and the first detection of a GW passing through Earth. Furthermore, GW150914 demonstrates the existence of stellar-mass black holes more massive than $25 M_{\odot}$, and establishes that binary black holes can form in nature and merge within a Hubble time. Thanks to the discovery of GWs, the 2017 Nobel Prize in Physics was awarded to Rainer Weiss, Barry Barish and Kip Thorne “[...] for decisive contributions to the LIGO detector and the observation of gravitational waves”.

Binary neutron star observation: GW170817. GW170817 is a signal observed by the LIGO and Virgo detectors on 17 August 2017, originating from the shell elliptical galaxy NGC 4993. The GW was produced by the last minutes of two neutron stars spiralling closer to each other and finally merging. The two stars had a mass of $\sim 1.4 M_{\odot}$ each

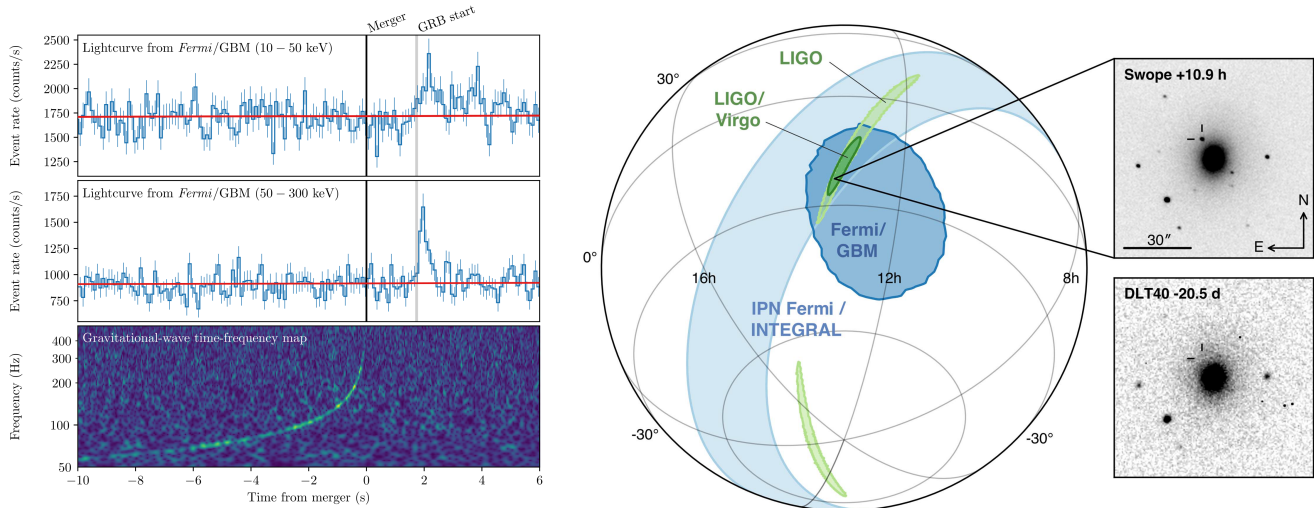


Figure 9.3: Left panel: the figure shows the coincident measurements of the gravitational signal GW170817 and the short gamma-ray burst GRB170817 observed ~ 2 s after the GW peak by Fermi spacecraft. Right panel: combined measurement of the sky location of GW170817 source. It is possible to see that using the information from three detector it is possible to strongly constraint the position of the source with GWs.

and the system was located at ~ 40 Mpc from the Earth. Neutron stars are very interesting objects since the matter that compose them has a density larger than the nuclear density, and its behaviour is currently not fully understood, and they represent natural laboratories in which we can test our theories.

The gravitational wave signal GW170817 lasted for approximately 100 seconds starting from a frequency of 24 Hz. It covered approximately 3000 cycles, increasing in amplitude and frequency to a few hundred hertz in the typical inspiral chirp pattern. An automatic computer search of the LIGO-Hanford datastream triggered an alert to the LIGO team about 6 minutes after the event. The gamma-ray burst (GRB) alert had already been issued at this point (16 seconds post-event), so the timing near-coincidence was automatically flagged. The LIGO/Virgo team issued a preliminary alert (with only the crude gamma-ray position) to astronomers in the follow-up teams at 40 minutes post-event. Sky localisation of the event requires combining data from the three interferometers; this was delayed by two problems. The Virgo data were delayed by a data transmission problem, and the LIGO Livingston data were contaminated by a brief burst of instrumental noise (called *glitch*) a few seconds before the peak of the signal. These required manual analysis and interpolation before the sky location could be announced about 4.5 hours post-event. The three detections localized the source to an area of ~ 30 square degrees in the southern sky at 90% probability. This narrow sky region allowed the astronomical telescopes to locate the source of the signal and they were able to observe the electromagnetic counterpart of this violent collision. This light emitted by this source was observed by many observatories on the Earth and in the space, and in many observational bands, allowing us to perform the first multi-messenger observation with GW and light.

Next-generation detectors. Current detectors perform measurements in the frequency range 20 Hz to 1 kHz and at the threshold of the quantum noise. On one hand, there is a push to accessing higher frequencies and improve the current ground-based interferometer sensitivity. Higher frequencies are interesting for example for signal from neutron star mergers (not only inspiral). On the other hand, other frequency ranges can explore a variety of new sources like supermassive black holes or white dwarf. Other interferometric antennas are being set up to start multi-band GW astronomy.

An ongoing project is LISA, acronym for Laser Interferometer Space Antenna. This is a mission led by the European Space Agency (ESA) and it would be the first dedicated space-based gravitational wave detector. The LISA concept has a constellation of three spacecraft arranged in an equilateral triangle with sides millions of kilometers long, flying along an Earth-like heliocentric orbit. Each satellite corresponds to a test mass and the distance between them is measured with interferometric techniques. The size of this interferometer allows one to observe GWs in the frequency range from 0.1 mHz to 1 Hz. In this band, we expect to find the signals from heavy mass black holes, like galactic collisions, and the inspirals from extreme mass ratio binary coalescences. On December 2015, the ESA launched the LISA Pathfinder. In this mission, two test masses were placed in a nearly perfect gravitational free-fall and their relative motion was monitored. The laser interferometer measured the relative position and orientation of the masses to an accuracy of less than 0.01 nanometres. This was a result beyond the expectation and this accuracy is estimated to be sensitive enough to detect GWs by the follow-on mission. LISA is expected to be launched in 2034.

