3+1 DECOMPOSITION OF GR

$$^{(4)}G_{ab} = {}^{(4)}R_{ab} - \frac{1}{2}\,{}^{(4)}R\, g_{ab} = 8\pi T_{ab}$$

HAMILTONIAN CONSTRAINT

Consider the projection of G_{ab} orthogonal to Σ:

$$n^a n^b\, {}^{(4)}G_{ab} = n^a n^b\, {}^{(4)}R_{ab} - \frac{1}{2}\,{}^{(4)}R\, \underbrace{n^a n^b g_{ab}}$$

$$n^a n^b (\gamma_{ab} - n_a n_b) = -\underbrace{n^a n_a}_{=-1}\, \underbrace{n^b n_b}_{=-1} = -1$$

$$= n^a n^b\, {}^{(4)}R_{ab} + \frac{1}{2}\,{}^{(4)}R$$

$E \equiv n^a n^b T_{ab} \stackrel{\text{def.}}{=}$ MATTER ENERGY DENSITY measured by Eulerian observers

Combine the projection with the contracted Gauss relation *

$$2 n^a n^b G_{ab} = 2 n^a n^b\, {}^{(4)}R_{ab} + {}^{(4)}R = R + K^2 - K_{ij}K^{ij}$$

$$= 16\pi\, n^a n^b T_{ab} \quad = 16\pi E$$

$$\boxed{R + K^2 - K_{ij}K^{ij} = 16\pi E}$$

Contains only spatial quantities and spatial derivatives.

MOMENTUM CONSTRAINT

Consider the mixed projection of G_{ab} once along Σ_t and once along n^a:

$$\gamma_a^{\ p} n^q G_{pq} = \gamma_a^{\ p} n^q\, {}^{(4)}R_{pq} - \frac{1}{2}\,{}^{(4)}R\, \gamma_a^{\ p} n^q g_{pq} =$$

$$= \gamma_a^{\ p} n^q\, {}^{(4)}R_{pq} - \frac{1}{2}\,{}^{(4)}R\, \underbrace{\gamma_a^{\ p} n^q (\gamma_{pq} - n_p n_q)} =$$

$$\gamma_a^{\ p} \underbrace{n^q \gamma_{pq}}_{=0} - n^q n_q \underbrace{\gamma_a^{\ p} n_p}_{=0}$$

$$= \gamma_a^{\ p} n^q\, {}^{(4)}R_{pq}$$

$P_a \equiv -\gamma_a{}^c n^d T_{cd}$ $\overset{\text{def.}}{=}$ MATTER MOMENTUM DENSITY of Eulerian observers

Combine the projection with the contracted Codazzi eq. ×

$$\gamma^p{}_a n^q G_{pq} = \gamma^p{}_a n^q \, {}^{(4)}R_{pq} = D_a K - D_p K^q{}_a$$

$$= 8\pi P_a$$

$$\boxed{D_j K^j{}_i - D_i K = 8\pi P_i}$$

Contains only spatial quantities and derivatives.
Only 3 equations

CURVATURE EVOLUTION

Consider full projection of Einstein's eq on Σ_t.
Use the trace-reversed version:

$${}^{(4)}R_{ab} = 8\pi\left(T_{ab} - \frac{1}{2} T g_{ab}\right)$$

project:

$$\gamma^p{}_a \gamma^q{}_b \, {}^{(4)}R_{pq} = 8\pi \left(\gamma^p{}_a \gamma^q{}_b T_{pq} - \frac{1}{2} T \gamma^p{}_a \gamma^q{}_b g_{pq}\right)$$

$S_{ab} \equiv \gamma^p{}_a \gamma^q{}_b T_{pq}$ $\overset{\text{def.}}{=}$ MATTER STRESS TENSOR as measured by Eulerian obs.

Note it is a purely spatial quantity defined on Σ_t.

Trace:
$$S = S_{ab} g^{ab} = S_{ij} \gamma^{ij}$$

Note that (E, P_a, S_{ab}) fully determine T_{ab}:

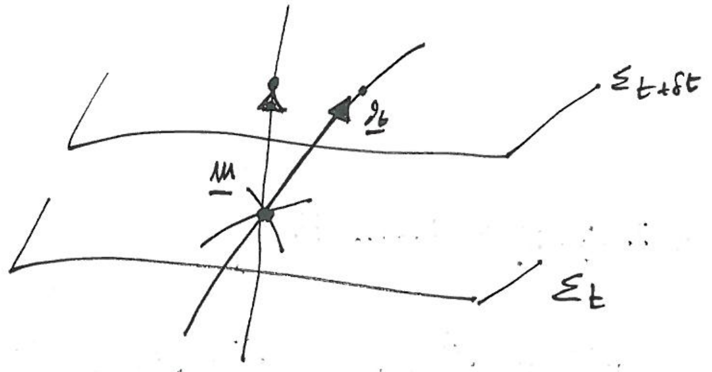
ADAPTED COORDINATES

The goal is to translate the tensorial eqs into PDEs.

