GAUGE CONDITIONS AND SINGULARITIES

SLICING CONDITIONS

Harmonic slicing

Let us consider the slicing condition given by the harmonic gauge: $\Box x^\alpha = 0$

$\alpha = 0 \rightarrow x^0 =: t \rightarrow \quad \Box t = 0$

$$(-g)^{-1/2} \partial_\mu \left[(-g)^{1/2} g^{\mu\nu} \underbrace{\partial_\nu t}_{\delta_\nu{}^0} \right] = 0$$

$$0 = (-g)^{-1/2} \partial_\mu \left[(-g)^{1/2} g^{\mu 0} \right] = (-g)^{-1/2} \partial_\mu \left(\alpha \sqrt{\gamma}\, g^{\mu 0} \right) \Rightarrow$$

$$\partial_t \left(\alpha \sqrt{\gamma}\, g^{00} \right) + \partial_i \left(\alpha \sqrt{\gamma}\, g^{i0} \right) = 0$$

$$g^{00} = \alpha^{-2} \;;\quad g^{i0} = +\alpha^{-2} \beta^i \Rightarrow \partial_t \left(\sqrt{\gamma}\, \alpha^{-1} \right) + \partial_i \left(\sqrt{\gamma}\, \alpha \beta^i \right) = 0$$

$$\partial_t \alpha - \beta^i \partial_i \alpha - \alpha \underbrace{\left[\frac{1}{\sqrt{\gamma}} \partial_t \sqrt{\gamma} \underbrace{- \frac{1}{\sqrt{\gamma}} \partial_i \left(\sqrt{\gamma} \beta^i \right)}_{- D_i \beta^i} \right]}_{- \alpha K} = 0$$

Hence:

$$\boxed{d_m \alpha = \left(\partial_t - \mathcal{L}_\beta \right) \alpha = -\alpha^2 K}$$

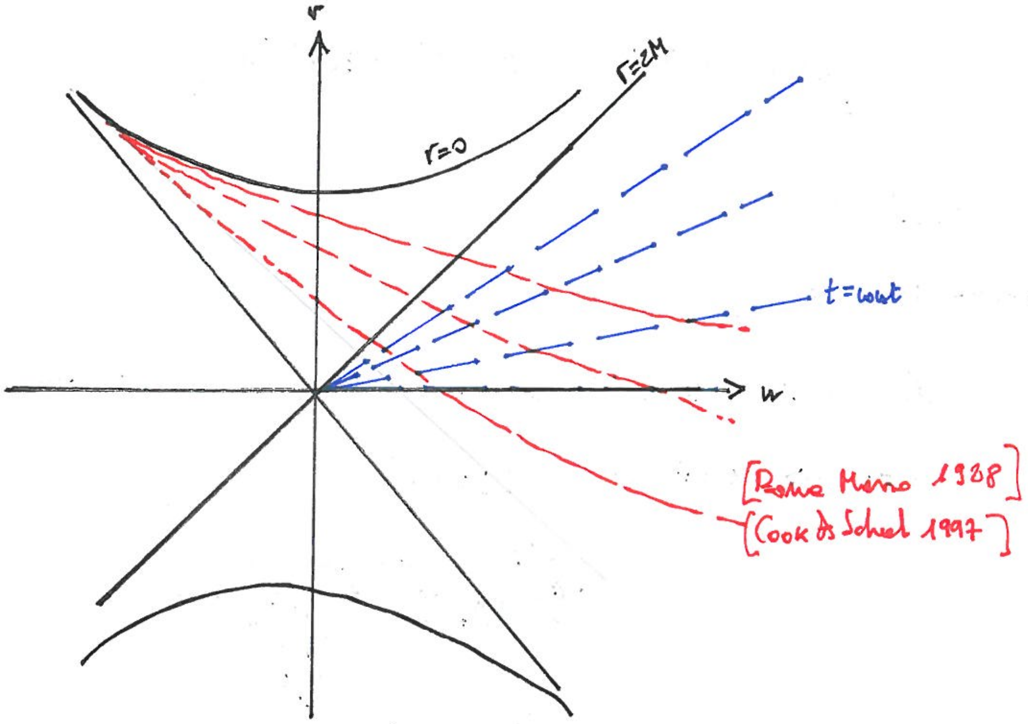
- Evolution equation for α
- Compare to maximal slicing: $D_i D^i \alpha = K_{ij} K^{ij}$

  "Similar" $\Rightarrow$ singularity avoidance property (but less strong than maximal slicing)
- Schwarzschild in Schwarzschild coordinate: $\partial_t \alpha = 0$, $\beta = 0$, $K = 0 \Rightarrow$

  $t = \text{const}$ are harmonic slices!

