3+1 GEOMETRY

$(\mathcal{M}, g)$ spacetime + abstract index notation

CAUCHY SURFACE $\overset{\text{def.}}{=}$ SPACELIKE HYPERSURFACE Σ in $\mathcal{M}$ SUCH THAT EACH CAUSAL (TIMELIKE OR NULL) CURVE WITHOUT END-POINT INTERSECT Σ ONLY ONCE.

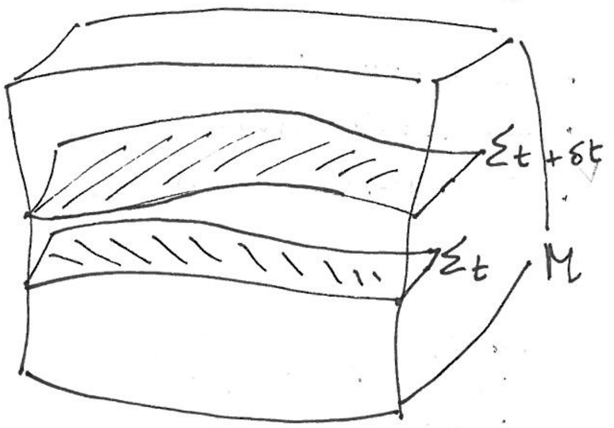
GLOBALLY HYPERBOLIC SPACETIME $\overset{\text{def.}}{=}$ ADMITS A CAUCHY SURFACE

Properties :
- TOPOLOGY is $\mathbb{R} \times \Sigma$
- SCALAR WAVE EQ. is WELL POSED
- CAN BE FOLIATED BY A FAMILY OF SPACELIKE HYPERSURFACES Σ_t THAT DO NOT INTERSECT :

$\hat{t}$ smooth and regular $(\nabla \hat{t} \neq 0)$ scalar field on $\mathcal{M}$

$$\forall t \in \mathbb{R} \qquad \Sigma_t \equiv \{ p \in \mathcal{M} : \hat{t}(p) = t \}$$

(in the following we do not distinguish between $\hat{t}$ and t)



NORMAL VECTOR n^a :
- Timelike
- future pointing
- Unit vector

$$\left.\begin{array}{l} \text{- Timelike} \\ \text{- future pointing} \\ \text{- Unit vector} \end{array}\right\} \quad n^a \equiv -\alpha (\nabla t)^a$$

where

α normalisation constant : $n^a n_a = -1$

$$\alpha = \left(- g^{ab} \nabla_a t \nabla_b t\right)^{1/2} > 0 \qquad \text{LAPSE FUNCTION}.$$

The normal vector n^a defines the EULERIAN OBSERVERS, that are orthogonal to Σ_t.
Physically, Σ_t is locally the set of events that are simultaneous in the Eulerian frame.

INDUCED METRIC AND PROJECTORS: The <u>ambient</u> metric g_{ab} induces a metric on Σ_t

$$\gamma_{ab} = g_{ab} + n_a n_b$$

Any tensor field can be projected into spacelike components (on Σ_t) and timelike components (normal to Σ_t). The projector operators are:

$$\gamma^a{}_b = g^{ac}\gamma_{cb} = g^a{}_b + n^a n_b = \delta^a{}_b + n^a n_b$$

$$\underline{N}^a{}_b = -n^a n_b$$

More, one can define a <u>connection</u> by projecting the 4D covariant derivative ∇:

$$D_a T^b{}_c \equiv \gamma_a{}^d \gamma_e{}^b \gamma_c{}^f \nabla_d T^e{}_f \, ,$$

CONNECTION AND INTRINSIC CURVATURE

The expression above
- is A CONNECTION
- is TORSION-FREE
- is COMPATIBLE WITH γ_{ab} : $D_a \gamma_{bc} = 0$

$$\gamma_a{}^{e}\gamma_b{}^{f}\gamma_c{}^{d}\nabla_d \gamma_{ef} = \gamma_a{}^e \gamma_b{}^f \gamma_c{}^d \nabla_d (\underbrace{g_{ef}}_{=0} + n_e n_f) =$$

