CONFORMAL DECOMPOSITION

Lichnerowicz 1944 : constraint equations and initial data problem.
York 1971 : evolution problem.

CONFORMAL TRANSFORMATION $\stackrel{\text{def}}{=}$ $\gamma_{ij} = \Psi^4 \tilde{\gamma}_{ij}$

- Ψ^4 : conformal factor (power "4" is arbitrary)
- $\tilde{\gamma}_{ij}$: conformal metric

ALL THE METRICS THAT CAN BE RELATED TO $\tilde{\gamma}_{ij}$ BY A CONFORMAL TRANSFORMATION FORM A CONFORMAL EQUIVALENCE CLASS.

York : The 2 degrees of freedom of the gravitational field are carried by the conformal equivalence classes of 3-metrics.

BACKGROUND METRIC

A common choice for conformal metric is to require determinant = 1 (1 "new" field included)

Since $\det(cA) = c^n \det A$ with $A = n \times n$ matrix, we get

$$\det \gamma_{ij} = \Psi^{12} \det \tilde{\gamma}_{ij},$$

$\gamma \equiv \det \gamma_{ij}$

$\tilde{\gamma} \equiv \det \tilde{\gamma}_{ij}$

Taking $\Psi = \gamma^{1/12}$ gives $\tilde{\gamma}_{ij} = \gamma^{-1/3} \gamma_{ij}$ and $\tilde{\gamma} = 1$

Problem : $\tilde{\gamma}_{ij}$ IS NOT a tensor!

→ Tensor transformation : $T_{ab} = J_a^{a'} J_b^{b'} T_{a'b'}$ with $J_a^{a'} = \dfrac{\partial x^{a'}}{\partial x^a}$

→ Tensor density of weight w : $\mathcal{T}_{ab} = J^w J_a^{a'} J_b^{b'} \mathcal{T}_{a'b'}$ with $J = \det(J_a^{a'})$

$\tilde{\gamma}_{ij}$ IS A TENSOR DENSITY OF WEIGHT $-2/3$, because $\det g' = J^2 \det g$.

Ψ IS NOT A SCALAR FIELD since γ depends on the coordinates.

IMPORTANT CONSEQUENCE : $\tilde{\gamma}_{ij}$ has <u>no</u> unique Levi-Civita connection associated !

How to associate a connection to $\tilde{\gamma}_{ij}$? The connection will be coordinate dependent.

<u>Want!</u>: write a formalism that can be used with any type of coordinates on Σ.

Introduce a <u>BACKGROUND metric</u> f_{ij} :

- Riemannian metric $(+,+,+)$
- Components obey $\frac{\partial}{\partial t} f_{ij} = \mathcal{L}_{\underline{\partial_t}} f_{ij} = 0$

  i.e. the metric f is Lie-dragged along coordinate time evolution vector $\underline{\partial_t}$
- Inverse metric f^{ij} : $f^{ik} f_{kj} = \delta^i{}_j$
- Associated Levi-Civita connection $\mathcal{D}_k f_{ij} = 0$

  with $\mathcal{D}^i \equiv f^{ij} \mathcal{D}_j$ and $\bar{\Gamma}^k{}_{ij} = \frac{1}{2} f^{kl} (\partial_i f_{lj} + \partial_j f_{il} - \partial_l f_{ij})$

The introduction of a background metric allows one to define a conformal metric with "unit determinant" in every coordinate system.

CONFORMAL METRIC WITH UNIT det $\stackrel{\text{def.}}{=}$ $\tilde{\gamma}_{ij} \equiv \Psi^{-4} \gamma_{ij}$

$$\text{with} \quad \Psi \equiv \left(\frac{\gamma}{f} \right)^{1/12}$$

With this definition Ψ is a tensor field since for a change of coords $(x^i) \longrightarrow (x^{i'})$ one has

$$\gamma' = J^2 \gamma$$
$$f' = J^2 f$$

and Ψ behaves as a <u>scalar</u> on Σ_t

$\tilde{\gamma}_{ij}$ transforms now as a tensor field on Σ_t and $\tilde{\gamma} = \det \tilde{\gamma}_{ij} = f$

Example : Schwarzschild

(t, R, θ, φ) Schw. coords.
(t, r, θ, φ) Isotropic coords. : $R = r\left(1+\frac{m}{2r}\right)^2 \Rightarrow$

$$ds^2 = -\left(\frac{1-\frac{m}{2r}}{1+\frac{m}{2r}}\right)^2 dt^2 + \left(1+\frac{m}{2r}\right)^4 \underbrace{\left[dr^2 + r^2 d\Omega^2\right]}$$

$f_{ij} = \mathrm{diag}(1, r^2, r^2 \sin^2\theta)$: flat metric in spherical coords.

$$\Psi = 1 + \frac{m}{2r}$$

and we prove that Schwarzschild is conformally flat.

