# ASYMPTOTIC FLATNESS & GLOBAL QUANTITIES

## ASYMPTOTICALLY FLAT SPACETIME

A globally hyperbolic spacetime is <u>asymptotically flat</u> iff

$\forall\, \Sigma_t \;\; \exists$ a Riemannian background metric $f_{ij}$ such that

- $f_{ij}$ is FLAT, except on a compact domain ("strong field" region)
- Given some coordinates $x^i$ and for $r \to +\infty$

$$\gamma_{ij} = f_{ij} + \mathcal{O}(r^{-1})$$

$$\frac{\partial \gamma_{ij}}{\partial x^k} = \mathcal{O}(r^{-2})$$

$$K_{ij} = \mathcal{O}(r^{-2})$$

$$\frac{\partial K_{ij}}{\partial x^k} = \mathcal{O}(r^{-3})$$

Notes:

- $r \to +\infty$ is <u>spatial infinity</u> $i_0$
- There exist coordinate-independent definitions [Wald book]
- AF spacetimes are an idealization in physical terms but:
  - they do describe approximately ISOLATED BODY spacetimes
  - they do <u>not</u> in general describe COSMOLOGICAL SPACETIMES
- No GW are present at $i_0$ in AF spacetimes.

GW: $$\gamma_{ij} = f_{ij} + \frac{F_{ij}(t-r)}{r} + \mathcal{O}(r^{-2})$$

$$\to \partial_k \gamma_{ij} = -\frac{F'_{ij}(t-r)}{r}\frac{x^k}{r} + \dots \qquad F'_{ij} \neq 0$$

the condition on the derivative is not fullfilled.

Note physically this is not a problem since GW ~~...~~ are defined at null infinity, which imply ~~...~~ we can assume they are emitted at finite $t<0$ and not yet arrived...

- THE DEF. OF AF DEPENDS ON $\Sigma_t$ AND ON $x^i$, the coordinate choice.

what are the coordinate transformations that preserve the properties of the definition?

$$x'^{\alpha} = \Lambda^{\alpha}_{\ \mu} x^{\mu} + c^{\alpha}(\theta,\varphi) + O(r^{-1})$$

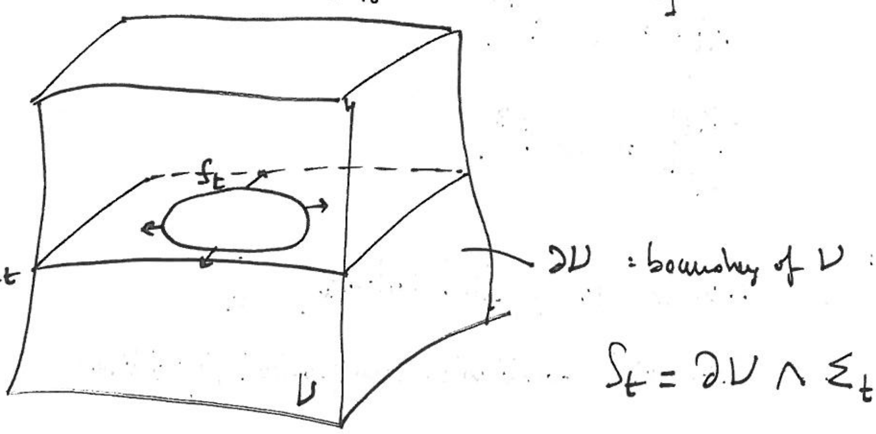
with $\Lambda^{\alpha}_{\ \mu}$ : Lorentz matrix.

ADM MASS

Action for GR

$$S_{GR} = \underbrace{\int_{\mathcal{U}} {}^4R\sqrt{-g}\,d^4x}_{\text{Einstein-Hilbert}} + \underbrace{2\oint_{\partial\mathcal{U}} (Y - Y_0)\sqrt{h}\,d^3y}_{\text{boundary term for A.F.}}$$

↳ [Regge & Teitelboim 1974]




$$S_t = \partial\mathcal{U} \wedge \Sigma_t$$

$Y = \mathrm{Tr}\,\underline{\underline{K}}$ $\quad(\partial\mathcal{U}, h)$ embedded in $(M, g)$
$Y_0 = \mathrm{Tr}\,\underline{\underline{K}}$ $\quad(\partial\mathcal{U}, h)$ embedded in $(M, \eta)$

$$H_{GR} = -\int_{\Sigma_t} (\alpha C_0 - 2\beta^i C_i)\sqrt{\gamma}\,d^3x - 2\oint_{S_t} \left[\alpha(\kappa - \kappa_0) + \beta_i(K_{ij} - K\gamma_{ij})s^j\right]\sqrt{q}\,d^2y$$