25.04.2012 : CHECK!

$$= \gamma_a{}^e \gamma_b{}^f \gamma_c{}^d \nabla_d (n_e n_f) =$$

$$= \gamma_a{}^e \gamma_b{}^f \gamma_c{}^d \, n_e \nabla_d n_f + \gamma_a{}^e \gamma_b{}^f \gamma_c{}^d \, \nabla_d n_e \cdot n_f =$$

$$= \underbrace{\gamma_a{}^e n_e}_{=0} \gamma_b{}^f \gamma_c{}^d \nabla_d n_f + \gamma_a{}^e \underbrace{\gamma_b{}^f n_f}_{=0} \gamma_c{}^d \nabla_d n_e = 0$$

Note: $\gamma^a{}_b n_a = \delta^a{}_b n_a + \underbrace{n^a n_a}_{=-1} n_b = n_b - n_b = 0$

(obviously because they are orthogonal)

The Riemann tensor associated to the connection D defines the <u>intrinsic curvature</u> of (Σ, γ) via the usual formula:

$$(D_a D_b - D_b D_a) V^c = R^c{}_{dab} V^d \qquad \forall\, V^d$$

with the usual contractions to form the Ricci scalar and tensor

$$R_{ab} = R^c{}_{acb} = \gamma^{cd} R_{acbd}$$
$$R = \gamma^{ab} R_{ab}$$

EXTRINSIC CURVATURE

The intrinsic curvature does <u>not</u> tell however how Σ_t is bended in $\mathcal{M}$. Stated differently one may ask: How does n^a changes when it is transported along Σ_t?

The <u>extrinsic curvature</u> measures this change.

$$K_{ab} \equiv -\gamma^c{}_a \nabla_c n_b$$

- Projection of $\nabla_c n_b$
- Spatial tensor

~~Properties~~:

* $K_{ab} = -\gamma^c{}_a \nabla_c n_b = -(\delta^c_a + n^c n_a) \nabla_c n_b = -\nabla_a n_b - n_a \underbrace{n^c \nabla_c n_b}_{\equiv a_b}$

$$= -\nabla_a n_b - n_a a_b$$

$a_b \stackrel{\text{def.}}{=}$ ACCELERATION OF EULERIAN OBSERVERS

Property:

$$n^b a_b = n^b n^c \nabla_c n_b = n^c \nabla_c (n^b n_b) - n_b \underbrace{n^c \nabla_c n^b}_{a^b} \Rightarrow$$

$$2 n^b a_b = n^c \nabla_c (\underbrace{n^b n_b}_{=-1}) = 0 \Rightarrow$$

orthogonal to n^a

K_{ab} has different and equivalent definitions.

How much the metric changes when transported along n^a ?

$\rightarrow$ Lie derivative !

$$\mathcal{L}_n \gamma_{ab} = \text{Lie derivative of } \gamma_{ab} \text{ along } n^a$$

$$\equiv n^c \nabla_c \gamma_{ab} + \gamma_{ac} \nabla_b n^c + \gamma_{bc} \nabla_a n^c$$

term-by-term:

$$n^c \nabla_c \gamma_{ab} = n^c \underbrace{\nabla_c (g_{ab}}_{=0} + n_a n_b) = n^c \nabla_c (n_a n_b) =$$

$$= \underbrace{n^c n_a}_{\gamma^c_a - g^c_a} \nabla_c n_b + n^c n_b \nabla_c n_a =$$

$$= (\gamma^c_a - g^c_a) \nabla_c n_b + (\gamma^c_b - g^c_b) \nabla_c n_a =$$

$$= \gamma^c_a \nabla_c n_b - \nabla_a n_b + \gamma^c_b \nabla_c n_a - \nabla_b n_a$$