Another EU project GW observations is the Einstein Telescope (ET). This proposal is for an interferometer located underground in order to reduce seismic noise and gravity gradient noise (the same technique is adopted by the current interferometer KAGRA). The arms will be 10 km long, and like LISA, there will be three arms in an equilateral triangle, with two detectors in each corner. In order to measure the polarization of incoming gravitational waves and

avoid having an orientation to which the telescope is insensitive, a minimum of two detectors are required. While this could be done with two 90° interferometers at 45° to each other, the triangular form allows the arms to be shared. The 60° arm angle reduces each interferometer's sensitivity, but that is made up for by the third detector, and the additional redundancy provides a useful cross-check. Each of the three detectors would be composed of two interferometers, one optimized for operation below 30 Hz and one optimized for operation at higher frequencies. The low-frequency interferometers will use cooled optics, with a beam power of about 18 kW in each arm cavity. The high-frequency ones will use room-temperature optics and a much higher recirculating beam power of 3 MW. The ASD is expected to be a couple of orders of magnitude below the design sensitivity of the current detectors. This might allow us to see the post-merger dynamics of binary neutron star mergers, currently hidden by the high-frequency noise. These sources are expected to emit radiation in the band from 2 kHz to 6 kHz and these phenomena could reveal unknown aspects of the nuclear matter in high-density regimes.

Another proposal for a ground-based detector is Cosmic Explorer (CE). This mission is planned to be funded by U.S. contribution and the project is a 40 km L-shaped observatory. As ET, this detector will be able to determine the nature of the densest matter in the universe and reveal the universes binary black hole and neutron star populations throughout cosmic time, providing an independent probe of the history of the expanding universe. Thanks to its large size, CE is expected to perform measurements with high accuracy and it will be able to extend the band coverage of the current detectors, especially in the low-frequency region.

9.2 Noise characterization

The output of a standard GW detector is a time series $s(t)$ which contains also the gravitational-wave time series $h(t)$ ². In general, $s(t)$ is not the signal detected at the photodiode: assuming an experimental apparatus with linear response, the output signal s_{out} is a linear function of the Fourier transform of the signal,

$$\tilde{s}_{\text{out}}(f) = T(f)\tilde{s}(f) , \quad (9.10)$$

where $T(f)$ is the transfer function. If the apparatus is composed by different components, the resulting transfer function is the product of the $T_i(f)$ of each component. In absence of noise, $s(t) = h(t)$ and thus one expects that

$$\tilde{h}_{\text{out}}(f) = T(f)\tilde{h}(f) . \quad (9.11)$$

In general $s(t) \neq h(t)$ because the detectors picks up many other signals that are not GW. All these other signals form the noise.

Power spectral density (PSD) and amplitude spectral density (ASD). To introduce the concept of *noise* imagine that each time the laser beam interacts with a detector component spurious signal fluctuations are generated. Let the output be a superposition of the gravitational signal and the noise,

$$s_{\text{out}}(t) = h_{\text{out}}(t) + n_{\text{out}}(t) , \quad (9.12)$$

and define a fictitious input noise,

$$\tilde{n}(f) = T^{-1}(f)\tilde{n}_{\text{out}}(f) . \quad (9.13)$$

as if all the noise is generated in input and propagates with the gravitational signal in the apparatus. Then we can write

$$s(t) = h(t) + n(t) , \quad (9.14)$$

and we can think that the detector output is generated by the signal $h(t)$ and the noise $n(t)$, both dimensionless quantities.

If the noise is stationary, i.e. its properties are time independent, the different Fourier components are uncorrelated, then

$$\langle \tilde{n}^*(f)\tilde{n}(f') \rangle = \frac{1}{2} \delta(f - f') S_n(f) , \quad (9.15)$$

where $\langle \dots \rangle$ denotes the average over a the ensemble³ and $S_n(f)$ is the *power spectral density (PSD)*. Since $n(t)$ is real $\tilde{n}(-f) = \tilde{n}^*(f)$, $S_n(-f) = S_n(f)$. If $n(t)$ is dimensionless, the PSD has dimensions Hz^{-1} . Moreover, we observe that in a real scenario we work with finite segment of data of a given period T , and we are dealing with discrete functions fo time and frequency. Therefore the integral of the δ function does not diverge for $f = f'$;

$$\delta(f = 0) \rightarrow \left[\int_0^T e^{2\pi i f t} dt \right] = T , \quad (9.16)$$

and from Eq. (9.15) we get

²The GW is described by a tensor, but reduces to a scalar time series once projected onto the detctor antenna pattern.

³The average over the ensemble is the average over the possible realizations of our system. But we have only one possible system, i.e. our detector, then the average over the ensemble is replaced with the time average over a period T . The noise contained in the data is considered as one of the possible realizations. This is equivalent to the assumption of ergodic system. We also observe that the resolution of the Fourier transform $\tilde{n}(f)$ is $\Delta f = 1/T$.

$$\langle |\tilde{n}(f)|^2 \rangle = \frac{T}{2} S_n(f). \quad (9.17)$$

The power spectral density can be introduced in a second, more rigorous, way. Assume the combination of all the noise sources in a detector produces a time series $n(t)$ that can be represented by a vector n_i , with components given by the discrete time samples $n_i = n(t_i)$. The noise is described as a stochastic process with statistical properties given by the probability distribution $p(n)$. This model can be used to define summary statistics such as the mean $\mu = E[n]$ (where E is defined as the expectation value) and covariance $C_{ij} = E[(n_i - \mu)(n_j - \mu)]$ where the expectation values are taken with respect to $p(n)$. The mean can be estimated from the data as

$$\mu = \frac{1}{N} \sum_{i=1}^N n_i, \quad (9.18)$$

where N is the total number of points. Note that, in practice, the analyses do not use all N samples at once, but rather use segments of contiguous data of various lengths from a few seconds up to hours depending on the application. Noise is referred to as *Gaussian* if the joint probability distribution follows a multi-variate normal distribution:

$$p(n) = \frac{1}{\sqrt{\det(2\pi C_{ij})}} \exp \left[-\frac{1}{2} \sum_{i,j} (n_i - \mu) C_{ij}^{-1} (n_j - \mu) \right], \quad (9.19)$$

where C_{ij}^{-1} is the inverse covariance matrix. The noise is referred to as *stationary* if C_{ij} depends only on the lag $|i - j|$. Recalling that for every index i exists a related time t_i , the condition of stationarity can be written as

$$C_{ii+k} = C_{jj+k}, \quad \forall i, j, k.$$

Stationary noise is characterized by the correlation function $C(\tau)$, where $\tau = |t_i - t_j|$ is the time lag, and then the mean value μ and the other statistical properties are constant with respect to time.

Transforming to the Fourier domain, where the labels i, j now refer to frequencies f_i, f_j , stationary noise has a diagonal covariance matrix $C_{ij} \rightarrow \delta_{ij} S_n(f)$, which defines the PSD $S_n(f)$. The power spectral density is given by the Fourier transform of the correlation function $C(\tau)$,

$$S_n(f) = 2 \int_{-\infty}^{+\infty} C(\tau) e^{+2\pi i f \tau} d\tau, \quad (9.20)$$

and, on the other hand,

$$C(\tau) = \int_0^{\infty} S_n(f) e^{-2\pi i f \tau} df. \quad (9.21)$$

The last integral is limited to positive frequencies $f > 0$ because $C(\tau)$ is a real function and $S_n^*(f) = S_n(-f)$. Sometimes the *amplitude spectral density (ASD)* is also used to characterize the noise, this quantity corresponds to the square root of the PSD, $\sqrt{S_n(f)}$.