$\kappa = \mathrm{Tr}\,\underline{\underline{K}}$ $\quad S_t$ embedded in $(\Sigma_t, \gamma)$
$\kappa_0 = \mathrm{Tr}\,\underline{\underline{K}}$ $\quad S_t$ embedded in $(\Sigma_t, f)$

$q$ : metric procedure induced on $S_t$

$$H_{GR}(\text{solution}) = -2\oint_{S_t} \left[\alpha(\kappa - \kappa_0) + \beta_i(K_{ij} - K\gamma_{ij})s^j\right]\sqrt{q}\,d^2y$$

because $C_0 = 0 = C_i$ for a GR solution. Mass conservation $\Longleftrightarrow$ time translations, set $\beta_i = 0$

ADM mass $\stackrel{\text{def}}{=}$ $\boxed{M_{ADM} \equiv -\frac{1}{8\pi} \lim_{r\to\infty} \oint_{S_t} (K - K_0) \sqrt{q}\, d^2y}$

Total energy contained in $\Sigma_t \leftrightarrow$ value of $H_{GR}$ on the solution.
Note that at infinity $\alpha = 1$ and $\beta^i = 0$.
In terms of the 3-metric the expression above becomes:

$$M_{ADM} = +\frac{1}{16\pi} \lim_{r\to+\infty} \oint_{S_t} \left[ D^j \gamma_{ij} - D_i (f^{kl} \gamma_{kl}) \right] s^i \sqrt{q}\, d^2y ,$$

that in Cartesian coordinates

$$D_i = \partial_i$$
$$f^{kl} = \delta^{kl}$$

becomes (original ADM expression):

$$M_{ADM} = +\frac{1}{16\pi} \lim_{r\to+\infty} \oint_{S_t} \left( \frac{\partial \gamma_{ij}}{\partial x^j} - \frac{\partial \gamma_{jj}}{\partial x^i} \right) s^i \sqrt{q}\, d^2y .$$

Let us now express $M_{ADM}$ in terms of the conformal variables.
We use as background metric the ~~same~~ metric used for the definition of AF,

$$\gamma_{ij} = \Psi^4 \tilde{\gamma}_{ij}$$

with $\tilde{\gamma} = \det \tilde{\gamma}_{ij} = 1$ and Cartesian coordinates $x^i$, so that
$f = \det f_{ij} = 1$ and $f_{ij} = \operatorname{diag}(1,1,1)$.

AF conditions give then

$$\Psi = 1 + O(r^{-1})$$
$$\frac{\partial \Psi}{\partial x^k} = O(r^{-2})$$
$$\tilde{\gamma}_{ij} = f_{ij} + O(r^{-1})$$
$$\frac{\partial \tilde{\gamma}_{ij}}{\partial x^k} = O(r^{-2}) .$$

$$D^j \gamma_{ij} - D_i (f^{kl} \gamma_{kl}) = \underbrace{\cancel{\ldots}}_{4\Psi^3 D^j \Psi \cdot \tilde{\gamma}_{ij}} + \Psi^4 D^j \tilde{\gamma}_{ij} - 4\Psi^3 D_i \Psi \cdot f^{kl} \tilde{\gamma}_{kl} - \Psi^4 D_i (f^{kl} \tilde{\gamma}_{kl})$$

$$= 4\psi^3 D^j\psi\,\tilde{\gamma}_{ij} + \psi^4 D^j\tilde{\gamma}_{ij} - 4\psi^3 D_i\psi\, f^{kl}\tilde{\gamma}_{kl} - \psi^4 D_i(f^{kl}\tilde{\gamma}_{kl}) =$$

asymptotically:
$$\psi^3 \sim 1$$
$$\tilde{\gamma}_{ij} \sim f_{ij}$$
$$f^{kl}\tilde{\gamma}_{kl} \sim 3$$

$$= 4 D^j\psi f_{ij} + D^j\tilde{\gamma}_{ij} - 4\cdot 3\, D_i\psi - D_i(f^{kl}\tilde{\gamma}_{kl}) =$$

$$= -8\underbrace{D_i\psi}_{O(r^{-2})} + \underbrace{D^j\tilde{\gamma}_{ij}}_{O(r^{-2})} - \underbrace{D_i(f^{kl}\tilde{\gamma}_{kl})}_{?}$$