$$\gamma_{ac} \nabla_b n^c = (g_{ac} + n_a n_c) \nabla_b n^c =$$

$$= \nabla_b n_a + n_a \underbrace{n_c \nabla_b n^c}_{= \frac{1}{2} \nabla_c (\underbrace{n^a n_a}_{=-1}) = 0}$$

$$= \nabla_b n_a$$

put together:

$$\mathcal{L}_n \gamma_{ab} = \gamma^c_a \nabla_c n_b + \gamma^c_b \nabla_c n_a - \nabla_a n_b - \nabla_b n_a + \nabla_b n_a + \nabla_a n_b =$$

$$= \gamma^c_a \nabla_c n_b + \gamma^c_b \nabla_c n_a = -K_{ab} - K_{ab}$$

$$\boxed{\mathcal{L}_n \gamma_{ab} = -2 K_{ab}}$$

"kinematical" or "evolution" eq for γ_{ab}

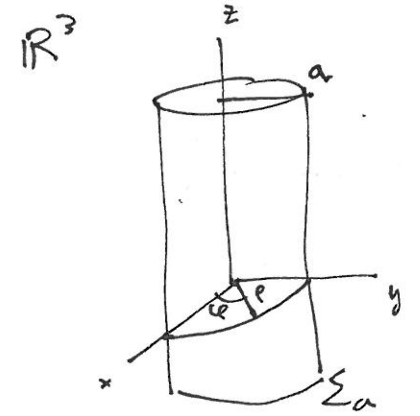
Alternative expression for extrinsic curvature

$$K_{ab} = -\gamma^c_a \gamma^d_b \nabla_c n_d = -\gamma^c_a \left(\delta^d_b + n^d n_b\right) \nabla_c n_d =$$

$$= -\gamma^c_a \nabla_c n_b - \gamma^c_a n_b \underbrace{n^d \nabla_c n_d}_{=0} = -\gamma^c_a \nabla_c n_b$$

<u>Example</u>

Extrinsic curvature of a cylinder embedded in $\mathbb{R}^3$



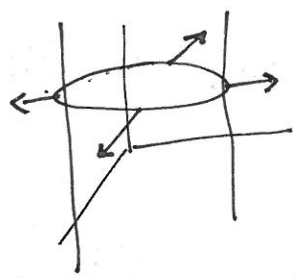
- Coordinates for the cylinder $x^\alpha = (\rho, \varphi, z)$
  $\rho^2 = x^2 + y^2$, $x^i = (\varphi, z)$
- Σ_a is defined by $\rho = a = \text{const}$
  or alternatively as the level set of
  $\hat{t} = \rho - a$

- INDUCED METRIC $\quad \gamma_{ij} dx^i dx^j = a^2 d\varphi^2 + dz^2 \qquad \gamma_{ij} = \begin{pmatrix} a^2 & \\ & 1 \end{pmatrix}$

  The induced metric is FLAT since

  $a = \text{const} \rightarrow \eta \equiv a\varphi \rightarrow \gamma_{ij} dx^i dx^j = d\eta^2 + dz^2$

- UNIT NORMAL VECTOR



$\hat{n}^a = (x, y, 0)$

normalise : $n^a = \rho^{-1}(x, y, 0)$

- COMPUTE $\nabla_b n^a$

  ∇ is the connection on $\mathbb{R}^3$ (flat metric) $\nabla_\alpha = \partial_\alpha$.

  In Cartesian coordinates (*)

$$\nabla_b n^a = \partial_b n^a = \rho^{-3/2} \begin{pmatrix} y^2 & -xy & 0 \\ -xy & x^2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

(*) $\dfrac{\partial}{\partial x}\left(\dfrac{x}{\sqrt{x^2+y^2}}\right) = \rho^{-1/2} - \frac{1}{2}\rho^{-3/2} x \cdot 2x = \rho^{-3/2} y^2$

$\dfrac{\partial}{\partial y}\left(\dfrac{x}{\sqrt{x^2+y^2}}\right) = \rho^{-1/2} - \frac{1}{2} x \dfrac{2y}{\rho^{-3/2}} = -\rho^{-3/2} xy$

