Numerical relativity — Exercise sheet # 4

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Asymptotia

Exercise 1.1: *Modified AF and Schwarzschild*
The notion of asymptotic flatness (AF) allows for a global description of quantities and is of particular use in working with isolated bodies. The AF conditions (§7.1 of the lecture notes) lead to a well-defined prescription for calculation of M_{ADM} and P_k^{ADM} via integration at spatial infinity i^0. Consider now calculation of the angular momentum J_k^{ADM} via integration as in Eq.(7.25) of the lecture notes. In order for this integral to converge the AF conditions must be modified by imposing stronger fall-off conditions. This can be achieved by working with the quasi-isotropic gauge.

Show that the standard representation of Schwarzschild does not satisfy the quasi-isotropic gauge condition.

Exercise 1.2: *Stokes theorem and Komar integral*
Stokes theorem is in general derived for every differential form in differential geometry language, see e.g. Wald's book. It generalizes the divergence (or Gauss) and the curl (Kelvin-Stokes) theorems. A simpler derivation specified for antisymmetric rank-2 tensors and using the divergence theorem has direct application to Komar integrals.

Show the following equations for any anti-symmetric tensor A^{ab}

$$\sqrt{g}\nabla_b A^{ab} = \partial_b(\sqrt{g}A^{ab}) \tag{1}$$

$$\sqrt{g}\nabla_a\nabla_b A^{ab} = \partial_a\partial_b(\sqrt{g}A^{ab}) = 0\ . \tag{2}$$

Note the second expression implies that

$$0 = \int_\Omega \nabla_a\nabla_b A^{ab} d\Omega = \int_{\partial\Omega} \partial_b(\sqrt{g}A^{ab})\hat{n}_a d^3x\ , \tag{3}$$

where Ω is a 4D spacetime region, $\partial\Omega$ the boundary, $d\Omega = \sqrt{g}d^4x$, and $\hat{n}^a$ the unit normal vector to $\partial\Omega$.

Assume that $\partial\Omega$ is composed two spatial hypersurfaces Σ_{t_1} and Σ_{t_2} and a time-like cylindrical tube σ, so that

$$0 = \int_{\Sigma_{t_1}} \partial_b(\sqrt{g}A^{ab})\hat{n}_a d^3x - \int_{\Sigma_{t_2}} \partial_b(\sqrt{g}A^{ab})\hat{n}_a d^3x + \int_{\sigma} \partial_b(\sqrt{g}A^{ab})\hat{n}_a d^3x \ . \tag{4}$$

Take $A^{ab} = \nabla^a \xi^b$, where ξ^a is a Killing vector; the anti-symmetric tensor is now given by the divergence of the Killing vector field. For this choice and using Einstein equations, show that the integral on σ is zero, $\int_\sigma (...) = 0$, for an isolated source. As a consequence $\int_{\Sigma_{t_1}} (...) = \int_{\Sigma_{t_2}} (...)$, which indicates that the integral

$$I(\xi) := \int_{\Sigma_t} \partial_b(\sqrt{g}\nabla^a\xi^b)\hat{n}_a d^3x = \int_{\Sigma_t} \sqrt{g}(\nabla^a\xi^b) dv_a \tag{5}$$

is independent on Σ, i.e. it is "conserved in time". Note that the second line uses again Eq. 1 and the definition $dv_a := \sqrt{g} n_a d^3x$, and $\hat{n}^a = n^a$ by definition.

Finally, choose coordinates $\hat{n}^a = \delta^a_0$ and show

$$I(\xi) = \int_{\partial\Sigma_t} \nabla^a \xi^b s_a n_b da \tag{6}$$

with s^a (spacelike) and n^a (timelike) unit vectors and da the proper area element on $\partial\Sigma$ and $s_a n_b da = \sqrt{g}\hat{n}_a \hat{s}_b d^2x$.