$$\tilde{\gamma}_{kl} \sim f_{kl} + \varepsilon_{kl} \quad \text{with} \quad \varepsilon_{ij} = O(r^{-1}) \text{ and } \partial_k\varepsilon_{ij} = O(r^{-2})$$

$$f^{kl}\tilde{\gamma}_{kl} = 3 + \varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz}$$

$$1 = \det\tilde{\gamma}_{ij} = 1 + \varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz} + \ldots + \varepsilon^2 + \ldots + \ldots + \varepsilon^3 + \ldots$$

$$\varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz} = -(\ldots\varepsilon^2\ldots + \varepsilon^3)$$

$$D_i(f^{kl}\tilde{\gamma}_{kl}) = D_i(\varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz}) = \partial_i(\varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz}) =$$

$$\approx \partial_i(\varepsilon^2 + \varepsilon^3) \approx \varepsilon\,\partial\varepsilon + \ldots = O(r^{-3})$$

$\Rightarrow$ the last term does not contribute to the integral!

$$\boxed{M_{ADM} = -\frac{1}{2\pi}\lim_{r\to\infty}\oint_{S_t} s^i\left(D_i\psi - \frac{1}{8}D^j\tilde{\gamma}_{ij}\right)\sqrt{q}\,d^2y}$$

Comments

- AF conditions guarantee the existence of the $M_{ADM}$ integral!

$$\left.\begin{array}{l}\dfrac{\partial \gamma_{ij}}{\partial x^k} = \mathcal{O}(r^{-2}) \\ \sqrt{q} \sim r^2\end{array}\right\} \Rightarrow M_{ADM} \text{ is finite}$$

- Newton (weak field) limit and justification of name "mass".

Consider the conformally decomposed expression and the weak-field metric

$$\tilde{\gamma}_{ij} = f_{ij} \quad ; \quad \Psi = 1 - \tfrac{1}{2}\Phi \quad , \quad \text{then}$$

$$\left.\begin{array}{l} \mathcal{D}^j \tilde{\gamma}_{ij} = 0 \\ \mathcal{D}_i \Psi = -\tfrac{1}{2}\mathcal{D}_i \Phi \end{array}\right\} \rightarrow M_{ADM} = + \frac{1}{8\pi} \lim_{r\to\infty} \oint_{S_t} s^i \mathcal{D}_i \Phi \sqrt{q}\, d^2y =$$

$$= \frac{1}{4\pi} \lim_{r\to\infty} \oint_{S_t} \mathcal{D}_i \mathcal{D}^i \Phi \sqrt{f}\, d^3x =$$

$$= \int_{\Sigma_t} \rho\, d^3x$$

- Schwarzschild spacetime and justification of "mass".

Consider the conformally decomposed formula and the Schwarzschild metric in isotropic coordinates:

$$\left.\begin{array}{l} \Psi = 1 + \dfrac{m}{2r} \\ \tilde{\gamma}_{ij} = f_{ij} \end{array}\right\} \rightarrow \mathcal{D}^j \tilde{\gamma}_{ij} = 0$$

NOTE WE NEED TO USE THE FORMULA WITH $\mathcal{D}$ AND NOT THE ONE WITH $\partial$ BECAUSE ISOTROPIC COORDINATES ARE NOT CARTESIAN.

$$f_{ij} = \mathrm{diag}(1, r^2, r^2 \sin\theta)$$

$S_t$ on the spheres $r = \text{const}$ on $\Sigma_t$ ; $y^a = (\theta, \varphi)$ ; $\sqrt{q} = r^2 \sin\theta$ ; and

$$s^i \sqrt{q}\, d^2y \sim r^2 \sin\theta\, d\theta\, d\varphi\, (\partial_r)^i \quad \text{at } i_0 \;;$$

$$M_{ADM} = -\frac{1}{2\pi} \lim_{r\to\infty} \oint_{r=\text{const}} \underbrace{\frac{\partial \Psi}{\partial r}}_{\partial_r\left(1+\frac{m}{2r}\right) = -\frac{m}{2} r^{-2}} r^2 \sin\theta\, d\theta\, d\varphi = + m \frac{1}{2} \frac{1}{\cancel{2\pi}} \cancel{4\pi} = m$$

• POSITIVE ENERGY THEOREM [Schoen & Yau 1981, Witten 1981]

$$\boxed{M_{ADM} \geq 0}$$

if the matter content of the spacetime obeys the

<u>dominant energy condition</u> : $E \geq \sqrt{P_a P^a}$ .