• COMPUTE K_{ij} IN THE BASIS ASSOCIATED TO COORDS (φ, z)

$$\partial_i = \left(\frac{\partial}{\partial\varphi}, \frac{\partial}{\partial z} \right)$$

$$(\partial_\varphi)^\alpha = (-y, x, 0)$$

$$(\partial_z)^\alpha = (0, 0, 1)$$

$$K_{ij} = -\nabla_a n_b \, (\partial_i)^a \, (\partial_j)^b = \ldots = \begin{pmatrix} -a & 0 \\ 0 & 0 \end{pmatrix}$$

25.04.2018

Trace :

$$\gamma^{ij} = \begin{pmatrix} \frac{1}{a^2} & \\ & 1 \end{pmatrix} \qquad K = \gamma^{ij} K_{ij} = -\frac{1}{a}$$

GAUSS – CODAZZI – RICCI EQUATIONS

Fundamental equations relating projections of the 4D Riemann and Ricci tensors to the 3D intrinsic and extrinsic curvature of Σ.

GAUSS EQ.

Spatial projection of the ${}^{(4)}R_{abcd}$

$$\boxed{\gamma_a^{\ p} \gamma_b^{\ q} \gamma_c^{\ r} \gamma_d^{\ s} \, {}^{(4)}R_{pqrs} = R_{abcd} + K_{ac} K_{bd} - K_{ad} K_{bc}}$$

$$\gamma_a^{\ p} \gamma_b^{\ q} \gamma_c^{\ r} \gamma_d^{\ s} \, {}^{(4)}R^{c}_{\ spq} = R^{c}_{\ dab} + K^{c}_{\ a} K_{db} - K^{c}_{\ b} K_{ad}$$

Contractions :

$$\gamma_a^{\ p} \gamma_b^{\ q} \, {}^{(4)}R_{pq} + \gamma_{ap} n^q \gamma_b^{\ r} n^s \, {}^{(4)}R^{p}_{\ qrs} = R_{ab} + K K_{ab} - K_{ap} K^{p}_{\ b}$$

$${}^{(4)}R + 2 \, {}^{(4)}R_{ab} n^a n^b = R + K^2 - K_{ij} K^{ij} \qquad *$$

CODAZZI EQ.

Three spatial projections and one contraction with n^a

$$\boxed{\gamma_a^{\ r}\,\gamma_b^{\ p}\,\gamma_c^{\ q}\, n^s\ {}^{(4)}R_{rpqs} = D_b K_{ac} - D_a K_{bc}}$$

$$\gamma^r_{\ e}\,\gamma^p_{\ b}\,\gamma^q_{\ c}\, n^s\ {}^{(4)}R^c_{\ srp} = D_b K^c_{\ a} - D_a K^c_{\ b}$$

Contractions:

$$\gamma^p_{\ a}\, n^q\ {}^{(4)}R_{pq} = D_a K - D_p K^q_{\ a} \qquad \times$$

RICCI EQ.

Two spatial projections and two contractions with n^a

$$\boxed{\gamma_a^{\ q}\,\gamma_b^{\ r}\, n^s\, n^d\ {}^{(4)}R_{rdqs} = \mathcal{L}_n K_{ab} + \alpha^{-1} D_a D_b \alpha + K^c_{\ b} K_{ac}}$$

Note that the term "$\gamma\gamma n n\ {}^{(4)}R$" appears also in the contracted Ricci eq.
The two equations can be combined eliminating "$\gamma\gamma n n\ {}^{(4)}R$" to obtain:

- $$\gamma^q_{\ a}\gamma^p_{\ b}\ {}^{(4)}R_{qp} = -\alpha^{-1}\mathcal{L}_m K_{ab} - \alpha^{-1} D_a D_b \alpha + R_{ab} + K K_{ab} - 2K_{ar}K^r_{\ b}$$

Note: Gauss and Codazzi equations are defined on Σ_t
Ricci equation contains $\mathcal{L}_n$, a normal derivative along n.