Example : Weak field

$$ds^2 = -(1+2\Phi)\,dt^2 + (1-2\Phi)\, f_{ij}\, dx^i dx^j$$

$$\Psi = (1-2\Phi)^{+1/4} \simeq 1 - \frac{1}{2}\Phi \qquad |\Phi| << 1$$

the conformal factor is essentially the gravitational potential.

By comparing with the above example :

$$1 - \frac{1}{2}\Phi \simeq 1 + \frac{m}{2r} \Rightarrow \Phi \sim -\frac{m}{r}$$

CONFORMAL CONNECTION

introduce a Levi-Civita connection for the conformal metric $\tilde{D}_k \tilde{\gamma}_{ij} = 0$
with associated Christoffel symbols $\tilde{\Gamma}^k{}_{ij}$.

Formula for generic tensors relating D_i and $\tilde{D}_i$:

$$D_k T^{i_1 \dots i_p}{}_{j_1 \dots j_q} = \tilde{D}_k T^{i_1 \dots i_p}{}_{j_1 \dots j_q} + \sum_{r=1}^{p} \Delta^{i_r}{}_{k\ell}\, T^{i_1 \dots \ell \dots i_p}{}_{j_1 \dots j_q} - \sum_{r=1}^{q} \Delta^{\ell}{}_{k j_r}\, T^{i_1 \dots i_p}{}_{j_1 \dots \ell \dots j_q}$$

with :

$$\Delta^k{}_{ij} = \Gamma^k{}_{ij} - \tilde{\Gamma}^k{}_{ij} \qquad \text{(is a tensor field)}$$

$$\Delta^k_{ij} = \frac{1}{2}\gamma^{kl}\left(\tilde{D}_i\gamma_{lj} + \tilde{D}_j\gamma_{il} - \tilde{D}_l\gamma_{ij}\right) =$$

$$= 2\left(\delta^k_i\,\tilde{D}_j \ln\Psi + \delta^k_j\,\tilde{D}_i \ln\Psi - \tilde{D}^k \ln\Psi\,\tilde{\gamma}_{ij}\right)$$

and

$$\Delta^k_{ki} = 6\,\tilde{D}_i \ln\Psi$$

EXPRESSIONS RELATING THE RICCI TENSORS

$$R_{ij} = \tilde{R}_{ij} + \tilde{D}_k\Delta^k_{ij} - \tilde{D}_i\Delta^k_{kj} + \Delta^k_{ij}\Delta^l_{lk} - \Delta^k_{il}\Delta^l_{kj}$$

General formula relating the Ricci of two connection

$$= \tilde{R}_{ij} - 2\tilde{D}_i\tilde{D}_j\ln\Psi - 2\tilde{D}_k\tilde{D}^k\ln\Psi\,\tilde{\gamma}_{ij} + 4\tilde{D}_i\ln\Psi\,\tilde{D}_j\ln\Psi - 4\tilde{D}_k\ln\Psi\,\tilde{D}^k\ln\Psi\,\tilde{\gamma}_{ij}$$

where in the 2nd line we explicitly use the fact that γ_{ij} and $\tilde{\gamma}_{ij}$ are related by the conformal transformation.

Taking the trace with respect to γ_{ij}, $R = \gamma^{ij}R_{ij} = \Psi^{-4}\tilde{\gamma}^{ij}R_{ij}$, one finds

$$R = \Psi^{-4}\tilde{R} - 8\Psi^{-5}\tilde{D}_i\tilde{D}^i\Psi$$

TRACELESS AND CONFORMAL DECOMPOSITION OF EXTRINSIC CURVATURE

$$K_{ij} \equiv A_{ij} + \frac{1}{3}K\gamma_{ij}$$

Verify traceless: $\underbrace{\gamma^{ij}K_{ij}}_{K} = \gamma^{ij}A_{ij} + \frac{1}{3}K\underbrace{\gamma^{ij}\gamma_{ij}}_{=3} \Rightarrow A = \gamma^{ij}A_{ij} = 0$

Note definitions of "up-up" components : $K^{ij} = \gamma^{ik}\gamma^{jl}K_{kl}$, same for A^{ij}.