And:

$$M_{ADM} = 0 \quad \Longleftrightarrow \quad \Sigma_t \text{ is a hypersurface of Minkowski}$$

Note that:

$$E \geq \sqrt{P_a P^a} \quad \Longleftrightarrow \quad \begin{cases} E^2 \geq P_a P^a \\ E \geq 0 \end{cases} \quad \text{because } P^a \text{ is spacelike.}$$

• The Hamiltonian $\mathcal{H}_{GR}$ depends only on $\alpha, \beta^i, \gamma_{ij}$ and conjugate momenta; it does not depend on time $t$ $\Rightarrow$

$$\boxed{\frac{d}{dt} M_{ADM} = 0}$$

ADM MOMENTUM

Consider again the Hamiltonian ~~expression~~ expression on the GR solution:

$$\mathcal{H}_{GR} = -2 \oint_{S_t} \left[ \alpha (K - K_0) + \beta^i (K_{ij} - K \gamma_{ij}) \right] \sqrt{q}\, d^2y$$

In terms of the Cartesian coordinates $(x^i)$, the 3 vectors $(\partial_i)_{i \in \{1,2,3\}}$ identify the possible directions for <u>spatial translations</u>.

Setting $\alpha = 0$ and $\beta^i_{(k)j} = \delta^i_j$ we define:

$$\boxed{P_i \equiv \frac{1}{8\pi} \lim_{r \to +\infty} \oint_{S_t} \left( K_{jk} - K \gamma_{jk} \right) (\partial_i)^j s^k \sqrt{q}\, d^2y \qquad i = 1,2,3}$$

Comments:

- The AF conditions guarantee the existence of the integral;
- $P_i$ behaves like a 1-form under a change of coordinates that corresponds to rotation and/or translation at $i_0$;
- Similarly the quantity $P^{ADM}_\alpha = (-M_{ADM}, P_1, P_2, P_3)$ behaves like a 4-dimensional form under the coordinate transformation that preserve AF:

$$P'^{ADM}_\alpha = (\Lambda^{-1})^\mu{}_\alpha \, P^{ADM}_\mu$$

[ADM PR 122, 3 997 (1961)]. The derivation uses Einstein equations.

ANGULAR MOMENTUM

Conserved quantity associated to invariance with respect to rotations.

Consider Cartesian coordinates $(x^i)$, the three vectors

$$\underline{\phi}_x \equiv -z\,\partial_y + y\,\partial_z$$

$$\underline{\phi}_y \equiv -x\,\partial_z + z\,\partial_x$$

$$\underline{\phi}_z \equiv -y\,\partial_x + x\,\partial_y$$

are Killing vectors of $f_{ij}$ corresponding to rotations about the axes.

One can then define:

$$\boxed{J_i \equiv \frac{1}{8\pi} \lim_{r\to+\infty} \oint_{S_t} (K_{jk} - K\,\gamma_{jk})\,(\underline{\phi}_i)^j\, s^k \sqrt{q}\, d^2y \qquad i=1,2,3}$$

i.e. we set $\alpha = 0$ and $\beta^i_{(\underline{\phi}_j)} = (\underline{\phi}_j)^i$.

NOTE THAT ASYMPTOTICALLY $(\underline{\phi}_j)^i \sim O(r)$.

Comments:

- $(K_{jk} - K\gamma_{jk})(\phi_i)^j \sim \mathcal{O}(r^{-1}) \Rightarrow$ AF conditions do <u>not</u> guarantee $J_i$ to be finite.

- However $J_i$ might be finite because there is the contraction with $\underline{s}$, e.g.

$$\gamma_{jk}(\phi_i)^j s^k = \langle \underline{\phi}_i, \underline{s} \rangle = 0 \quad \Leftarrow \quad \underline{s} \sim \frac{x}{r}\partial_x + \frac{y}{r}\partial_y + \frac{z}{r}\partial_z$$

- $J_i$ DO NOT TRANSFORM as a vector components at $i_0$ under a change of coordinate that preserve AF.

  $\Rightarrow$ $J_i$ are coordinate dependent quantities.

  One can still use the $J_i$ expression by ~~expressing that~~ <u>defining</u> ang. momentum as the quantity that remains invariant under a <u>sub-class</u> of transformation at $i_0$.