Important note

- The definitions of induced metric and extrinsic curvature are general concepts in differential geometry and in the treatment of hypersurfaces embedded in M.
  → There is no need for the assumption of globally hyperbolic spacetime

- Similarly, Gauss - Codazzi equations do <u>not</u> need that assumption, but just require the definition of spacelike hypersurfaces with γ_{ij} and K_{ij} embedded in M

NORMAL EVOLUTION VECTOR OR TIME VECTOR

Definitions used so far:

- $\hat{t}$ scalar function
- ∇t gradient (1-form)
- $\alpha \equiv -\left[(\nabla t)_a (\nabla t)^a\right]^{-1/2} > 0$ lapse function
- $n^a \equiv -\alpha (\nabla t)^a$ normal vector

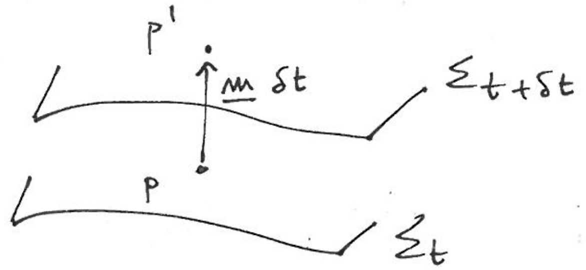
Normal ev. vector or "time" vector:

$$m^a \equiv \alpha\, n^a$$

Properties:

- $m^a m_a = -\alpha^2$
- $\nabla_{\underline{m}} t = m^a (\nabla t)_a = \alpha\, n^a (\nabla t)_a = -\alpha^2 \underbrace{(\nabla t)_a (\nabla t)^a}_{-\alpha^{-2}} = 1$

The vector $\underline{m}$ is adapted to the scalar field t



$$t(p') = t(p + \underline{m}\,\delta t) = t(p) + \delta t \underbrace{\nabla_{\underline{m}} t}_{=1} = t(p) + \delta t$$

ALL VECTORS $\underline{m}$ ORIGINATING FROM Σ_t WILL END ON $\Sigma_{t+\delta t}$

→ Σ_t are Lie-dragged by $\underline{m}$

$\forall\, \underline{v} \in T(\Sigma_t)\quad \mathcal{L}_m v \in T(\Sigma_t)$

Using the property of the Lie derivative

$$\mathcal{L}_n \gamma_{ab} = \phi^{-1} \mathcal{L}_{\phi n} \gamma_{ab} \qquad \forall \phi \text{ (scalar)}$$

the kinematical relation

$$\mathcal{L}_n \gamma_{ab} = -2 K_{ab}$$

reads

$$\boxed{\mathcal{L}_m \gamma_{ab} = -2 \alpha K_{ab}}$$

Also one can prove that : $\mathcal{L}_m \gamma^a{}_b = 0 \Rightarrow$

for any tensor tangent to Σ_t, $\mathcal{L}_m T$ is tangent to Σ

USEFUL RELATIONS FOR EULERIAN OBSERVERS ACCELERATION

$$a_a = n^p \nabla_p n_a = - n^p \nabla_p (\alpha \nabla_a t) = - n^p \nabla_p \alpha \cdot \underbrace{\nabla_a t}_{= -\alpha^{-1} n_a} - \alpha n^p \underbrace{\nabla_p \nabla_a t}_{= \nabla_a \underbrace{\nabla_p t}_{= -\alpha^{-1} n_p}} =$$

TORSION FREE

$$= + \alpha^{-1} n_a n^p \nabla_p \alpha + \alpha n^p \nabla_a (\alpha^{-1} n_p) =$$

$$= \alpha^{-1} n_a n^p \nabla_p \alpha + \alpha \underbrace{n^p n_p}_{= -1} \nabla_a (\alpha^{-1}) + \underbrace{n^p \nabla_a n_p}_{= 0} =$$

$$= \alpha^{-1} n_a n^p \nabla_p \alpha + \alpha^{-1} \nabla_a \alpha =$$

$$= \alpha^{-1} \underbrace{(g^p{}_a + n_a n^p)}_{= \gamma^p{}_a} \nabla_p \alpha = \alpha^{-1} \underbrace{\gamma^p{}_a \nabla_p \alpha}_{D_a \alpha} = D_a \ln \alpha$$

This relation is used to derive Ricci equation.