Conformal decomposition of traceless part:

$$A^{ij} = \Psi^{\alpha}\,\tilde{A}^{ij}$$

Different possible choices of α depending on the use of the equations: time evolution or initial data.

Time evolution making $\boxed{\alpha=-4}$

Consider $\mathcal{L}_m \gamma_{ij} = -2\alpha K_{ij}$ and take the trace:

$$\mathcal{L}_m(\Psi^4\tilde\gamma_{ij}) = -2\alpha A_{ij} - \frac{2}{3}\alpha K \gamma_{ij}$$

$$\mathcal{L}_m \tilde\gamma_{ij} = -2\alpha \Psi^{-4} A_{ij} - \frac{2}{3}\alpha K \tilde\gamma_{ij} - 4\tilde\gamma_{ij}\underbrace{\Psi^{-4}\Psi^{3}\mathcal{L}_m\Psi}_{\mathcal{L}_m \ln\Psi}$$

Tr: $$\tilde\gamma^{ij}\mathcal{L}_m\tilde\gamma_{ij} = -2\alpha\Psi^{-4}\underbrace{\tilde\gamma^{ij}A_{ij}}_{=0} - \frac{2}{3}\alpha K\underbrace{\tilde\gamma^{ij}\tilde\gamma_{ij}}_{=3} - 4\underbrace{\tilde\gamma^{ij}\tilde\gamma_{ij}}_{3}\mathcal{L}_m\Psi =$$

$$= -2\alpha K - 12\,\mathcal{L}_m\Psi$$

Now use the identity on the LHS:

$$\delta(\ln\det M) = \mathrm{tr}(M^{-1}\cdot\delta M)$$

valid for every derivative operator "δ" and matrix "M".

Here: $\delta \to \mathcal{L}_m$

$M \to \tilde\gamma_{ij}$, we get:

$$\tilde\gamma^{ij}\mathcal{L}_m\tilde\gamma_{ij} = \mathcal{L}_m \ln\det\tilde\gamma_{ij} = (\partial_t - \mathcal{L}_\beta)\ln f = -\mathcal{L}_\beta \ln f =$$

$$= -\tilde\gamma^{ij}\mathcal{L}_\beta\tilde\gamma_{ij} = -\tilde\gamma^{ij}\big(\underbrace{\beta^k\tilde D_k\tilde\gamma_{ij}}_{=0} + \tilde\gamma_{kj}\tilde D_i\beta^k + \tilde\gamma_{ik}\tilde D_j\beta^k\big) =$$

$$= -\delta^i_k\tilde D_i\beta^k - \delta^j_k\tilde D_j\beta^k = -2\tilde D_i\beta^i$$

Combine the two expressions for $\tilde{\gamma}^{ij} \mathcal{L}_m \tilde{\gamma}_{ij}$:

$$\alpha K + 6 \mathcal{L}_m \ln \psi = \tilde{D}_i \beta^i$$

which implies an evolution equation for the conformal factor:

$$\boxed{\mathcal{L}_m \ln \psi = \frac{1}{6}\left(\tilde{D}_i \beta^i - \alpha K\right)}$$

and the following eq for the conformal metric:

$$\boxed{\mathcal{L}_m \tilde{\gamma}_{ij} = -2\alpha \psi^{-4} A_{ij} - \frac{2}{3} \tilde{D}_k \beta^k \tilde{\gamma}_{ij}}$$

The above equation suggests $\alpha = -4$: $\bar{A}^{ij} = \psi^4 A^{ij}$.

9.05.2018 chose sign.

Momentum constraints scaling : $\boxed{\alpha = -10}$

Consider the momentum equation. It contains the divergence of the extr. curvature:

$$D_j K^{ij} = D_j A^{ij} + \frac{1}{3} D^i K$$

Compute:

$$D_j A^{ij} = \tilde{D}_j A^{ij} + \Delta^i{}_{jk} A^{kj} + \Delta^j{}_{jk} A^{ik} = \ldots =$$

$$= \tilde{D}_j A^{ij} + 10 A^{ij} \tilde{D}_j \ln \psi - 2 \tilde{D}^i \ln \psi \underbrace{\tilde{\gamma}_{jk} A^{jk}}_{= \psi^{-4} \gamma_{jk} A^{jk} = 0} =$$

$$= \psi^{-10} \tilde{D}_j (\psi^{10} A^{ij}).$$

Formula is valid for any symmetric, traceless tensor : $\left\{ \begin{array}{l} A^{ij} = \psi^{\alpha} \bar{A}^{ij} \\ D_j A^{ij} = \psi^{10} \tilde{D}_j (\psi^{10+\alpha} \bar{A}^{ij}) \end{array} \right.$