  Note: They do not penetrate the horizon (singular at $r = 2M$)

Kruskal coords :



- [Bona & Masso 1988] [Cook & Scheel 1997] found a slicing of Kerr-Newman metric that is fully harmonic.
  - Harmonic slicing of Schwarzschild : extend to $r=0$ (and penetrate the horizon)
  - Harmonic slicing of Kerr : $\alpha=0$ at $r=r_-$, and $\alpha<0$ for $r<r_-$
  - Harmonic spatial gauge : singularity at $r=M$

  Note : $r_- < M < r$, might be problematic for highly spinning BH $a\to 0$ $r_+ \simeq M$

Bona-Masso slicing

General form of evolution equation for α :

$$\boxed{\mathcal{L}_m \alpha = -\alpha^2 K f(\alpha)}$$

$$f(\alpha) = \begin{cases} 0 & \text{Geodesic slicing} \\ 1 & \text{Harmonic slicing} \\ \frac{2}{\alpha} & \text{"1+log"} \quad \text{[Bernstein 93, Anninos 95]} \end{cases}$$

$$(\partial_t - \mathcal{L}_\beta)\alpha = -2\alpha K = \partial_t \ln\gamma - 2 D_i\beta^i$$

$$\beta^i \equiv 0 \Rightarrow \partial_t \alpha = \partial_t \ln\gamma \Rightarrow \quad \alpha = 1 + \ln\gamma$$

Comments :

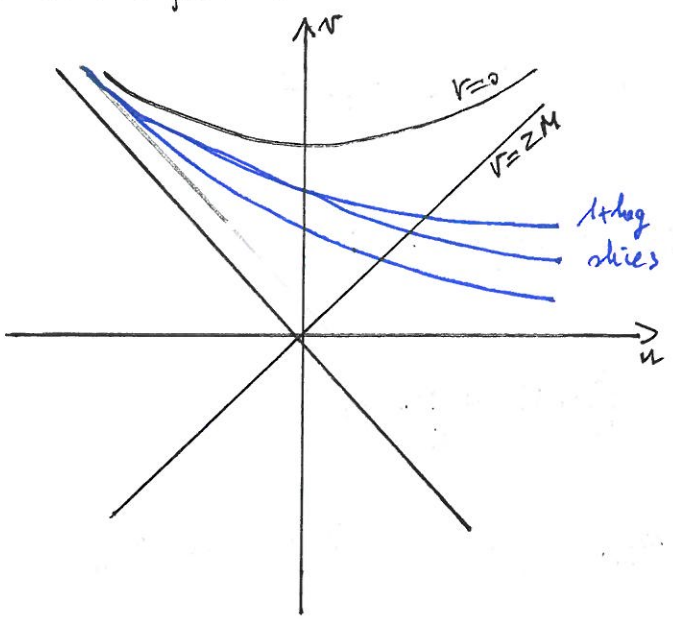
- Often called "1+log" even if $\beta^i \neq 0$
- Singularity avoidance property ~ max. slicing (and stronger than harmonic slicing)
- REM : Max. slicing refers to a hypersurface Σ

  Geodesic, harmonic, 1+log, ... refer to a foliation Σ_t

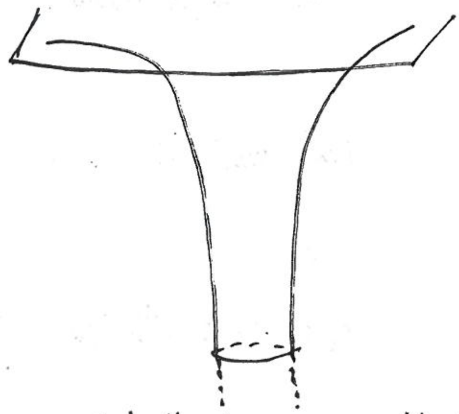
Stationary Schwarzschild slices in 1+log