Introduce coords. on Σ $\quad x^i = (x^1, x^2, x^3)$

Basis of the tangent space $T(\mathcal{M})$ $\quad (\underline{\partial_t}, \underline{\partial_i}) \stackrel{\text{def.}}{=}$ NATURAL BASIS

$\underline{\partial_t}$ TIME VECTOR $\stackrel{\text{def.}}{=}$ tangent to $x^i = (\text{const})^i$ lines



Properties

- $(\nabla t)_a (\underline{\partial_t})^a = 1 \Rightarrow$ as $\underline{m}$, $\underline{\partial_t}$ Lie-drags Σ_t

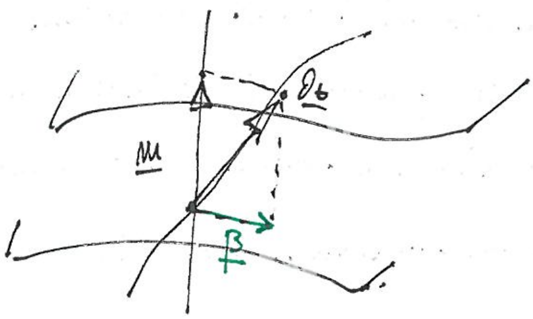
- However, in general $\underline{\partial_t}$ is <u>not</u> parallel to $\underline{m}$ (depends on x^i)

$$\underline{\partial_t} = \underline{m} + \underline{\beta} = \alpha\, \underline{n} + \underline{\beta}$$

- SHIFT VECTOR $\stackrel{\text{def.}}{=}$ defined by the above formula

$$1 = (\nabla t)_a (\underline{\partial_t})^a = \underbrace{(\nabla t)_a m^a}_{=1} + (\nabla t)_a \beta^a \Rightarrow (\nabla t)_a \beta^a = \alpha\, n_a \beta^a = 0$$

In components (natural basis): $\quad \beta^i = (0, \beta^1, \beta^2, \beta^3)$



$$S_{ab} = \gamma^p{}_a \gamma^q{}_b T_{pq} = (\delta^p_a + n^p n_a)(\delta^q{}_b + n^q n_b) T_{pq} =$$

$$= \delta^p_a \delta^q_b T_{pq} + \underbrace{\delta^p_a n^q n_b T_{pq}}_{?} + n^p n_a \delta^q{}_b T_{pq} + n_a n_b \underbrace{n^p n^q T_{pq}}_{=E} =$$

$$= T_{ab}$$

$$\boxed{T_{ab} = S_{ab} + 2 n_{(a} P_{b)} + n_a n_b E}$$

Trace: $g^{ab} T_{ab} = T = g^{ab} S_{ab} + \underbrace{g^{ab} n_a P_b}_{n_a P^a = 0} + g^{ab} n_b P_a + \underbrace{n^a n_a}_{=-1} E$

$$= S - E$$

Combine projection with the combined Gauss + Ricci eq:

$$\gamma^p{}_a \gamma^q{}_b \, {}^{(4)}R_{pq} = -\alpha^{-1} \mathcal{L}_m K_{ab} - \alpha^{-1} D_a D_b \alpha + R_{ab} + K K_{ab} - 2 K_{ac} K^c_b$$

$$= 8\pi \left[S_{ab} - \frac{1}{2} (S - E) \gamma_{ab} \right]$$

$$\boxed{\mathcal{L}_m K_{ij} = -D_i D_j \alpha + \alpha \left\{ R_{ij} + K K_{ij} - 2 K_{ik} K^k{}_j + 4\pi \left[(S-E) \gamma_{ij} - 2 S_{ij} \right] \right\}}$$

Einstein equation is equivalent to the system composed of the constraints (1 scalar and one rank-1 tensorial equation with 3 independent components) and of the curvature evolution. The latter is a rank-2 tensorial equation involving symmetric tensors ⇒ 6 independent components. Thus $1 + 3 + 6 = 10$ independent components.