From the above argument, we understand that the analysis in the frequency-domain is better suited for our task, since the covariant $N \times N$ matrix is reduced in a diagonal form. Stationary noise is uncorrelated between frequency bins, and the noise $\tilde{n}(f)$ in each bin follows a Gaussian distribution with random phase and amplitude $S_n^{1/2}(f)$. However, the reality is more complex and LIGO-Virgo data analysis pipelines must take into account the effect of non-stationarity and non-Gaussian deviations.

The PSD is a key quantity in GW analyses but it is not known a priori, it must be estimated from the data. In order to estimate PSD of a noise segment of data, one can perform a complex Fourier transform of the entire data stream around some time to be searched for signals, but that yields to a result strongly affected by noise fluctuations. To overcome this, it is common to use some form of averaging and these methods can be used to reduce the variance in the estimated power spectrum, but at the cost of either reducing the frequency resolution or requiring longer stretches of data. An example is the *Welch averaging method*. If we have a noise time series of duration T , the method consists in dividing the strain in M chunks with a fixed duration of T/M and estimate the Fourier spectrum for all of them, then the final PSD is the average of the square amplitudes of the spectra of the different chunks.

Remark 9.2.1. *In order to estimate the PSD, the time-domain series should not include, in principle, a GW signal.*

Remark 9.2.2. *The technical tool used to estimate the Fourier spectrum of a data segment is the fast Fourier transform (FFT) ⁴, a fast algorithm that reduces the complexity of computing the discrete Fourier transform (DFT)*

⁴https://en.wikipedia.org/wiki/Fast_Fourier_transform

of a time series of N points from $\mathcal{O}(N^2)$ (matrix-vector multiplication) to $\mathcal{O}(N \log(N))$. The DFT of a finite time series $a_i(t_i)$ $i = 1, \dots, N$ is defined as

$$\tilde{a}_j(f_j) = \sum_{i=0}^N a_i e^{-\frac{2\pi i}{N} i j}, \quad (9.22)$$

where $N = T \cdot F_s$ and F_s defines the sampling frequency. The DFT $\tilde{a}_j$ is thus defined on a range of frequencies $f_j \in [-\frac{F_s}{2}, +\frac{F_s}{2}]$ with a step of $\Delta f = \frac{1}{T}$. Thus, the length of the time-series T defines the resolution in the frequency domain. Note that the Fourier transform assumes a periodical signal, but in practical applications the DFT/FFT is always applied to a finite time series. A related issue in signal processing with the FFT is windowing, that is essential in order to avoid spectral leakage effects. This will be further discussed in Rem. 9.4.1.

9.3 Antenna pattern

The tensor $h_{ij}^{\text{TT}}(t)$ is reduced to a scalar quantity $h(t)$ when it is projected onto the detector. The scalar $h(t)$ depends on the geometry of the detector and on the direction of origin of the GW. An interferometer has not the same sensitivity in all the directions.

From linearized GR, we know that we decompose write a GW signal in its polarizations,

$$h_{ij}(t, \mathbf{x}) = \sum_{A=+, \times} e_{ij}^A(\mathbf{n}) h_A(t, \mathbf{x}), \quad (9.23)$$

where e_{ij}^A are the polarization matrix defined in Eq. (3.11), $\mathbf{n}$ is the unitary vector that defines the direction of propagation of the GW and $\mathbf{x} = (r, \theta, \phi)$ are the polar coordinates centred in the position of the interferometer. First of all, we note that we assume that the observed GWs have wavelengths much larger than the size of the detector (like it is for the ground-based interferometers). Then, we are allowed to neglect the spatial dependency of the GW.

In general the output of a GW detector is a scalar quantity $h(t)$. The measured strain is proportional to the emitted GW (which is a tensorial quantity) through the detector tensor D_{ij} as

$$h(t) = D_{ij} h_{ij}(t) = \sum_{A=+, \times} D_{ij} e_{ij}^A(\mathbf{n}) h_A(t, \mathbf{x}), \quad (9.24)$$

where $D_{ij}(\mathbf{n})$ is a constant tensor which depends only the geometry of the detector. It is then convenient to define the *antenna pattern functions* $F_A(\mathbf{n})$,

$$F_A(\mathbf{n}) = D_{ij} e_{ij}^A(\mathbf{n}), \quad \text{for } A = +, \times. \quad (9.25)$$

The antenna pattern functions depend on the direction of propagation $\mathbf{n} = (\theta, \phi)$ of the incoming GW and thanks to them we can rewrite Eq. (9.24) as

$$h(t) = F_+(\mathbf{n}) h_+(t) + F_\times(\mathbf{n}) h_\times(t). \quad (9.26)$$

The above equation implicitly assumes that we fixed a system of axes $(\mathbf{u}, \mathbf{v})$ in the plane orthogonal to the propagation direction $\mathbf{n}$ of the wave, with respect to which the polarizations h_+ and $h_\times$ are defined. It is interesting to generalize this expression for any arbitrary set of axes, performing a rotation of ψ around $\mathbf{n}$. The axes $(\mathbf{u}, \mathbf{v})$ are rotated to new axes according to

$$\mathbf{u}' = \mathbf{u} \cos \psi - \mathbf{v} \sin \psi, \quad (9.27)$$

$$\mathbf{v}' = \mathbf{u} \sin \psi + \mathbf{v} \cos \psi. \quad (9.28)$$

In the new frame, the two polarizations change their amplitudes $h_{+, \times} \rightarrow h'_{+, \times}$ according to

$$h_+ = h_+ \cos(2\psi) - h_\times \sin(2\psi), \quad (9.29)$$

$$h_\times = h_\times \cos(2\psi) + h_+ \sin(2\psi). \quad (9.30)$$

Indeed, in the new frame, the polarization tensors are given by

$$(e_{ij}^+)' = u'_i u'_j - v'_i v'_j, \quad (9.31)$$

$$(e_{ij}^\times)' = u'_i v'_j + v'_i u'_j. \quad (9.32)$$

Combining the last equation with Eq. (9.27), it is possible to get the transformation law Eq. (9.29). This equation tells us how the polarization angle affects the observed strain.