DERIVATION OF GAUSS EQ.

$$D_a D_b V^c - D_b D_a V^c = R^c{}_{dab} V^d$$

$$D_a D_b V^c = D_a (D_b V^c) = \gamma^e_a \gamma^f_b \gamma^c_r \nabla_e (D_f V^r) = \gamma^e_a \gamma^f_b \gamma^c_r \nabla_e (\gamma^s_f \gamma^r_g \nabla_s V^g) =$$

$$= \gamma\gamma\gamma \left(\underbrace{\nabla_e \gamma^s_f}_{\cdot} \cdot \gamma \nabla V + \gamma \underbrace{\nabla_e \gamma^r_g}_{\cdot} \cdot \nabla V + \gamma\gamma \nabla_e \nabla_s V^g \right) =$$

$$\cdot \ \nabla_e \gamma^s_f = \nabla_e (\delta^s_f + n^s n_f) = \nabla_e (n^s n_f) = \nabla_e n^s \cdot n_f + n^s \nabla_e n_f$$

$$= \gamma^e_a \gamma^f_b \gamma^c_r \left[\gamma^r_g (\nabla_e n^s \cdot n_f + n^s \nabla_e n_f) \nabla_s V^g + \gamma \nabla_e \gamma^r_g \cdot \nabla V + \gamma\gamma \nabla_e \nabla_s V^g \right] =$$

$$\gamma^f_b n_f = 0$$

$$= \gamma^e_a \gamma^f_b \gamma^c_r \gamma^r_g n^s \nabla_e n_f \nabla_s V^g + \gamma\gamma\gamma \left[\gamma \nabla_e \gamma^r_g \cdot \nabla V + \gamma\gamma \nabla_e \nabla_s V^g \right] =$$

$$= \gamma^e_a \gamma^f_b \gamma^c_g n^s \nabla_e n_f \nabla_s V^g +$$

$$+ \gamma^e_a \gamma^f_b \gamma^c_r \gamma^s_f (\nabla_e n^r \cdot n_g + n^r \nabla_e n_g) \nabla_s V^g + \gamma\gamma\gamma \left[\gamma\gamma \nabla_e \nabla_s V^g \right] =$$

$$\gamma^c_r n^r = 0$$

$$= \gamma^e_a \gamma^f_b \gamma^c_g \nabla_e n_f \cdot n^s \nabla_s V^g + \gamma^e_a \gamma^f_b \gamma^c_r \gamma^s_f n_g \nabla_e n^r \nabla_s V^g + \gamma\gamma\gamma \left[\gamma\gamma \nabla_e \nabla_s V^g \right] =$$

$$= \underbrace{\gamma^e_a \gamma^f_b \nabla_e n_f}_{-K_{ab}} \cdot \gamma^c_g n^s \nabla_s V^g + \gamma^e_a \gamma^f_b \gamma^c_r \gamma^s_f \nabla_e n^r \underbrace{n_g \nabla_s V^g}_{= -V^g \nabla_s n_g \ (\Leftarrow V^g n_g = 0)} + \gamma\gamma\gamma \left[\gamma\gamma \nabla_e \nabla_s V^g \right]$$

$$= -K_{ab} \gamma^c_g n^s \nabla_s V^g + \gamma^e_a \underbrace{\gamma^f_b \gamma^s_f}_{\gamma^s_b} \gamma^c_r \nabla_e n^r (-V^g \nabla_s n_g) + \gamma\gamma\gamma \left[\gamma\gamma \nabla_e \nabla_s V^g \right]$$