Note: in general $\underline{\partial_t}$ is NOT timelike:

$$(\underline{\partial_t})_a (\underline{\partial_t})^a = -\alpha^2 + \beta_a \beta^a = -\alpha^2 + \beta^2$$

$$\underline{\partial_t} \text{ is } \begin{cases} \text{TIMELIKE} & \Longleftrightarrow \quad \beta^2 < \alpha^2 \\ \text{NULL} & \Longleftrightarrow \quad \beta^2 = \alpha^2 \\ \text{SPACELIKE} & \Longleftrightarrow \quad \beta^2 > \alpha^2 \end{cases}$$

it depends on the choice of the coordinates! In the latter case one has a <u>superluminal</u> shift.

$$\underline{\partial_t} = \alpha \underline{n} + \underline{\beta} \longmapsto \text{components in adapted coordinates: } \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \alpha n^a + \begin{pmatrix} 0 \\ \beta^1 \\ \beta^2 \\ \beta^3 \end{pmatrix}$$

$$\Longrightarrow n^a = (\alpha^{-1}, \beta^i)$$

<u>Metric components</u>: $\quad g_{\alpha\beta} = g(\partial_\alpha, \partial_\beta)$

$$g_{00} = (\underline{\partial_t})^a (\underline{\partial_t})_a = -\alpha^2 + \beta_i \beta^i$$

$$g_{0i} = (\underline{\partial_t})^a (\underline{\partial_i})_a = (\underline{n} + \underline{\beta}) \cdot \underline{\partial_i} = \beta_i \qquad (\underline{n} \cdot \underline{\partial_i} = 0)$$

$$g_{ij} = \gamma_{ij}$$

$$g_{\alpha\beta} = \begin{pmatrix} g_{00} & g_{0i} \\ g_{i0} & g_{ij} \end{pmatrix} = \begin{pmatrix} -\alpha^2 + \beta_k \beta^k & \beta_i \\ \beta_i & \gamma_{ij} \end{pmatrix}$$

$$g^{\alpha\beta} = \begin{pmatrix} -\alpha^{-2} & \alpha^{-2}\beta^j \\ \alpha^{-2}\beta^j & \gamma^{ij} - \frac{\beta^i\beta^j}{\alpha^2} \end{pmatrix} \qquad \text{Note: } g^{ij} \neq \gamma^{ij} \,!$$

Determinant (Cramer inversion formula):

$$g^{00} = \frac{1}{\det g_{\alpha\beta}} \det \gamma_{ij} = -\frac{1}{\alpha^2} \Longrightarrow \sqrt{g} = \alpha \sqrt{\gamma}$$

where $\quad g \equiv \det g_{\alpha\beta}, \quad \gamma \equiv \det \gamma_{ij}$

they depend on the coordinate!

Finally consider the Lie derivative:

$$\mathcal{L}_{\alpha n} = \mathcal{L}_{\partial_t} - \mathcal{L}_{\beta}$$

$$\mathcal{L}_{\partial_t} = \partial_t$$

$$\mathcal{L}_{\beta} K_{ij} = \beta^k \partial_k K_{ij} + K_{kj} \partial_i \beta^k + K_{ik} \partial_j \beta^k \qquad \text{(def. of Lie-derivative)}$$

Note that one can substitute $\partial_k \leftrightarrow D_k$ in the formula above.
That is convenient for the equation:

$$\mathcal{L}_{\beta} \gamma_{ij} = \underbrace{\beta^k D_k \gamma_{ij}}_{=0} + D_i \beta_j + D_j \beta_i$$

Now one can insert all these formulas, specified to the adapted coordinates, into the 3+1 GR equations:

$$\mathcal{L} \mapsto \partial_t , D_i$$

$$D_i \mapsto \partial_t \text{ and Christoffel } \Gamma^i_{jk}$$

and the equations reduce to PDE.
The equation determine (γ_{ij}, K_{ij}) but NOT (α, β^i) which are related to the choiche and evolution of the coordinates.
For the special case $\alpha = 1$, $\beta^i = 0$ one obtains in PRINCIPAL PART:

$$\begin{cases} -\dfrac{\partial^2 \gamma_{ij}}{\partial t^2} + \gamma^{kl}\left(\dfrac{\partial^2 \gamma_{ij}}{\partial x^k \partial x^l} + \dfrac{\partial^2 \gamma_{kl}}{\partial x^i \partial x^j} - \dfrac{\partial^2 \gamma_{lj}}{\partial x^i \partial x^k} - \dfrac{\partial^2 \gamma_{il}}{\partial x^j \partial x^k}\right) + \text{terms with lower derivatives} = 0 \\[2ex] \gamma^{ik}\gamma^{jl} \dfrac{\partial^2 \gamma_{ij}}{\partial x^k \partial x^l} - \gamma^{ij}\gamma^{kl} \dfrac{\partial^2 \gamma_{ij}}{\partial x^k \partial x^l} + \text{terms with lower derivatives} = 0 \\[2ex] \gamma^{jk} \dfrac{\partial^2 \gamma_{ki}}{\partial x^j \partial t} - \gamma^{kl} \dfrac{\partial^2 \gamma_{kl}}{\partial x^i \partial t} + \text{terms with lower derivatives} = 0 \end{cases}$$