The pattern functions $F_{+, \times}$ depends on the polarization tensors $e_{ij}^{+, \times}$ through Eq. (9.25). Thus, it is convient to include the dependency on the polarization angle in the antenna functions,

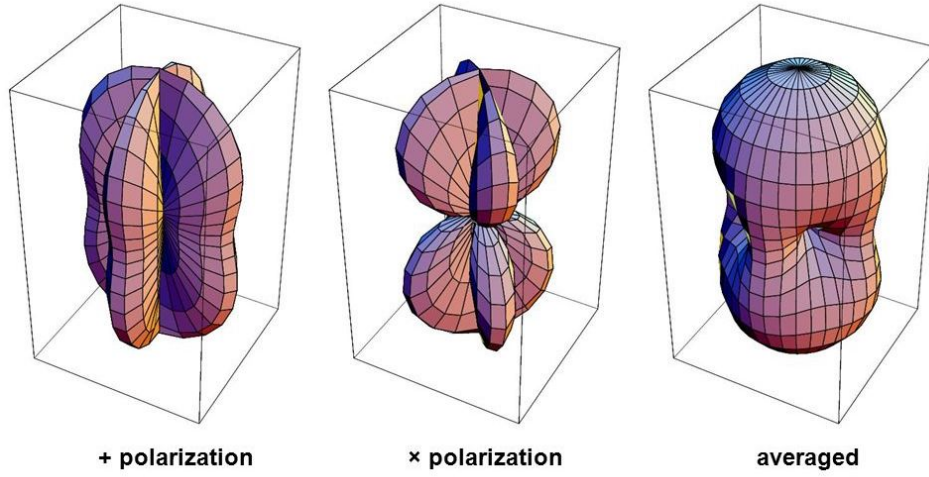


Figure 9.4: Antenna pattern functions for an L-shaped interferometer detector. The figures show the value of $F_A(\theta, \phi, \psi = 0)$ for $A = +, \times$ in cartesian coordinate, where the detector is located at the centre of the box. The average corresponds to the root-mean-square (rms) sensitivity, $F_{\text{rms}} = (F_+^2 + F_\times^2)^{1/2}$.

$$F_+(\mathbf{n}, \psi) = F_+(\mathbf{n}, 0) \cos(2\psi) - F_\times(\mathbf{n}, 0) \sin(2\psi), \quad (9.33)$$

$$F_\times(\mathbf{n}, \psi) = F_\times(\mathbf{n}, 0) \sin(2\psi) + F_+(\mathbf{n}, 0) \cos(2\psi), \quad (9.34)$$

where $F_{+,\times}(\mathbf{n}, 0)$ are the functions previously defined. Of course, fixing the axes $(\mathbf{u}, \mathbf{v})$, the polarization angle ψ is constant for an incoming GW signal and the antenna functions depend only on the direction of propagation. However, it is useful to keep the general notation in order to denote how the measured strain is modified by a rotation of ψ . A useful identity satisfied by the antenna functions, independently from the specific form of the detector tensor D_{ij} , is

$$\int F_\times(\mathbf{n}) F_+(\mathbf{n}) d\Omega = 0, \quad (9.35)$$

where $d\Omega = \sin\theta d\theta d\phi$ is the integral over the solid angle. Moreover, it is interesting to observe that it is always possible to choose a frame (i.e. a value of ψ) where so that $F_\times$ vanishes and F_+ is non-zero, or viceversa. Note that integrating over the angle ψ we get

$$\int_0^{2\pi} F_+^2(\mathbf{n}, \psi) d\psi = \int_0^{2\pi} F_\times^2(\mathbf{n}, \psi) d\psi. \quad (9.36)$$

By studying in detail the interaction between the GWs and a laser interferometer using the equation of the geodesic deviation, it is possible to prove that the antenna pattern functions for such a detector have the following forms,

$$F_+(\mathbf{n}, \psi = 0) = \frac{1}{2} (1 + \cos^2 \theta) \cos(2\phi), \quad (9.37)$$

$$F_\times(\mathbf{n}, \psi = 0) = \cos \theta \sin(2\phi), \quad (9.38)$$

where we expanded the dependency on the direction $\mathbf{n}$ using the polar angles θ, ϕ .

As it is shown in Fig. 9.4, the antenna functions are relatively smooth functions of the position of the source in the sky. On one hand, this is an advantage because the detector has a good coverage of the sky, of almost 4π , except for some blind spots (differently from a conventional astronomic telescope). On the other hand, however, with a single detector it is impossible to locate the source in the sky. Indeed, Eq. (9.26) depends on four variables: the two polarizations $h_{+,\times}$ and the two angles θ, ϕ . Then, if we assume to use two interferometers, we have only three measurements: the strains $s_1(t)$, $s_2(t)$ observed by the detectors and the time delay between them τ_{12} . These quantities are not enough to solve Eq. (9.26). However, if we use three detectors, we will get five measured quantities (the three strains $s_i(t)$, $i = 1, 2, 3$, and two independent delay times) and we are able to solve Eq. (9.26). Then it is possible to understand that a global network of interferometers is necessary to fully understand the nature of these signals.

9.4 Data analysis & Match filtering

We have seen that we can write the detector output as $s(t) = h(t) + n(t)$. For the typical GW event expected by astrophysical events and the sensitivity of the current ground-based detectors it holds that

$$|h(t)| \ll |n(t)|. \quad (9.39)$$

Then, we ask ourselves: How can we distinguish a signal in a recorded strain of data?

This is a problem of signal processing in which the noise is louder than our signal; an optimal technique to solve the problem is to use data *filtering*. To understand the idea of filtering, suppose that we know the form of $h(t)$ that we are looking for. Then we can average the quantity $s(t) \cdot h(t)$ over a period T of time,

$$\frac{1}{T} \int_0^T s(t) \cdot h(t) dt = \frac{1}{T} \int_0^T h^2(t) dt + \frac{1}{T} \int_0^T n(t) \cdot h(t) dt. \quad (9.40)$$

If $h(t)$ and $n(t)$ are uncorrelated oscillating functions, then, the second integral will grow as $T^{-1/2}$ (as it is for systems behaving as random-walks),

$$\frac{1}{T} \int_0^T n(t) \cdot h(t) dt \propto \frac{n_0 h_0}{\sqrt{T}}, \quad (9.41)$$

where h_0 and n_0 are respectively the characteristic amplitudes of the signal and the noise. On the other hand, the first integral is positive definite and it contributes as

$$\frac{1}{T} \int_0^T h^2(t) dt \propto h_0^2. \quad (9.42)$$

Thus, for $T \rightarrow \infty$, the second term of Eq. (9.40) is averaged to zero and we have *filtered out* the contribution of the noise from the output. In a realistic scenario, we cannot use an infinite number of data points and we are limited to the observation time. Still, denoting with τ_0 the time-scale of a GW event, we see that this method is able to identify a signal against a given background noise in the regime $n_0 \sqrt{\tau_0/T} \lesssim h_0$, digging below the noise threshold. To give an idea, if we consider a binary black hole coalescence with characteristic time-scale $\tau_0 \sim 0.02$ s, with a segment of $T = 16$ s we can observe GW amplitudes of 3 orders of magnitude below the ASD limit.