  In particular one can require stronger decay properties for $\tilde{\gamma}_{ij}$ and $K_{ij}$:

$$\frac{\partial \tilde{\gamma}_{ij}}{\partial x^j} = \mathcal{O}(r^{-3}) \quad \stackrel{\text{def}}{=} \quad \text{QUASI-ISOTROPIC GAUGE}$$

$$K = \mathcal{O}(r^{-3}) \quad \stackrel{\text{def}}{=} \quad \text{ASYMPTOTICALLY MAXIMAL GAUGE}$$

  Under these conditions $J_i$ transforms as a vector.

- Note that the ADM mass in quasi-isotropic gauge reads:

$$M_{ADM} = -\frac{1}{2\pi} \lim_{r \to +\infty} \oint_{S_t} s^i D_i \psi \sqrt{q}\, d^2y$$

since $D^j \tilde{\gamma}_{ij} \approx 0$ for $r \gg 1$.

# KOMAR CURRENTS : MASS AND ANGULAR MOMENTUM IN PRESENCE OF KILLING VEC.

## General argument.

Consider a Killing vector $K^a$, $\nabla_b K_a + \nabla_a K_b = 0 \leftarrow \mathcal{L}_K g_{ab} = 0$.

For <u>any</u> symmetric and conserved tensor $T^{ab}$ ($\nabla_a T^{ab} = 0$) one has:

$$\nabla_a (T^{ab} K_b) = \underbrace{\nabla_a T^{ab}}_{=0} \cdot K_b + T^{ab} \nabla_a K_b = \frac{1}{2}\left(T^{ab}\nabla_a K_b + T^{ab}\nabla_a K_b\right) \underset{T^{ab}=T^{ba}}{=}$$

$$= \frac{1}{2} T^{ab}\left(\nabla_a K_b + \nabla_b K_a\right) = 0$$

Thus, the current $J^a \equiv T^{ab} K_b$ is <u>conserved</u>.

Take now the Einstein tensor $G_{ab}$, and construct the conserved current

$$J_1^a \equiv G^{ab} K_b = {}^4R^{ab} K_b - \frac{1}{2} {}^4R K^a .$$

Note that using Einstein equations the above expression defines a conserved quantity related to the stress-energy of the matter...

More in general the current:

$$J^a_{(\alpha)} \equiv {}^4R^{ab} K_b - \frac{1}{2} \alpha {}^4R K^a \qquad \alpha \in \mathbb{R} \qquad \text{(not the lapse!)}$$

is conserved since

$$\nabla_a J^a_{(\alpha)} = \nabla_a ({}^4R^{ab} K_b) - \frac{1}{2} \alpha \nabla_a ({}^4R K^a) =$$

$$= \nabla_a ({}^4R^{ab} K_b) - \frac{1}{2} \alpha {}^4R \underbrace{\nabla_a K^a}_{\substack{=0 \\ \text{from antisym.}}} - \frac{1}{2} \alpha K^a \nabla_a R =$$

$$= \nabla_a ({}^4R^{ab} K_b) - \frac{1}{2} \alpha \underbrace{K^a \nabla_a R}_{\equiv 0 \text{ invariance of curvature along Killing vec.}} =$$

$$= 0 \quad , \text{ because}$$

$${}^4R^{ab}K_b = \nabla_c \nabla^c K^a$$

$$\nabla_a \left( {}^4R^{ab} K_b \right) = \nabla_c \nabla^c \left( \nabla_a K^a \right) = 0$$

[ Remember for Killing vectors: $\nabla_a \nabla_b K^a = {}^4R_{ab} K^b \iff \nabla_c \nabla^c K^a = -{}^4R^{ab} K_b$ ]

the $\alpha = 0$ case is special: $J^a_{(0)} = {}^4R^{ab} K_b = \nabla_b \left( \nabla^a K^b \right) = -\nabla_b \left( \nabla^b K^a \right)$

it can be written as the divergence of an <u>asymmetric</u> tensor !

For <u>any</u> antisymmetric tensor $A^{ab}$ <u>Stokes theorem</u> apply:

$$\int_{\Sigma_t} dV_a \, \nabla_b A^{ab} = \int_{\partial \Sigma} dS_{ab} \, A^{ab}$$

In gen.

$$\boxed{\int_M d\underline{\omega} = \oint_{\partial M} \underline{\omega}}$$

where:

$dV_a \equiv \sqrt{\gamma}\, n_a \, d^3x$ is volume element on $\Sigma_t$

$dS_{ab} \equiv \left( S_e n_a - n_a S_b \right) \sqrt{q}\, d^2y$ is the surface element of $\partial\Sigma = S_t$.