$$= -K_{ab} \gamma^c_g n^s \nabla_s V^g - \gamma^e_a \gamma^s_b \gamma^c_r \nabla_e n^r \nabla_s n_g \cdot V^g +$$

$$+ \gamma_a^e \gamma_b^f \underbrace{\gamma_r^c \gamma_f^s \gamma_g^r}_{\gamma_g^c} \nabla_e \nabla_s v^g =$$

$$\underbrace{\gamma_a^e \gamma_b^f \gamma_f^s \gamma_g^c \nabla_e \nabla_s v^g}$$

$$\gamma_a^e \gamma^s_b \gamma_g^c \nabla_e \nabla_s v^g$$

$$= -K_{ab} \gamma^c_g n^s \nabla_s v^g - \gamma_a^e \underbrace{\gamma_b^s \gamma_r^c \nabla_e n^r \nabla_s n_g}_{\gamma_b^s \nabla_s n_g = -K_{bg}} v^g + \gamma_a^e \gamma_b^s \gamma_g^c \nabla_e \nabla_s v^g =$$

$$-K_{bg} \gamma_a^e \gamma_r^c \nabla_e n^r \cdot v^g$$

$$-K_{bg} \cdot \gamma_r^c \underbrace{\gamma_a^e \nabla_e n^r}_{-K_a^r} \cdot v^g \qquad ⊛$$

$$= -K_{ab} \gamma^c_g n^s \nabla_s v^g - K^c_a K_{bg} v^g + \gamma_a^e \gamma_b^s \gamma_g^c \nabla_e \nabla_s v^g$$

$$⊛ \quad \gamma_r^c \gamma_a^e \nabla_e n^r = \gamma_r^c \gamma_a^e \nabla_e (g^{rh} n_h) = \gamma_r^c g^{rh} \underbrace{\gamma_a^e \nabla_e n_h}_{-K_{ah}} = \gamma_r^c (\gamma^{rh} + n^r n^h)(-K_{ah})$$

$$= -K_{ah} \Big(\underbrace{\gamma_r^c \gamma^{rh}}_{\gamma^{ch}} + \underbrace{\gamma_r^c n^r}_{=0} n^h \Big) = -K_{ah} \gamma^{ch} = -K_a^c$$

Result of the calculation:

$$D_a D_b V^c = -K_{ab}\gamma^c_d n^s \nabla_s V^d - K^c_a K_{bd} V^d + \gamma^e_a \gamma^f_b \gamma^c_d \nabla_e \nabla_f V^d$$

Compute the Riemann:

$$D_a D_b V^c - D_b D_a V^c = -K_{ab}\cancel{\gamma^c_d n^s} \nabla_s V^d - K^c_a K_{bd} V^d + \gamma^e_a \gamma^s_b \gamma^c_d \nabla_e \nabla_s V^d +$$

$$+ K_{ba}\cancel{\gamma^c_d n^e} \nabla_s V^d + K^c_b K_{ad} V^d - \gamma^e_b \gamma^s_a \gamma^c_d \nabla_e \nabla_s V^d =$$

$$= -\left(K^c_a K_{bd} - K^c_b K_{ad}\right) V^d + \gamma^e_a \gamma^s_b \gamma^c_d \underbrace{\left(\nabla_e \nabla_s V^d - \nabla_s \nabla_e V^d\right)}_{{}^{(4)}R^d{}_{fes} V^f}$$

$$\gamma^p_a \gamma^q_b \gamma^c_d \, {}^{(4)}R^d{}_{rpq} V^r = R^c{}_{rab} V^r + \left(K^c_a K_{bd} - K^c_b K_{ad}\right)$$

Since $V^r = \gamma^r_e V^e$

$$\gamma^p_a \gamma^q_b \gamma^c_d \gamma^r_e \, {}^{(4)}R^d{}_{rpq} V^e = R^c{}_{rab} V^r + \left(K^c_a K_{bd} - K^c_d K_{ed}\right)$$

TO DO:

Contraction a c to obtain

Contracted terms