Matched filtering. Once clarified the basic idea, let us introduce a mathematical framework. We are interested in a quantify that defines the *loudness* of a recorded signal $s(t)$. Then, we write

$$S = \int_{-\infty}^{+\infty} s(t) K(t) dt = \int_{-\infty}^{+\infty} \tilde{s}^*(f) \tilde{K}(f) df. \quad (9.43)$$

where $K(t)$ is a filter function that has to *match* our signal s . For this equality we made use of the convolution product. The noise can be quantified as the standard deviation of the recorded strain within the assumption $h(t) = 0$,

$$\begin{aligned} \mathcal{N}^2 &= \left[\langle S^2 \rangle - \langle S \rangle^2 \right]_{h=0} = \langle S^2 \rangle_{h=0} \\ &= \int_{-\infty}^{+\infty} K(t) K(t') \langle n(t) n(t') \rangle dt dt' \\ &= \int_{-\infty}^{+\infty} K(t) K(t') \int_{-\infty}^{+\infty} e^{2\pi i t(f-f')} \langle \tilde{n}^*(f) \tilde{n}(f') \rangle df df' dt dt' \\ &= \frac{1}{2} \int_{-\infty}^{+\infty} S_n(f) |\tilde{K}(f)|^2 df, \end{aligned} \quad (9.44)$$

where we used $\langle n(t) \rangle = 0$ and Eq. (9.15).

Now, we assume to know how to describe the GW signal we are looking for (i.e. we have a model for $h(t)$) and we ask ourselves: Which is the best filter function $K(t)$ that maximises amount of signal S with respect to N ? Averaging S over a time period T we get

$$\langle S \rangle = \int_{-\infty}^{+\infty} \tilde{h}^*(f) \tilde{K}(f) df, \quad (9.45)$$

since $\langle n(t) \rangle = 0$. Then computing the ratio between them we obtain

$$\frac{\langle S \rangle^2}{\mathcal{N}^2} = \frac{\left| \int_{-\infty}^{+\infty} \tilde{h}^*(f) \tilde{K}(f) df \right|^2}{\frac{1}{2} \int_{-\infty}^{+\infty} S_n(f) |\tilde{K}(f)|^2 df} = \frac{\left| \int_{-\infty}^{+\infty} \left(\frac{\tilde{h}^*(f)}{\sqrt{S_n(f)}} \right) \left(\tilde{K}(f) \sqrt{S_n(f)} \right) df \right|^2}{\frac{1}{2} \int_{-\infty}^{+\infty} S_n(f) |\tilde{K}(f)|^2 df}. \quad (9.46)$$

Using the Cauchy-Schwarz inequality, the ratio in Eq. (9.46) can be maximised as

$$\frac{\langle S \rangle^2}{\mathcal{N}^2} = 2c \frac{\left(\int_{-\infty}^{+\infty} \frac{|\tilde{h}(f)|^2}{S_n(f)} df \right) \left(\int_{-\infty}^{+\infty} |\tilde{K}(f)|^2 S_n(f) df \right)}{\int_{-\infty}^{+\infty} S_n(f) |\tilde{K}(f)|^2 df} = 2c \int_{-\infty}^{+\infty} \frac{|\tilde{h}(f)|^2}{S_n(f)} df, \quad (9.47)$$

where c is a constant factor. This equation leads to the value of the optimal filter

$$\tilde{K}(f) = 4c \frac{\tilde{h}(f)}{S_n(f)}. \quad (9.48)$$

The above argument suggests to introduce the *Wigner scalar product* between two real functions $a(t)$ and $b(t)$

$$(a|b) = 4 \int_0^\infty \frac{\tilde{a}^*(f) \tilde{b}(f)}{S_n(f)} df, \quad (9.49)$$

where the constant c has been fixed to $c = 1$ and the factor 4 comes from the limitation of the integral to positive frequencies $f \geq 0$. In general $(a|b)$ is a complex number, while $(a|a)$ is always real. Using this definition we can rewrite Eq. (9.43) as

$$S = (s|h), \quad (9.50)$$

and the same for Eq. (9.47),

$$\frac{\langle S \rangle^2}{\mathcal{N}^2} = (h|h), \quad (9.51)$$

An interesting interpretation of Eq. (9.49) is to split the PSD into two ASD terms that can be associated with the two time series, i.e.

$$(a|b) = 4 \int_0^\infty \frac{\tilde{a}^*(f)}{\sqrt{S_n(f)}} \frac{\tilde{b}(f)}{\sqrt{S_n(f)}} df.$$

This ends up weighting both functions by the inverse of the ASD, which are the “whitened forms” of these functions: the integrand of scalar product weights each Fourier’s component of the functions $a(t)$ and $b(t)$ with the intensity of the noise in that specific frequency bin. In other words, the Wigner scalar product is weighting the signals more at the frequencies observed at higher sensitivity, while it weights less the signals at frequencies at which the noise is loud. It is interesting to observe that, if we take a generic GW signal such that

$$h_+(t) = A(t) \cos(\phi(t)), \quad h_\times(t) = A(t) \sin(\phi(t)),$$

then it is possible to prove that

$$(h_+|h_\times) = 0. \quad (9.52)$$

This result leads to the conclusion that the product $(s|h)$ does not mix the contributions from different polarizations and its result is a linear combination of the projections of each component, i.e. if $h = a_+ h_+ + a_\times h_\times$, where $a_{+,\times}$ are real constant values, then $(s|h) = a_+(s|h_+) + a_\times(s|h_\times)$.

The Wigner scalar product allows one to quantify the match (overlap) between a template and a signal and the concept of signal-to-noise ratio in the context of filtered analysis. The *overlap* between two time series $a(t)$ and $b(t)$ is defined as

$$\mathcal{O}[a, b] = \frac{|(a|b)|}{\sqrt{(a|a)(b|b)}}. \quad (9.53)$$

The quantity $\mathcal{O}$ takes value from 0 to 1 and it measures the agreement between the two time series. We observe that $(a|a)$ is proportional to the power spectrum of the whitened series $\tilde{a}(f)/\sqrt{S_n(f)}$, and the same for $b(t)$.

How can we estimate the loudness of the recorded signal assuming a given template $h(t)$? The quantity defined in Eq. (9.43) is able to quantify the agreement between the data and the GW template. If one is interested in the strength of the signal that matches the template h in the data s , one needs to normalize the overlap $(s|h)$ with the “length” of the template. The *signal-to-noise ratio* (*SNR*) is thus defined as

$$\rho := \frac{(s|h)}{\sqrt{(h|h)}}. \quad (9.54)$$

The SNR ρ is proportional to amplitude of the signal and independent from the amplitude of the template. The SNR is a fundamental quantity for GW data analysis since it estimates the amount of signal deposited in the recorded data s with respect to a given template h .