$\rightarrow$ Using this general scheme one can construct conserved quantities on $\Sigma$ in presence of symmetries (Killing vectors). The conserved quantities do <u>not</u> depend on surface $\partial\Sigma = S_t$ as long as the latter is outside the matter content (sufficiently far away).

<u>Komar mass</u>

Assume <u>stationarity</u>, the Killing vector is $K^a = \underline{\partial_t}$ in adapted coordinate,

so: $\underline{K} = \alpha \underline{n} + \underline{\beta}$.

$$\nabla^a K^b dS_{ab} = \nabla^a K^b \left( S_e n_a - n_a S_b \right) \sqrt{q}\, d^2y =$$

$$= \nabla_a K_b \left( s^a n^b - n^a s^b \right) \sqrt{q}\, d^2y =$$

$$= \nabla_a K_b \, s^a n^b - \underbrace{\nabla_a K_b}_{(-\nabla_b K_a) \text{ and rename indexes}} n^a s^b \sqrt{q}\, d^2y =$$

$$= 2\, \nabla_a K_b\, s^a n^b \sqrt{q}\, d^2y =$$

$$= 2\left(\nabla_a \alpha \cdot n_b + \alpha \nabla_a n_b + \nabla_a \beta_b\right) s^a n^b \sqrt{q}\, d^2y =$$

$$= 2\Big(-\nabla_a \alpha \cdot s^a + \alpha\, s^a \underbrace{n^b \nabla_a n_b}_{=0} + s^a \underbrace{n^b \nabla_a \beta_b}\Big)\sqrt{q}\, d^2y =$$

$$n^b \beta_b = 0 \Rightarrow n^b \nabla_a \beta_b = -\beta_b \nabla_a n^b$$

$$= 2\Big(-s^i D_i \alpha - s^a \beta^b \underbrace{\nabla_a n_b}\Big)\sqrt{q}\, d^2y$$

$$+\nabla_a n_b = -K_{ab} - n_a a_b$$

$$= 2\Big(-s^i D_i \alpha + s^a \beta^b K_{ab} + \underbrace{n_a s^a}_{=0} \beta^b a_b\Big)\sqrt{q}\, d^2y =$$

$$= -2\left(s^i D_i \alpha - s^i \beta^j K_{ij}\right)\sqrt{q}\, d^2y .$$

The Komar mass is the current associated to $k^a = \partial_t$ :

$$\boxed{M_K \overset{\text{def.}}{=} \frac{1}{4\pi} \oint_{S_t} \left(s^i D_i \alpha - K_{ij}\, s^i \beta^j\right)\sqrt{q}\, d^2y}$$

Example : Weak field

$$\alpha = 1 + \phi \quad ; \quad K_{ij} = 0 \quad ; \quad D_i \to \partial_i$$

$$\int_{S_t} s^i D_i \Phi \sqrt{q}\, d^2y = \int_V \partial^i \partial_i \phi\, d^3x = 4\pi \int_V \rho\, d^3x$$

Example : Schwarzschild (in Schwarzschild coords)

$K_{ij} = 0$

$S_t$ : surface $r = \text{const}$

$y^a = (\theta, \varphi)$ , $\sqrt{q} = r^2 \sin\theta$

$$s^i = \left[\left(1 - \frac{2m}{r}\right)^{1/2}, 0, 0\right]$$

$$\alpha = \left(1 - \frac{2m}{r}\right)^{1/2}$$

$$M_K = \frac{1}{4\pi} \int_{r=\text{const}} \left(1-\frac{2m}{r}\right)^{1/2} \frac{\partial}{\partial r}\left[\left(1-\frac{2m}{r}\right)^{1/2}\right] r^2 \sin\theta \, d\theta \, d\varphi =$$

$$= \frac{1}{4\pi} \int (\;)^{1/2} (\;)^{-1/2} \left(+\frac{1}{2}\right) \frac{2m}{r^2} r^2 \sin\theta \, d\theta \, d\varphi = m$$

Note • does not depend on the choice of the spheres $r=\text{const}$.

• $\boxed{M_K = M_{ADM}}$ THIS IS A GENERAL RESULT FOR STATIONARY SPACETIMES AND FOR ANY FOLIATION SUCH THAT

$$\underline{n} \doteq \underline{k} \quad \text{at } i_0$$

(normal vector = Killing vector, at infinity)

## Angular momentum

Assume <u>axisymmetry</u>, the Killing vector is $k^a = \phi^a$.

