Numerical relativity — Exercise sheet # 2

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3+1 decomposition of GR

Exercise 1.1: 3+1 decomposition of the Z4 system and constraints evolution

The Z4 equations are obtained by extending GR with terms involving a new vector field Z_a,

$$G_{ab} + \nabla_a Z_b + \nabla_b Z_a - \kappa(t_a Z_b + t_b Z_a + g_{ab} t^c Z_c) = 8\pi T_{ab} \ , \tag{1}$$

where we have also included "damping terms" proportional to a real parameter $\kappa > 0$ and built with a time-like vector t^a (set $t^a = n^a$, the future pointing unit normal vector on the time slicing). A solution of Z4 is also as solution of GR iff $Z_a = 0$.

Use the contracted Bianchi identity $\nabla^a G_{ab} = 0$ and show that Z_a satisfies the equation

$$\nabla^a \nabla_a Z_b + R_{ab} Z^a - \kappa \nabla^a (t_a Z_b + t_b Z_a + g_{ab} t^c Z_c) = 0 \ . \tag{2}$$

This homogeneous second order equation ensures that any deviation from GR propagates through light cones and is eventually damped by the κ terms. In the 3+1 Cauchy probelm (see below), sufficient conditions for the initial data ($t = 0$) to provide GR solutions *at all times* are given by $Z_a(0, x^i) = \partial_t Z_a(0, x^i) = 0$ (constraints are preserved).

Compute the 3+1 decomposition of the Z4 system, and show that

$$\mathcal{L}_m \theta = \frac{1}{2} C_0 + ... \tag{3}$$

$$\mathcal{L}_m Z_i = C_i + ... \ , \tag{4}$$

where $\theta = -n_a Z^a = \alpha Z^0$ (last expression is in adapted coordinates), $C_0 = 0$ and $C_i = 0$ are the 3+1 GR constraint equations and "..." indicate terms involving θ and Z_i. Finally, calculate the evolution equations for the constraints

$$\mathcal{L}_m C_0 = -4 C_i D_i (\log \alpha) + 2 K C_0 - 2 D_i C_i + ... \tag{5}$$

$$\mathcal{L}_m C_i = -D_i C_0 + K C_i - D_i (\log \alpha) C_0) + ... \ , \tag{6}$$

and observe again that if $C_0 = C_i = \theta = Z_i = 0$ at all times iff they all vanish at $t = 0$.

Exercise 1.2: ADM Hamiltonian

Compute the expression of the ADM Hamiltonian,

$$\mathcal{H} := \pi^{ij}\dot{\gamma}_{ij} - \mathcal{L} = \sqrt{\gamma}\left(\alpha\mathcal{C}_0 + 2\beta^i\mathcal{C}_i + 2D_j(K\beta^j - K^j_i\beta^i)\right) \ , \tag{7}$$

using the kinematical equation

$$K_{ij} = -\frac{1}{2\alpha}\mathcal{L}_m\gamma_{ij} \tag{8}$$

expressed in terms of 3-covariant derivatives of the shift and time derivative of the 3-metric.