So far we have defined the overlap between s and h where we have assumed the template to be of the same length as our data. However, what we are actually interested in is the matched filter of s using h where the data $s(t)$ is an extended time series whose length is longer than the template time series $h(t)$. Then we are to choose a reference time t_0 and a reference phase angle ϕ_0 for $h(t; t_0, \phi_0)$ such that the SNR amplitude is maximized. Recalling the rules of Fourier functions, we can define the template with respect to arbitrary reference values $t_0 = \phi_0 = 0$ and then write in Eq. (9.54) the explicit dependency from these parameters as

$$\rho(t_0, \phi_0) = \frac{4}{\sqrt{\langle h|h \rangle}} \int_0^\infty \frac{\tilde{s}^*(f) \tilde{h}(f) e^{-i(2\pi f t_0 + \phi_0)}}{S_n(f)} df. \quad (9.55)$$

Moreover, it is interesting to note that for the case $n(t) = 0$, i.e. $s(t) = h(t)$, Eq. (9.54) returns the case of Eq. (9.51). For this reason, the square root of Eq. (9.51) is usually labelled as optimal SNR, $\rho_{\text{opt}} = \sqrt{\langle h|h \rangle}$. This computation is valid for in the case of a single detector; for the multiple detectors it is possible to compute the network SNR summing the squared SNRs of every detector,

$$\rho_{\text{net}}^2 = \sum_{k=L,H,V,\dots} \rho_k^2. \quad (9.56)$$

Remark 9.4.1. The Fourier transform is a key mathematical tool for signal processing. The discrete realization of the FFT is the DFT introduced in Rem. 9.2.2 that can be computed using the FFT algorithm. The FFT technique allows us to transform time series in frequency series, but it assumes that the input time series $a_i = a(t_i)$ is a periodic function of time. If we perform the FFT of a non-periodic signal, the computation will be biased by the squared window function implicitly assumed using a finite time segment. Then, the resulting $\tilde{a}_j$ will be the convolution product between a_i and the squared window function, which contributes in the spectrum as a $1/f^2$ factor (the Fourier transform of a step function, see Fig. 9.5). In order to avoid this problem, the time series must be multiplied by a smooth function that has zeros at the beginning and of the end of the series. The Fourier contribution of a smoother function decreases faster than the $1/f^2$ and then it does not affect the estimation of the FFT as much as the step function. estimation of the FFT gives us better results. This method is called windowing and an example is shown in Fig. 9.5.

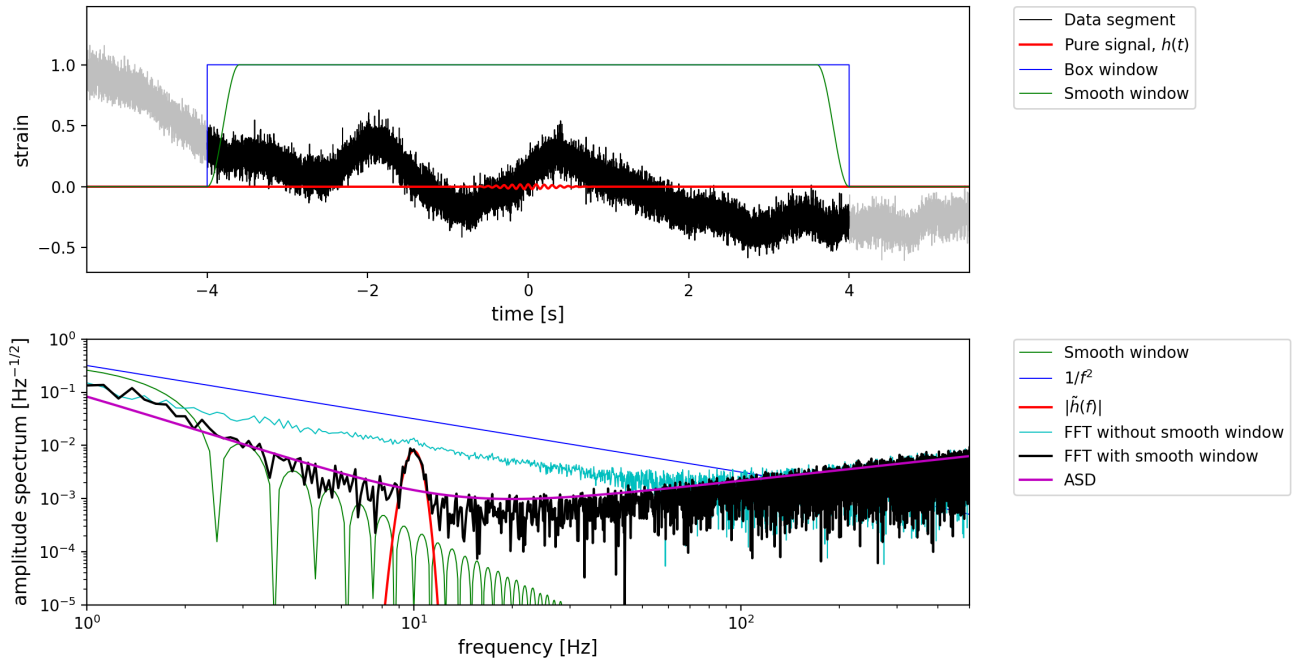


Figure 9.5: Top panel: the black line is a strain segment of artificial noise in which an fictitious signal (red) is injected. However, the signal is too small to be directly observed in the time-domain data. The blue function is the squared window implicitly applied in the choice of the segment, while the green function is a cosine-window, largely used in signal processing and known as Tukey window. Bottom panel: The purple line is the analytical ASD used to generate the noise and the red line is the amplitude spectrum of the pure injected signal. The cyan line is the spectrum of the data strain without windowing, i.e. the Tukey window is not applied. The black line is the spectrum of the same strain with the Tukey window. For the non-windowed strain, the low-frequency region is dominated by the *leakage*, proportional to $1/f^2$, due to not appropriate usage of the FFT and the contribution of the signal is not observable. On the other hand, the black line shows a clear deviation from the noise ASD. The spectrum of the Tukey window is plotted in green: it is possible to see that its contribution decrease much rapidly than $1/f^2$.