[Hannam + 2006, 2008; Brown 2006] (and refs. therein)

Kruskal diagram :



Embedding diagram :



infinite-long 3D cylinder

"trumpet"

$r = r_0 \cong 1.3\ldots$

$\alpha_{\text{horizon}} = \alpha(2M) \simeq 0.3\ldots$

- Maximal slicing : $\alpha^2 = 1 - \frac{2M}{r} + \frac{C^2}{r^4}$
- 1+log : $\alpha^2 = 1 - \frac{2M}{r} + \frac{C^2}{r^4} e^{\alpha}$

SINGULARITY AVOIDANCE PROPERTIES

$$\left.\begin{aligned} \partial_t \ln\sqrt{\gamma} &= -\alpha K \\ d\tau &= \alpha\, dt \end{aligned}\right\} \quad \partial_\tau \ln\sqrt{\gamma} = -K \qquad (*)$$

Gravitational collapse:

$$\partial_t K \simeq \alpha\left(R + K^2\right) + \text{matter terms}$$

if $K^2 \gg R$, matter terms $\rightarrow \partial_\tau K \simeq K^2 \rightarrow K \sim \dfrac{1}{\tau - \tau_s}$

A singularity $K \rightarrow +\infty$ forms at $\tau \sim \tau_s$.

$(*) \Rightarrow$ proper volume $\underset{\tau\to\tau_s}{\sqrt{\gamma} \rightarrow 0}$ as $\sqrt{\gamma} \sim (\tau_s - \tau)$

What must the lapse do to avoid the singularity?

$$\left.\begin{aligned} &(*) \\ &\partial_t \alpha = -2\alpha^2 K f(\alpha) \end{aligned}\right\} \rightarrow \quad \frac{2}{\sqrt{\gamma}}\partial_t\sqrt{\gamma} = \frac{2}{\alpha f(\alpha)}\partial_t \alpha .$$

let us suppress spatial dependency $\partial_t \rightarrow \dfrac{d}{dt}$

$$\frac{d\sqrt{\gamma}}{\sqrt{\gamma}} = \frac{d\alpha}{\alpha f(\alpha)}$$

$$\boxed{\sqrt{\gamma} \sim \exp\int \frac{d\alpha}{\alpha f(\alpha)}} \qquad \odot$$

The time to reach the singularity is:

$$\Delta t = \int_0^{\tau_s} \frac{d\tau}{\alpha} ,$$

Hence, if $\sqrt{\gamma} \rightarrow 0$ and $\alpha \rightarrow$ finite $\neq 0$, then the singularity is reached in finite simulation time and ~~the~~ cannot be avoided.

For $f(\alpha)>0$, equation $\odot$ shows that $\sqrt{\gamma} \not\to 0$ if $\alpha \to$ finite $\neq 0$, in other terms the Bona-Masso slicing with $f(\alpha)>0$ implies that:

$$\sqrt{\gamma} \to 0 \quad \Rightarrow \quad \alpha \to 0 \qquad \underline{\text{COLLAPSE OF THE LAPSE}}.$$

However, the lapse needs to collapse sufficiently fast: the time slices must stop advancing in time <u>before</u> the singularity is reached!

Let us assume $f(\alpha) = c\,\alpha^n$:

$$\odot \Rightarrow \begin{cases} \sqrt{\gamma} \sim \exp\left(-\dfrac{1}{n c \alpha^n}\right), & n \gtrsim 0 \\ \sqrt{\gamma} \sim \alpha^{1/c}, & n = 0 \end{cases}$$

Case $\boxed{n<0}$: $\sqrt{\gamma}$ is <u>finite</u> for $\alpha \to 0$, strong singularity avoidance

Example: "1+log"

Cases $\boxed{n \geq 0}$:

$$\Delta t = \int_0^{\tau_s} \frac{d\tau}{\alpha} = \int_{\alpha_0}^{0} \frac{d\tau}{d\alpha} \frac{d\alpha}{\alpha}$$

$\dfrac{d\tau}{d\alpha} \neq 0$ as $\alpha \to 0 \Rightarrow \Delta t \to +\infty$

$\dfrac{d\tau}{d\alpha} \approx \alpha^p$ $p>0$ as $\alpha \to 0$ (or faster)

$\Rightarrow \Delta t$ is finite and singularity is not avoided.