The last equation suggest to scale with $a=-10$, $\hat{A}^{ij} = \psi^{10} A^{ij}$.
With this scaling the momentum constraint became

$$\boxed{\tilde{D}_j \hat{A}^{ij} - \frac{2}{3}\psi^6 \tilde{D}^i K = 8\pi \underbrace{\psi^{10} p^i}_{\tilde{p}^i \equiv \psi^{10} p^i}}$$

CONFORMAL 3+1 GR

<u>Hamiltonian constraint</u>

$$R + K^2 - K_{ij}K^{ij} = 16\pi E \quad \text{, decompose:}$$

$$R = \psi^{-4}\tilde{R} - 8\psi^{-5}\tilde{D}_i\tilde{D}^i\psi$$

$$K_{ij}K^{ij} = \left(A_{ij} + \frac{1}{3}K\gamma_{ij}\right)\left(A^{ij} + \frac{1}{3}K\gamma^{ij}\right) = \underbrace{A_{ij}A^{ij}}_{\tilde{A}_{ij}\tilde{A}^{ij}} + \frac{1}{3}K\underbrace{\gamma^{ij}A_{ij}}_{=0} + \frac{1}{3}K\underbrace{\gamma_{ij}A^{ij}}_{=0} + \frac{1}{9}K^2\underbrace{\gamma^{ij}\gamma_{ij}}_{=3} =$$

$$= \tilde{A}_{ij}\tilde{A}^{ij} + \frac{1}{3}K^2$$

$$\tilde{D}_i\tilde{D}^i\psi - \frac{1}{8}\psi\tilde{R} + \psi^5\Big(\underbrace{-\frac{1}{8}K^2 + \frac{1}{8\cdot 3}K^2}_{-\frac{1}{12}K^2} + \frac{1}{8}\tilde{A}_{ij}\tilde{A}^{ij} + 2\pi E\Big) = 0$$

$$\boxed{\tilde{D}_i\tilde{D}^i\psi - \frac{1}{8}\tilde{R}\psi + \psi^5\left(\frac{1}{8}\tilde{A}_{ij}\tilde{A}^{ij} - \frac{1}{12}K^2 + 2\pi E\right) = 0}$$

With the alternative scaling $\hat{A}^{ij} = \psi^6 \tilde{A}^{ij}$:

$$\boxed{\tilde{D}_i\tilde{D}^i\psi - \frac{1}{8}\tilde{R}\psi + \frac{1}{8}\hat{A}_{ij}\hat{A}^{ij}\psi^{-7} + \left(2\pi E - \frac{1}{12}K^2\right)\psi^5 = 0}$$

<u>Lichnerowicz equation.</u>

These equations can be seen as elliptic-type equations for the conformal factor ψ.
They are non linear and it is useful to consider the problem of well-posedness and uniqueness of the solutions.

- Consider first the linear equation (toy model problem):

$$\Delta u = f\, u \quad \text{on } \Omega \quad \text{with} \quad u = 0 \text{ on } \partial\Omega, \text{ then}$$

$$f \geq 0 \Rightarrow u \equiv 0 \text{ on } \Omega.$$

Indirect proof using the maximum principle:

- if $u \neq 0$, u has a maximum, say positive, $\max u = u_* > 0$

$\Rightarrow \Delta u|_{u_*} < 0$, but this is in contradiction with $f \cdot u_* > 0$.

Similarly, the argument works for $u_* < 0$.

- Consider now the nonlinear equation

$$\Delta u = f u^n \quad \text{on } \Omega, \qquad n > 1$$

and 2 different solutions $u_1 = u_2 + \epsilon$ to the boundary problem $u_1 = u_2$ on $\partial\Omega$

$$\Delta u_1 = f u_1^n \rightarrow \Delta(u_2 + \epsilon) = f(u_2 + \epsilon)^n,$$

expanding to first order in ϵ:

$$\underbrace{\Delta u_2 - f u_2^n}_{=0} + \underbrace{\Delta \epsilon - n f u_2^{n-1} \epsilon}_{\text{linear equation of the same type as above.}} = 0$$

$\Rightarrow$ the solution is unique ($\epsilon = 0$, linear analysis) only if $\boxed{n f \geq 0}$

i.e. exponent and coefficient have the same sign.