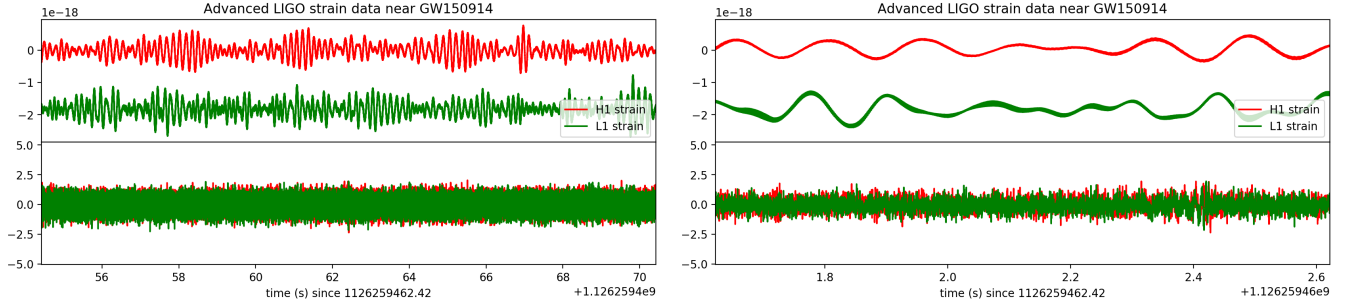


Figure 9.6: Left figure: top panel shows segments of 16 s extracted from the raw data of LIGO Hanford (red) and LIGO Livingston (green) centred around GW150914 trigger time. The bottom panel shows the same strain whitened with the respective PSDs. Right figure: The same as right, but zoomed on a period of 1 s. It is possible to see a correlation between the recorded strain of the two detectors.

Signal processing. Let us discuss the signal processing analysis of the first GW observation recorded by the LIGO detectors, GW150914. Data were recorded by two ground-based interferometers, labelled as H and L, and each of them returns a strain of data $s_{H,L}(t)$. The data are time series segments with a finite length T and with a constant sampling frequency F_s , i.e. the inverse of the time difference between each point of the series. Then, our series contains $T \cdot F_s$ elements. A power excess in the data indicates that a signal $h(t)$ might be contained in $s_{H,L}(t)$: How can we extract the GW signal?

The data are dominated by the noise contributions and their fluctuations are much larger than the ones due to a GWs. First of all, we have to estimate the PSD (or the ASD) of our detectors. The spectrum of the noise is estimated using data segments that do not contain the signal. The computation is performed with the Welch's method, previously discussed, and computing the Fourier transform using the FFT (See Rem. 9.2.2 and 9.4.1.) Second, we have to reweight the frequency components of the signal $s(t)$ on the ASD, in order to minimize the effects of the noise. This method is called *whitening*: the signal is rescaled in the frequency domain by the noise amplitude, in order to clean the time series and obtain constant noise contribution for each frequency bin (white noise, $S_n(f) = 1$). The whitened signal $s_w(r)$ can be computed as

$$s_w(t) = \int_{-\infty}^{+\infty} \frac{\tilde{s}(f) e^{+i2\pi ft}}{\sqrt{S_n(f)}} df, \quad (9.57)$$

which is the anti-Fourier transform of $\tilde{s}(f)/\sqrt{S_n(f)}$.

The application of this procedure to GW150914 is shown in Fig. 9.6. The bottom-right panel of Fig. 9.6 shows a strong correlation between the fluctuations of the two detectors. In order to better visualise the signal, we shift the L signal of a time delay equal to the time taken by the light to travel between the two locations. Then we have also to invert L signal since the two LIGO detectors are oriented in opposite directions. In the end, we apply a band-pass filter (Butterworth of the fourth order) in order to remove the high-frequency and low-frequency noise contributions, since the spectrogram of GW150914 shows evidence of signal in the range between 30 Hz and 350 Hz. It is important to observe that usually LVC does not apply any further filter to the observed data, we are doing this now only for visualisation scopes. The final result is shown in Fig. (9.7): the two strains match perfectly and they agree with GR predictions.

Parameter estimation. After introducing the concepts of matched filtering and signal processing, let us focus on the methods used to estimate the parameters of the source using the matched filtering technique. The working assumption here is that the GW matches a waveform template $h(t; \theta)$, where θ are the parameter of the source. In the case of a binary black hole merger we have 15 independent parameters: the two masses $m_{1,2}$ and the spins $\mathbf{s}_{1,2}$ of the two black holes, the inclination angle ι between the line of sight and the orbital plane, the luminosity distance D_L of the system from the Earth, the two angles for the sky location (α, δ) , the polarization angle ψ , and the reference time t_0 and phase ϕ_0 .

The parameter estimation methods are based on the Bayesian theory of probability. This framework is necessary in order to perform *inference* based on the measured data, recovering the posterior distribution for our parameters and quantifying the goodness of the template with respect to the data. The most important input quantities are the prior distribution of θ and the likelihood function $p(s|\theta, H)$. The prior distribution encodes the information or the assumptions that we have or make on the system (source) before making the measurement. The simplest and more agnostic prior for a parameter is a uniform distribution over a finite range. Then we have to define a likelihood function: this term quantifies the agreement between the experimental data and a template and it has to take into account the deviations due to instrumental noise. We recall that for every detector k we have the respective recorded strain of data $s_k(t)$ and the associated PSD $S_{n,k}(f)$ that describe the noise fluctuations. We can start imposing the assumption of Gaussian and stationary noise and rewrite Eq. (9.19) as