The Komar angular momentum is the associated current:

$$\boxed{J_K \equiv \frac{1}{16\pi} \oint_{S_t} \nabla^a \phi^b \, dS_{ab}}$$

which become:

$$\nabla^a \phi^b \, dS_{ab} = D_a \phi_b \left(s^a n^b - n^a s^b\right) \sqrt{q} \, d^2y =$$

$$= 2 \nabla_a \phi_b n^b s^a \sqrt{q} \, d^2y = \qquad (\phi_b \cdot n^b = 0)$$

$$= -2 \phi_b \nabla_a n^b \cdot s^a \sqrt{q} \, d^2y =$$

$$= 2 K_{ij} s^i \phi^j \sqrt{q} \, d^2y$$

$$\boxed{J_K = \frac{1}{8\pi} \oint_{S_t} K_{ij}\, s^i \phi^j \sqrt{q}\, d^2y}$$

Example : Kerr in Boyer-Lindquist coordinates

$S_t$ : sphere of radius $r = \text{const}$

$y^\alpha = (\theta, \varphi)$

$$\left.\begin{aligned} \phi^i &= (0, 0, 1) \\ s^i &= (s^r, 0, 0) \end{aligned}\right\} \to \text{select "} r\varphi \text{" component of } K_{ij}$$

$$J_K = \frac{1}{8\pi} \oint_{r=\text{const}} K_{r\varphi}\, s^r \sqrt{q}\, d\theta d\varphi$$

Using $2\alpha K_{ij} = \mathcal{L}_\beta \gamma_{ij}$ $(\partial_t \gamma_{ij} = 0)$ one can evaluate $K_{r\varphi}$ easily.
The shift is read-off from the metric (as the values of $\alpha$, etc.)

$$(\beta^r, \beta^\theta, \beta^\varphi) = \left(0, 0, -\frac{2amr}{(r^2+a^2)(r^2+a^2\cos^2\theta) + 2a^2mr\sin^2\theta}\right)$$

so :

$$K_{r\varphi} = \frac{1}{2\alpha}\Big(\underbrace{\beta^\varphi \partial_\varphi \gamma_{r\varphi}}_{=0} + \gamma_{\varphi\varphi}\frac{\partial \beta^\varphi}{\partial r} + \underbrace{\gamma_{r\varphi}\partial_\varphi \beta^\varphi}_{=0}\Big) = \frac{1}{2\alpha}\gamma_{\varphi\varphi}\frac{\partial \beta^\varphi}{\partial r}$$

axisymmetry

For $r \to +\infty$ :

$$s^r \sim 1 \;;\; \alpha \sim 1 \;;\; \gamma_{\varphi\varphi} \sim r^2 \sin^2\theta \;;\; \sqrt{q} \sim r^2 \sin\theta \;;\; \beta^\varphi \sim -2amr^{-3}$$

$\Rightarrow$

$$J_K = \frac{1}{16\pi}\oint_{r=\text{const}} r^2 \sin^2\theta \,\frac{6am}{r^4}\, r^2 \sin\theta\, d\theta d\varphi = \frac{3am}{8\pi} \cdot 2\pi \int_0^\pi \sin^3\theta\, d\theta = am$$

Further readings on mass definitions in GR :

- Wald book Chap. 11
- P. Chruściel lecture notes on "Energy in General Relativity".

# Euristic derivation of ADM mass

$$M = \int \rho\, dv = \underbrace{\frac{1}{4\pi}\int D_i D^i \Phi\, dv}_{\rho = \frac{1}{4\pi}\Delta\Phi = \frac{1}{4\pi} D_i D^i \Phi} = \frac{1}{4\pi}\underbrace{\int D^i \Phi \cdot s_i\, ds}_{\text{surface integral with flux} \sim \text{metric derivatives}}$$

Consider the Hamiltonian constraint : $\boxed{R + K^2 - K_{ij}K^{ij} = 16\pi E}$

Asymptotically : $K^2 \simeq K_{ij}K^{ij} \approx 0 + \dots$

$$\gamma_{ij} = f_{ij} + h_{ij}$$

$$R \simeq \partial_i(\partial_j h_{ij} + \partial_i h)$$

$$E \approx \rho$$

so one can think that the "flux" is given by $R$ :

$$M = \int \rho\, dv \approx \int R\, dv \approx \int dv\, \partial_i(\partial_j h_{ij} + \partial_i h) = \underbrace{\int (\partial_j h_{ij} + \partial_i h)\, s_i\, ds}_{\text{ADM formula with other notations}}$$