- let us apply these concepts to the Lichnerowicz equation.

Focus for simplicity to the case $K = 0$:

$$\tilde{D}_i \tilde{D}^i \psi - \frac{1}{8} \tilde{R} \psi + \frac{1}{8} \hat{A}^{ij} \hat{A}_{ij} \psi^{-7} + 2\pi E \psi^5 = 0$$

Set $\psi = \psi_0 + \epsilon$, and get:

$$\tilde{D}_i \tilde{D}^i \epsilon - a \epsilon = 0$$

with :

$$a = \frac{1}{8}\tilde{R} + \frac{7}{8}\hat{A}_{ij}\hat{A}^{ij}\psi_0^{-8} \underline{- 12\pi E\psi_0^{4}}$$

Uniqueness is given by $a>0$, but "a" has a negative term!

A way around to this problem is to rescale the energy term by defining :

$$\tilde{E} \equiv \psi^{s} E \quad \text{with } s>5 ,$$

this gives

$$a = \frac{1}{8}\tilde{R} + \frac{7}{8}\hat{A}_{ij}\hat{A}^{ij}\psi_0^{-8} + (s-5)2\pi\tilde{E}\,\psi_0^{4-s} \quad , \quad s>5$$

such that $a>0$.

Note: the solution of the Ham. constraint with $\tilde{E}$ requires iterative methods.

A useful choice for s is $s=8$. This choice allows one to implement the dominant energy condition :

$$E^2 \geq P^i P_i = \gamma_{ij} P^i P^j \qquad (\text{DEC})$$

directly on the conformally rescaled variables :

$$\tilde{E}^2 \geq \tilde{P}^i \hat{P}_i = \tilde{\gamma}_{ij}\tilde{P}^i\tilde{P}^j \qquad (\widetilde{\text{DEC}})$$

With natural (see later) choice :

$$P^i = \psi^{-10}\tilde{P}^i$$

one has that $(\widetilde{\text{DEC}}) \Rightarrow (\text{DEC})$ iif $s=8$ since :

$$\psi^{-2s} E^2 \geq \psi^4\psi^{-20} P^i P_i .$$

Let us finally repeat the analysis for the scaling $\alpha=-4$. The Ham. constraint for $k=0$ reads :

$$\tilde{D}_i\tilde{D}^i\psi - \frac{1}{8}\tilde{R}\psi + \frac{1}{8}\tilde{A}^{ij}\tilde{A}_{ij}\psi^5 + 2\pi\tilde{E}\psi^{-3} = 0$$

The linear perturbation equation has :

$$a = \frac{1}{8}\tilde{R} \underline{- \frac{5}{8}\hat{A}^{ij}\hat{A}_{ij}\psi_0^{4}} + 6\pi\tilde{E}\psi_0$$

the same argument as before shows that this scaling does ~~not~~ lead to a well-posed problem for the Ham. constraint.

Momentum constraint

It was already derived before, note that one could define:

$$\tilde{P}^i \equiv \psi^{10} P^i .$$

Alternative form with the $\alpha = -4$ scaling is:

$$\boxed{\bar{D}_j \bar{A}^{ij} + 6 \bar{A}^{ij} D_j \ln\psi - \frac{2}{3} \bar{D}^i K = 8\pi \psi^4 p^i}$$

Dynamical equation

In the conformal decomposition the dynamical eq.

$$\mathcal{L}_m K_{ij} = -D_i D_j \alpha + \alpha R_{ij} + \alpha K K_{ij} - 2 K_{ik} K^k{}_j \alpha + 4\pi\alpha\left[(S-E)\gamma_{ij} - 2 S_{ij}\right]$$

is split into 2 equations for

$$\mathcal{L}_m K = \dots$$

$$\mathcal{L}_m \bar{A}_{ij} = \dots$$

Let us derive the equation for the trace and leave as exercise the derivation of the $\mathcal{L}_m \bar{A}_{ij}$ which is rather lengthy. LHS of $\mathcal{L}_m K_{ij}$ is:

$$\mathcal{L}_m K_{ij} = \mathcal{L}_m \left(A_{ij} + \frac{1}{3} K \gamma_{ij}\right) = \mathcal{L}_m A_{ij} + \frac{1}{3} \mathcal{L}_m K \cdot \gamma_{ij} + \frac{1}{3} K \underbrace{\mathcal{L}_m \gamma_{ij}}_{-2\alpha K_{ij}} =$$

$$= \mathcal{L}_m A_{ij} + \frac{1}{3} \mathcal{L}_m K \cdot \gamma_{ij} - \frac{2}{3} \alpha K K_{ij}$$

