Numerical relativity — Exercise sheet # 3

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Maximal slicing

Exercise 1.1: Maximal Slicing of Schwarzschild

The 3+1 decomposition can be performed with different choices for the slicing of spacetime. The choice of time slicing amounts to making a choice for the lapse α by either prescribing a function or an equation. Since α can vary depending on the position on the spatial slice, the proper time is allowed to advance at different rates at different points on a given slice. A common choice is the *maximal slicing* condition:

$$K = 0. \tag{1}$$

If we want maximal slicing to hold at all times, then we should impose $\partial_t K = 0$ as well. In this case, the evolution equation for the extrinsic curvature scalar reduces to an elliptic equation for the lapse function

$$D_i D^i \alpha = \alpha \left(K_{ij} K^{ij} \right). \tag{2}$$

Step 1: In Schwarzschild coordinates, consider $\overline{t} = t + h(r)$ and enforce K = 0. Find the family of time independent slicings.

Step 2: From the general result in Step 1, recover the specific solutions for Schwarzschild and isotropic metric.

Step 3: Draw the Kruskal-Szekeres diagrams for the Schwarzschild and the isotropic slicings.

Step 4: Draw the embedding diagram.

Exercise 1.2: Extremum of K

In the lecture notes it was demonstrated (see $\S6.6$) that imposing the maximal slicing condition of Eq.(1) extremises volume. Argue that in the case of a Lorentzian manifold the extremum constitutes a maximum.

Approximations

Exercise 1.3: Newtonian limit

Recall that if the gravitational field is weak and static then it is always possible to find a coordinate system $(x^{\alpha}) = (x^0 = t, x^i)$ such that the metric components take the form:

$$g_{\mu\nu} dx^{\mu} dx^{\nu} = -(1+2\Phi) dt^2 + (1-2\Phi) f_{ij} dx^i dx^j, \qquad (3)$$

where Φ is the gravitational potential and $|\Phi| \ll 1$.

In performing 3 + 1 and conformal decompositions, we have found that the mean curvature evolves according to:

$$\mathcal{L}_m[K] = -D_i D^i[\alpha] + \alpha (4\pi (E+S) + K_{ij} K^{ij}), \qquad (4)$$

Working within the regime of validity for Eq.(3) show that Eq.(4) reduces to the Poisson equation:

$$\mathcal{D}_i \mathcal{D}^i [\Phi] = 4\pi\rho,\tag{5}$$

where \mathcal{D} is the derivative operator associated with the flat metric f_{ij} .

Exercise 1.4: *IWM approximation*

The Isenberg-Wilson-Mathews (IWM) approximation to GR follows from the hypotheses of a conformally flat space-time and fixing a foliation by restricting hypersurfaces Σ_t to be of maximal volume.

Making use of the kinematical equation involving $\mathcal{L}_m[\tilde{\gamma}_{ij}]$ verify that:

$$2\alpha \tilde{A}_{ij} = f_{ik} \mathcal{D}_j[\beta^k] + f_{kj} \mathcal{D}_i[\beta^k] - \frac{2}{3} \mathcal{D}_k[\beta^k] f_{ij}.$$
 (6)

Simplify this result making use of the conformal Killing operator associated with f_{ij} .