Study $\dfrac{d\tau}{d\alpha}$:

$$[\alpha f(\alpha)]^{-1} \frac{d\tau}{d\alpha} = \frac{d \ln\sqrt{\gamma}}{d\tau} = -\frac{1}{\tau_s - \tau} \Rightarrow \tau = \tau_s - \exp\int \frac{d\alpha}{\alpha f(\alpha)}$$

for $f(\alpha) = c\alpha^n$:

$$\tau = \tau_s - \begin{cases} \alpha^{1/c}, & n=0 \\ \exp\left(-\dfrac{1}{nc\alpha^n}\right), & n>0 \end{cases}$$

$$\Rightarrow \frac{d\tau}{d\alpha} = \begin{cases} -\dfrac{1}{c}\,\alpha^{1/c - 1}, & n=0 \\ -\dfrac{1}{c\alpha^{n+1}} \exp\left(-\dfrac{1}{nc\alpha^n}\right), & n>0 \end{cases}$$

$\underline{n=0}$ $\quad c < 1 \rightarrow \frac{d\tau}{d\alpha} \sim \alpha^p \rightarrow$ singularity reached at finite Δt

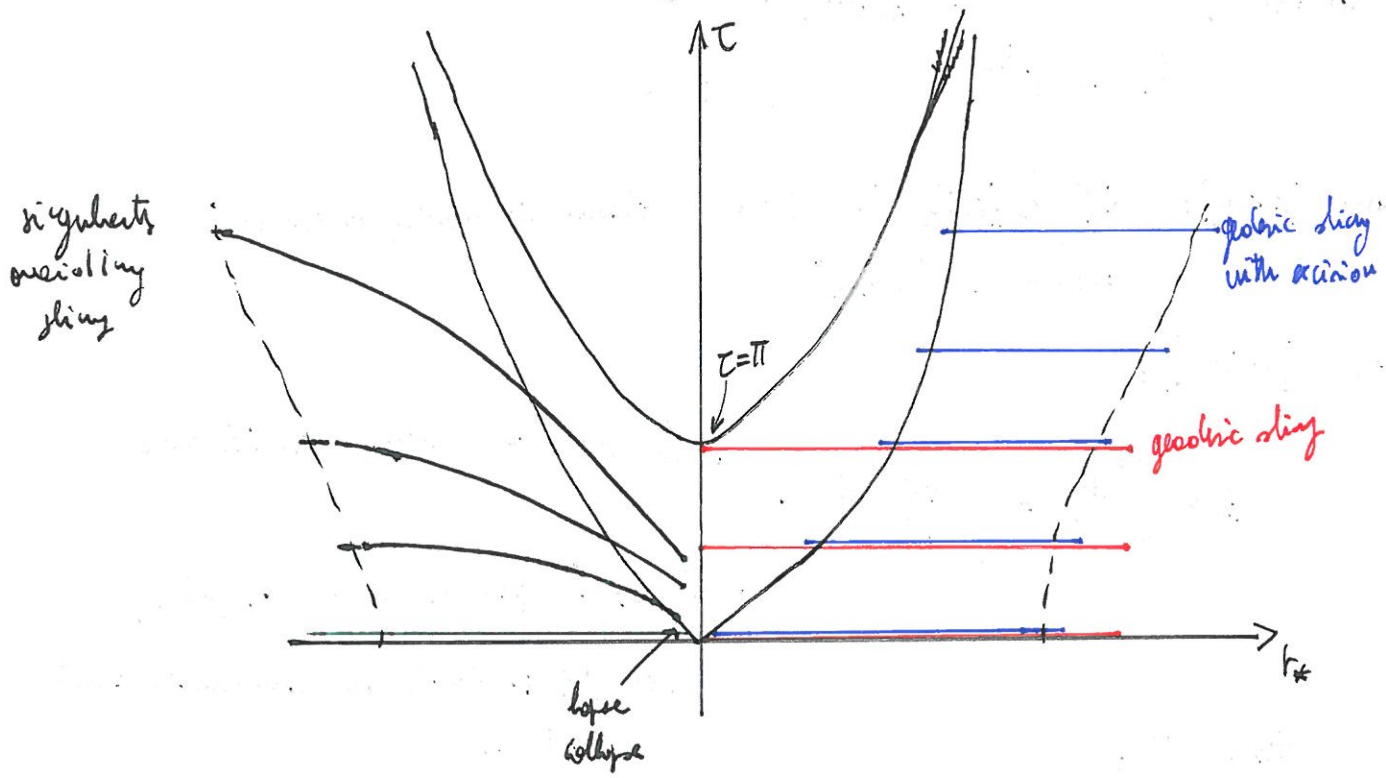
$c \geq 1 \rightarrow$ singularity reached at $\Delta t \rightarrow +\infty$