Take the trace $\gamma^{ij} \cdot (\quad)$ of $\mathcal{L}_m K_{ij}$ ~~LHS~~

$$\gamma^{ij} \mathcal{L}_m A_{ij} + \frac{3}{3} \mathcal{L}_m K - \frac{2}{3} \alpha K^2 = \underbrace{\mathcal{L}_m \gamma^{ij} A_{ij}}_{=0} - A_{ij} \underbrace{\mathcal{L}_m \gamma^{ij}}_{= 2\alpha K^{ij}} + \mathcal{L}_m K - \frac{2}{3} \alpha K^2$$

Take the trace of $\mathcal{L}_m K_{ij}$ eq is then:

$$-A_{ij}\,2\alpha K^{ij} + \mathcal{L}_m K - \frac{2}{3}\alpha K^2 = -D_iD^i\alpha + \alpha R + \alpha K^2 - 2\alpha K_{ij}K^{ij} + 4\pi\alpha(S-3E)$$

$$-2\alpha K^{ij}\underbrace{\left(A_{ij} + \frac{1}{3}K\gamma_{ij} - K_{ij}\right)}_{=0} + \mathcal{L}_m K = -D_iD^i\alpha + \underbrace{\alpha R + \alpha K^2} + 4\pi\alpha(S-3E)$$

$$\alpha(R+K^2) = \alpha[16\pi(E + K_{ij}K^{ij})]$$

Ham. constraint

$$\boxed{\mathcal{L}_m K = -D_iD^i\alpha + \alpha\left[4\pi(E+S) + K_{ij}K^{ij}\right]}$$

Newtonian limit:

- $\alpha = 1 + \Phi^{*}$, $|\Phi| \ll 1$
- $\beta = 0$
- $\gamma_{ij} = (1-2\Phi)f_{ij}$
- $K_{ij} = 0$
- $|S^i{}_j| \ll E$
- $E \approx \rho$, fluid-rest-frame energy density
- $D_i \to \mathcal{D}_i$

the dynamical equation reduces to

$$\boxed{0 = -\mathcal{D}^i\mathcal{D}_i\Phi - 4\pi\rho}$$

$(*\ \alpha = \sqrt{1+2\Phi} \approx 1 + \Phi)$

To express this in the conformal variables, consider the generic relation relating the divergence of vector field

$$D_iV^i = \psi^{-6}\tilde{D}_i(\psi^6 V^i) \qquad \forall V^i$$

Easily verified in Cartesian coordinates using the formula:

$$D_iV^i = \frac{1}{\sqrt{\gamma}}\partial_i(\sqrt{\gamma}V^i) = \psi^{-6}\frac{1}{\sqrt{f}}\partial_i(\psi^6\sqrt{f}V^i) = \ldots = \psi^{-6}\tilde{D}_i(\psi^6V^i)$$

$$\sqrt{\gamma} = \psi^6\sqrt{f}$$

So:

$$D_iD^i\alpha = \psi^{-6}\tilde{D}_i(\psi^6 D^i\alpha) = \psi^{-6}\tilde{D}_i[\psi^6(\gamma^{ij}D_j\alpha)] = \psi^{-6}\tilde{D}_i[\psi^6(\psi^{-4}f^{ij})\tilde{D}_j\alpha)] =$$

$$= \psi^{-6}\tilde{D}_i(\psi^2\tilde{D}^i\alpha) = \psi^{-4}\tilde{D}_i\tilde{D}^i\alpha + 2\psi^{-5}\tilde{D}_i\psi\tilde{D}^i\alpha =$$

$$= \psi^{-4}(\tilde{D}_i\tilde{D}^i\alpha + 2\tilde{D}_i\ln\psi\,\tilde{D}^i\alpha)$$

$$K_{ij}K^{ij} = (A_{ij} + \frac{1}{3}K\gamma_{ij})(A^{ij} + \frac{1}{3}K\gamma^{ij}) = A_{ij}A^{ij} + \frac{1}{3}K^2 = \bar{A}_{ij}\bar{A}^{ij} + \frac{1}{3}K^2$$

leading to :

$$\boxed{\mathcal{L}_m K = -\psi^{-4}(\tilde{D}_i\tilde{D}^i\alpha + 2\tilde{D}_i \ln\psi \tilde{D}^i\alpha) + \alpha\left[4\pi(E+S) + \bar{A}_{ij}\bar{A}^{ij} + \frac{K^2}{3}\right]}$$