# IDENTITIES INVOLVING RIEMANN TENSORS AND KILLING VECTORS

$R_{abcd} = R_{cdab}$ Riemann symmetry

1) $(\nabla_a\nabla_b - \nabla_b\nabla_a)V_c = -R^e{}_{cab}V_e$
2) $\nabla_a K_b + \nabla_b K_a = 0$

$\Rightarrow \boxed{\nabla_c\nabla_a k_b = R^d{}_{cab}K_d}$ (a.)

Proof.

$\nabla_a\nabla_b K_c + \nabla_b\nabla_c K_a + \nabla_c\nabla_a K_b = 0$ total antisymm. ciclic permutation $(a\ b\ c)$

$\nabla_c\nabla_a K_b = -\nabla_a\nabla_b K_c - \nabla_b\underbrace{\nabla_c K_a}_{\overset{2)}{=}-\nabla_a K_c} = -(\nabla_a\nabla_b - \nabla_b\nabla_a)K_c \overset{1)}{=} -R^e{}_{cab}K_e$.

$$\boxed{\nabla^c\nabla_a K_c = K^c R_{ac}}$$ (b.)

Proof:

(a.) $\Rightarrow$

$g^{cb}\nabla_c\nabla_a K_b = g^{cb}R^d{}_{cab}K_d$

$\nabla^c\nabla_a K_c = R^d{}_a K_d = R_{da}K^d$ .

$$\boxed{R_{ab}k^a k^b = (\nabla^a k^b)(\nabla_a k_b) + \nabla_b(K^a\nabla_a K^b)}$$ (c.)

Proof:

(b.) $\Rightarrow$ $R_{ab}k^a k^b = K^b\nabla^c(\nabla_b K_c) = \nabla^c(K^b\nabla_b K_c) - \nabla^c K^b\cdot\nabla_b K_c =$

$= \nabla^c(K^b\nabla_a K_c) + \nabla^c K^b\cdot\nabla_c K_b$ .

$$\boxed{\mathcal{L}_{\underline{K}}\, g_{ab} = 0 \quad \Rightarrow \quad \mathcal{L}_{\underline{K}}\, R = 0}$$

$$\nabla_a K_b + \nabla_b K_a = 0 \quad \Rightarrow \quad K^a \nabla_a R = 0$$

Proof:

$\nabla^a G_{ab} = 0$ Bianchi identity

Contract:

$$0 = (\nabla^a G_{ab}) K^b = (\nabla^a R_{ab}) K^b - \frac{1}{2} K^c \nabla_c R =$$

$$= \nabla^a (R_{ab} K^b) - \underbrace{R_{ab} \nabla^a K^b}_{=0} - \frac{1}{2} K^c \nabla_c R$$

because:

$$R_{ab} \nabla^a K^b = \frac{1}{2}\left(R_{ab} \nabla^a K^b + R_{ab} \nabla^a K^b\right) =$$

$$= \frac{1}{2}\left(R_{ab} \nabla^a K^b - R_{ab} \nabla^b K^a\right) =$$

$$= \frac{1}{2} R_{ab} \left(\nabla^a K^b - \nabla^b K^a\right) = 0$$

$$= \nabla^a (R_{ab} K^b) - \frac{1}{2} K^c \nabla_c R$$

$$2 \nabla^a (R_{ab} K^b) = K^c \nabla_c R$$

$$K^c \nabla_c R = 2 \nabla^a (R_{ab} K^b) = 2 \nabla^a (\nabla^c \nabla_a K_c) = \underbrace{\frac{2}{2} [\nabla^a, \nabla^c] \underbrace{(\nabla_a K_c)}_{\text{antisymm}}}_{\text{property of Ricci Tensor:}} = 0$$

$$[\nabla_a, \nabla_b] T^{ab} = R_{ab} (T^{ab} - T^{ba}) = 0$$

Similarly one can prove:

$$\boxed{\mathcal{L}_{\underline{K}}\, g_{ab} = 0 \quad \Rightarrow \quad \mathcal{L}_{\underline{K}}\, R_{abcd} = 0 \quad \wedge \quad \mathcal{L}_{\underline{K}}\, R_{ab} = 0}$$