Notes:

- 2nd order equations, QUASI-LINEAR (2nd derivatives)
- 10 eqs for 6 unknowns $\rightarrow$ overdetermined
- $\gamma^{kl} = \gamma^{kl}(\gamma_{ij})$ via the inversion formula

CLOSURE OF THE CONSTRAINTS

Consider the generalized GR :

$$G_{ab} + \nabla_a Z_b + \nabla_b Z_a - g_{ab} \nabla_c Z^c - \kappa \left(t_a Z_b + t_b Z_a + g_{ab} t^c Z_c \right) = 0$$

Known as the Z4 system, with t^a a time-like vector an $\kappa > 0$.

GR is obtained for $\quad Z_a = 0 \quad$ (or t^a Killing vector)

In adapted coordinates one has :

$$\begin{cases} \mathcal{L}_m \gamma_{ij} = -2\alpha K_{ij} \\ \mathcal{L}_m K_{ij} = (\dots) + \alpha \left(\nabla_i Z_j + \nabla_j Z_i \right) \\ \mathcal{L}_m \Theta = \frac{1}{2} H - \Theta K + D_k Z^k - Z^k D_k \ln \alpha - 3\kappa\Theta \\ \mathcal{L}_m Z_i = M_i + D_i \Theta - \Theta D_i \ln \alpha - 2K^k{}_i Z_k - \kappa Z_i \end{cases}$$

where :

$$\Theta \equiv n_a Z^a = \alpha Z^0$$

$$\left. \begin{aligned} 0 = H &= \text{Hamiltonian constraint} \\ 0 = M_i &= \text{Momentum constraint} \end{aligned} \right\} \text{constraints are "time derivatives" of } Z^a$$

All the equations are now <u>evolution equations</u>.

Consider the Bianchi identities :

$$\nabla^a G_{ab} = 0 \quad \Longrightarrow \quad 0 = \Box Z_a + R_{ab} Z^b - \kappa \nabla^a \left(t_a Z_b + t_b Z_a + g_{ab} t^c Z_c \right)$$

the last equation indicates that :

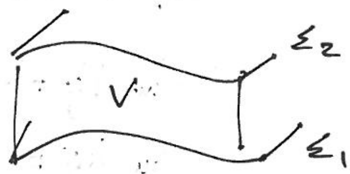
- $t=0$: $Z_a = 0$ and $H = M_i = 0 \iff$ CONSTRAINTS ARE SATISFIED <u>ALL</u> TIMES
- Constraint subsector is propagated and damped ($\kappa > 0$)

HISTORICAL REMARKS ON 3+1 DEC.

1927	Darmois	Gaussian coordinates $(\alpha = 1 , \beta^i = 0)$
1939	Lichnerowicz	$(\alpha \neq 1 , \beta^i = 0)$
1948	Choquet-Bruhat	Generic α , β^i
1962	Arnowitt + Deser + Misner	Hamiltonian formulation (ADM)

ADM

$$S_{GR} = \int_V {}^{(4)}R \sqrt{-g}\, d^4x + \text{boundary terms} \qquad \text{ACTION}$$



3+1 decomposition $\Rightarrow$

$$= \int_{t_1}^{t_2} \int_{\Sigma_t} \underbrace{\alpha \left(R + K_{ij}K^{ij} - K^2 \right) \sqrt{\gamma}}_{\mathcal{L} \ \text{LAGRANGIAN DENSITY}} \, d^3x \, dt$$

Conjugate momenta : $\pi^{ij} = \dfrac{\partial \mathcal{L}}{\partial \dot{\gamma}_{ij}} = \sqrt{\gamma} \left(K \gamma^{ij} - K^{ij} \right)$

Hamiltonian density : $\mathcal{H} = \pi^{ij} \dot{\gamma}_{ij} - \mathcal{L}$

Hamiltonian : $H = \int_{\Sigma_t} \mathcal{H} \, d^3x$

Equations :

$$\frac{\delta H}{\delta \pi^{ij}} = \dot{\gamma}_{ij} = -2\alpha K_{ij} + D_i \beta_j + D_j \beta_i$$

$\dfrac{\delta H}{\delta \gamma_{ij}} = -\dot{\pi}^{ij} = \ldots$ equivalent to $\mathcal{L}_m K_{ij}$ UP TO A TERM $\propto$ HAM. CONSTR.

$\dfrac{\delta H}{\delta \alpha} = 0 =$ Hamiltonian constr.

$\dfrac{\delta H}{\partial \beta^i} = 0 =$ momentum constr.

Note : in this formulation α, β^i appear as <u>lagrangian multipliers</u> to enforce constraints.