$$p(n) \propto e^{-\frac{1}{2}(n|n)}. \quad (9.58)$$

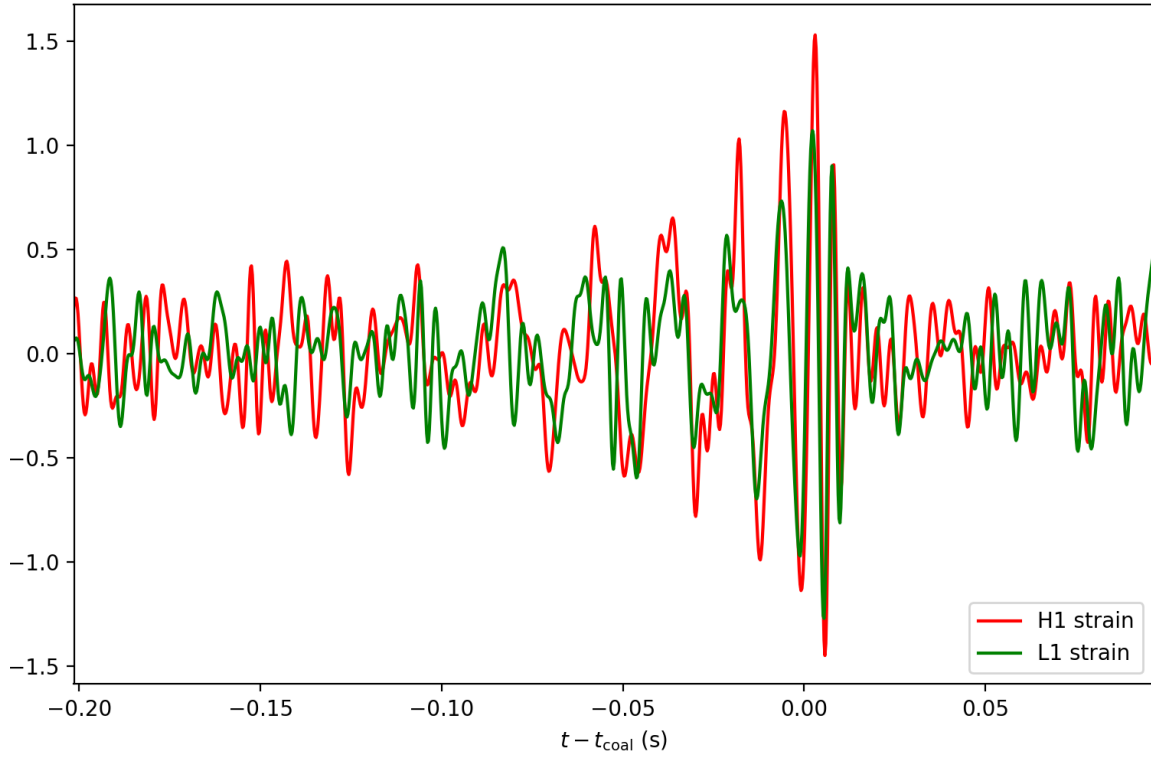


Figure 9.7: Whitened and band-pass-filtered strains of 0.3 s recorded by LIGO Hanford (red) and LIGO Livingston (green) centred around GW150914 trigger time. It is possible to see the chirp-like evolution of the frequency before merger (i.e. the amplitude peak) and then the signal is dissipated almost with constant frequency in the post-merger stage. The signal had a network SNR equal to 25, which is a very large value compared with the other detected transients.

Then it is natural to substitute $n(t) = s(t) - h(t; \boldsymbol{\theta})$, obtaining

$$p(s|\boldsymbol{\theta}, H, S_n) := A \cdot \exp \left[-\frac{1}{2} (s - h(\boldsymbol{\theta}) | s - h(\boldsymbol{\theta})) \right] \rightarrow A \cdot \exp \left(-\frac{2}{T} \sum_{i=1}^N \frac{|\tilde{s}(f_i) - \tilde{h}(f_i; \boldsymbol{\theta})|^2}{S_n(f_i)} \right), \quad (9.59)$$

where A is the normalization constant. Eq. (9.59) defines the likelihood function of the observed data for a single detector and its approximation for discrete time series (where T is the period of the analysed strain and $N = T \cdot F_s$ is the number of elements of the series). In the discrete case, we can write the normalization constant of the likelihood function as

$$A = \prod_{i=1}^N \sqrt{\frac{2}{\pi T S_n(f_i)}}.$$

The likelihood function in Eq. (9.59) is defined real and positive for every value of the parameters' space and its logarithm is proportional to the inverse of the residual between our data and the template $h(t; \boldsymbol{\theta})$, which is the quantity we want to maximise. In the case of multiple detectors, the resulting likelihood is the product of the likelihood of the single detectors,

$$p(s_{\{H,L,\dots\}}|\boldsymbol{\theta}, H, S_{n,\{H,L,\dots\}}) = \prod_{k=H,L,\dots} p(s_k|\boldsymbol{\theta}, H, S_{n,k}). \quad (9.60)$$

Once obtained the prior distribution and the likelihood function, we can apply the Bayes' theorem and compute the posterior distribution for the parameters $\boldsymbol{\theta}$ with the model $h(t; \boldsymbol{\theta})$. Once we have likelihood function and prior distribution, we can apply the Bayes' theorem and get the posterior distribution for our parameters $\boldsymbol{\theta}$ as

$$p(\boldsymbol{\theta}|s_{\{H,L,\dots\}}, S_{n,\{H,L,\dots\}}, H) = \frac{p(s_{\{H,L,\dots\}}|\boldsymbol{\theta}, S_{n,\{H,L,\dots\}}, H) \cdot p(\boldsymbol{\theta}|H)}{p(s_{\{H,L,\dots\}}|S_{n,\{H,L,\dots\}}, H)}. \quad (9.61)$$

Exploring a parameter space of dimension $\gtrsim 15$ can be computationally challenging. Moreover, both the GW template the likelihood function are usually not analytically defined. For the reasons above, the posterior distributions are

usually calculated using Markov Chain Monte Carlo (MCMC) techniques. MCMC are methods for obtaining samples from a target probability distributions using iterative chains. An introduction on these numerical methods can be found on dedicated lecture notes on the course webpage.

Assuming a “best match” templates that maximize the posterior is found, it is interesting to study the associated uncertainty. Let us assume that θ^* are the best combination of parameters such that $h(t; \theta^*)$ is the best template matching $s(t)$. For a small variation $\Delta\theta$ of the parameter, $\theta^* \rightarrow \theta^* + \Delta\theta$, the waveform varies as

$$h(t; \theta^*) \rightarrow h(t; \theta^*) + \left. \frac{\partial h(t)}{\partial \theta_i} \right|_{\theta=\theta^*} \Delta\theta_i. \quad (9.62)$$

and Eq. (9.59) becomes

$$\log p(s|\theta^*, \Delta\theta, H, S_n) \propto -\frac{1}{2} \Gamma_{ij} \Delta\theta_i \Delta\theta_j. \quad (9.63)$$

Assuming a uniform prior distribution for $\Delta\theta$, we can apply the Bayes' theorem and compute the posterior probability for the deviation parameters, $p(\Delta\theta|s, \theta^*, H, S_n)$, which will be identical to Eq. (9.63) unless a normalization constant. This probability density represents a Gaussian function with zero mean for each parameter $\Delta\theta_i$ with covariance matrix such that

$$\Gamma_{ij}(\theta^*) = \left(\frac{\partial h}{\partial \theta_i} \middle| \frac{\partial h}{\partial \theta_j} \right)_{\theta=\theta^*} = 4 \int_0^\infty \frac{1}{S_n(f)} \left[\left(\frac{\partial \tilde{h}(f)}{\partial \theta_i} \right)^* \left(\frac{\partial \tilde{h}(f)}{\partial \theta_j} \right) \right]_{\theta=\theta^*} df. \quad (9.64)$$

The matrix Γ_{ij} is usually called *Fisher information matrix* and its elements are functions of the chosen combination of parameters θ^* though the dependencies of the template $h(t; \theta)$. Then it follows that the variance associated with the uncertainty $\Delta\theta_i$ can be computed as

$$\text{Var}(\Delta\theta_i) = (\Gamma^{-1})_{ii}. \quad (9.65)$$

On the other hand, the integrals in Eq. (9.64) of the diagonal terms Γ_{ii} measure the amount of information that we gain for that particular parameter in every frequency bin. The integral of this quantity, i.e. Γ_{ii} itself, is then proportional to the cumulative information gained for the parameter θ_i .

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