Similar but lenghty calcultions lead to :

$$\begin{aligned}
\mathcal{L}_m \bar{A}_{ij} = & -\frac{2}{3}\tilde{D}_k\beta^k \bar{A}_{ij} + \alpha\left[K\bar{A}_{ij} - 2\tilde{\gamma}^{kl}\bar{A}_{ik}\bar{A}_{jl} - 8\pi(\psi^{-4}S_{ij} - \frac{1}{3}S\tilde{\gamma}_{ij})\right] \\
& + \psi^{-4}\Big\{ -\tilde{D}_i\tilde{D}_j\alpha + 2\tilde{D}_i\ln\psi\tilde{D}_j\alpha + 2\tilde{D}_j\ln\psi\tilde{D}_i\alpha + \frac{1}{3}(\tilde{D}_k\tilde{D}^k\alpha + \\
& - 4\tilde{D}_k\ln\psi\tilde{D}^k\alpha)\tilde{\gamma}_{ij} + \alpha\Big[\tilde{R}_{ij} - \frac{1}{3}\tilde{R}\tilde{\gamma}_{ij} - 2\tilde{D}_i\tilde{D}_j\ln\psi + \\
& + 4\tilde{D}_i\ln\psi\tilde{D}_j\psi + \frac{2}{3}(\tilde{D}_k\tilde{D}^k\ln\psi - 2\tilde{D}_k\ln\psi\tilde{D}^k\ln\psi)\tilde{\gamma}_{ij}\Big]\Big\}
\end{aligned}$$

ISENBERG – WILSON – MATHEWS (IWM) APPROXIMATION OF GR

Under the hypotesis :

i) $\tilde{\gamma}_{ij} = f_{ij}$ CONFORMALLY FLAT SPACETIME *

ii) $K = 0$ MAXIMAL SLICING CONDITION

The conformal 3+1 GR equations simplify to :

$$\text{(IWM)} \quad \begin{cases}
\Delta\alpha + 2\mathcal{D}_i\ln\psi\mathcal{D}^i\alpha = \alpha\left[4\pi(E+S) + \bar{A}_{ij}\bar{A}^{ij}\right] \\
\Delta\psi + \psi^5\left(\frac{1}{8}\bar{A}_{ij}\bar{A}^{ij} + 2\pi E\right) = 0 \\
\Delta\beta^i + \frac{1}{3}\mathcal{D}^i\mathcal{D}_j\beta^j + 2\bar{A}^{ij}\left(6\alpha\mathcal{D}_j\ln\psi - \mathcal{D}_j\alpha\right) = 16\pi\alpha\psi^4 p^i
\end{cases}$$

where, ~~[scribbled]~~ as a consequence of i) $\Rightarrow$ $\left.\begin{array}{l}\bullet\ D_i \longrightarrow \mathcal{D}_i \text{ flat derivative} \\ \bullet\ \tilde{R} = 0\end{array}\right\}$ *

- Δ is the flat Laplacian
- $\bar{A}_{ij}$ must be considered a function of α and β^i given by the relation:

$$2\alpha \bar{A}_{ij} = f_{kj} \mathcal{D}_i \beta^k + f_{ik} \mathcal{D}_j \beta^k - \frac{2}{3} \mathcal{D}_k \beta^k f_{ij}$$

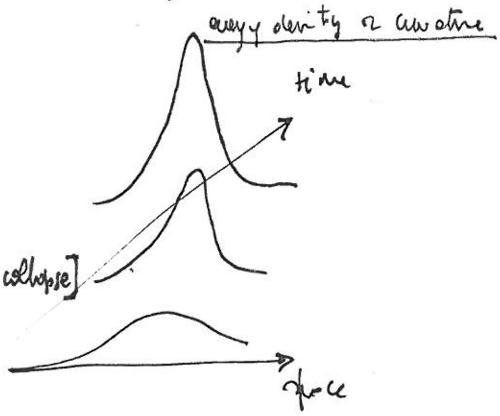
Properties of the IWM system:

- 3 elliptic equations for α, Ψ, β^i
- Key approximation: conformally flat $\Rightarrow$ no GW content (cotton tensor $\equiv 0$)
- $K=0$ is essentially a coordinate choice (see later) $\Rightarrow$ eq. for α, β^i
- Exact for spherically symmetric spacetimes, e.g. Schwarzschild
- Correct to 1-PostNewtonian order, expansion of GR

REMARK ON GEODESIC AND MAXIMAL SLICING

Consider <u>geodesic slicing</u>: $\alpha \equiv 1$

- Worldlines of Eulerian observers are GEODESICS, $a_a = D_a \ln\alpha \equiv 0$ (acceleration)
- Proper time of Eulerian observers = coordinate time t
- $a_a = 0 \Rightarrow$ free falling; due to the attractive nature of gravity, observers can focus. Singularities form during collapse



The $\mathrm{Tr} K_{ij}$ evolution equation gives

$$\partial_t K = \overset{\circ}{\hat{K}}_{ij} K^{ij} + 4\pi(E+S) \geq 0 \Rightarrow K \text{ grows [e.g. grav. collapse]}$$

the coord. singularity can form because, according to the equation,

$$\partial_t \ln\sqrt{\gamma} = -K$$

the coordinate volume element of these observers goes to 0 while $K \to +\infty$.

Consider maximal slicing : $K=0$

The $\mathcal{L}_m K$ equation reduces to: $\Delta \alpha = \alpha K_{ij} K^{ij}$

and one has 4-acceleration of the Eulerian observers $a_a = D_a \ln \alpha \neq 0$

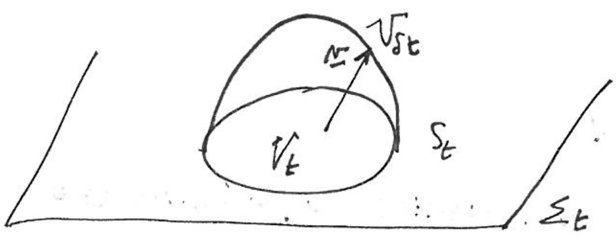
Moreover, the slicing condition gives

$$\boxed{0 = K = -D_a n^a}$$

"incompressible condition"

which $\underbrace{\cancel{says}}_{\text{suggests}}$ that Eulerian observers are accelerated in such a way to prevent them to focus and converge.

Maximal slicing $\Rightarrow$ hypersurfaces with maximal volume



S_t : closed curve on Σ_t
V_t : ~~volume~~ region inside S_t.

V = volume inside S_t = $\int_{V_t} \sqrt{\gamma}\, d^3x$

let us now deform V_t in such a way:

- $\underline{v}$ vector field that generate a infinitesimal displacement inside S_t
  i.e. $\underline{v}|_S = 0$

  (Note: $V_{\delta t}$ is not on a foliation $\Sigma_{t+\delta t}$; since the latter would intersect Σ_t ...)

- use adapted coordinates (t, x^i) such that S_t ~~is~~ time independent and remain on fixed coordinate values : $S_t(t, x^i) = S_t(x^i)$.

~~compute~~ $\Rightarrow$ $\underline{v} = \delta t\, \underline{\partial}_t = \delta t\, (\alpha \underline{n} + \underline{\beta})$

$\underline{v}|_S = 0 \Rightarrow \alpha|_S = 0 \wedge \underline{\beta}|_S = 0$

Compute the volume variation:

$$\frac{dV}{dt} = \int_{V_t} \frac{\partial \sqrt{\gamma}}{\partial t}\, d^3x$$ since we are using adapted coordinates

Now calculate the $\partial_t \sqrt{\det \gamma_{ij}}$:

$$\left.\begin{aligned} \gamma^{ij}\partial_t \gamma_{ij} &= -2\alpha K + 2 D_i\beta^i \\ &= \partial_t \ln\gamma = \frac{2}{\sqrt{\gamma}}\partial_t\sqrt{\gamma} \end{aligned}\right\} \quad \partial_t\sqrt{\gamma} = \frac{1}{\cancel{2}}\sqrt{\gamma}\left(-\cancel{2}\alpha K + \cancel{2} D_i\beta^i\right)$$

thus :

$$\frac{dV}{dt} = \int_{V_t} \sqrt{\gamma}\left(-\alpha K + D_i\beta^i\right) d^3x = -\int_{V_t} \sqrt{\gamma}\,\alpha K\, d^3x + \underbrace{\oint_{S_t} \beta^i s_i \sqrt{q}\, d^2y}_{=0 \;\Leftarrow\; \beta|_S = 0}$$

$K = 0 \Rightarrow V$ is extremal (if the boundary of V_t remains S_t)