"marginal singularity avoidance"

$\underline{n>0}$ $\quad$ singularity is <u>not</u> avoided

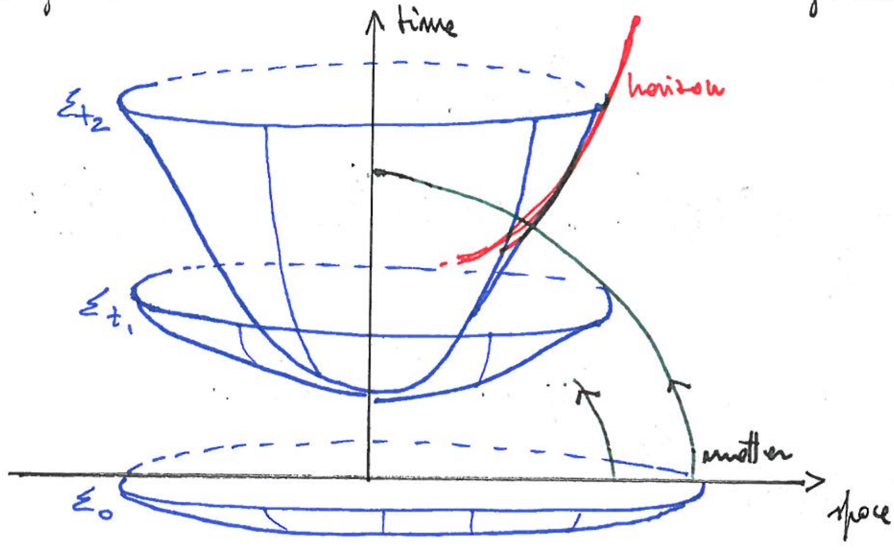
Note: harmonic slicing has $n=1$...

<u>Example</u>: Schwarzschild in Novikov coordinates (geodesic slicing)



SPATIAL GAUGE : FROM MINIMAL DISTORTION TO GAMMA - DRIVERS

let us consider a gravitational collapse simulation with a singularity avoiding slicing:



one expects that the slices Σ_t will become spatially very distorted with many grid points necessary to resolve large gradients. In simulations one also observe that the <u>area</u> of the horizon keep on growing in coordinate size if $\beta^i \simeq 0$.

These considerations lead us to require a spatial gauge that "minimize distortions" in space. In order to construct this gauges it is useful to remember that the <u>extrinsic curvature</u> K_{ab} essentially tells us how the hypersurface is embedded in the manifold, and in particular the <u>trace</u> $K = g^{ab} K_{ab}$ tells how the volume in Σ_t change. let us define:

$$Q_{ij} \equiv \partial_t \gamma_{ij} = -2\alpha K_{ij} + \mathcal{L}_\beta \gamma_{ij}$$

and the

<u>DISTORTION TENSOR</u> $\stackrel{\text{def.}}{=}$ $\Sigma_{ij} \equiv Q_{ij} - \frac{1}{3} Q \gamma_{ij} =$

$$= -2\alpha A_{ij} + (L\beta)_{ij} =$$

$$= \psi^4 \dot{\tilde{\gamma}}_{ij}$$

To find an equation for the shift we <u>minimize</u> the functional:

$$I[\beta^i] \equiv \int_{\Sigma_t} \Sigma_{ij}\Sigma^{ij}\sqrt{\gamma}\,d^3x$$

by keeping all the other fields <u>fixed</u>.

$$I[\beta^i] = \int_{\Sigma_t}\left[4\alpha^2 A_{ij}A^{ij} + 4\alpha A_{ij}(L\beta)^{ij} + (L\beta)_{ij}(L\beta)^{ij}\right]\sqrt{\gamma}\,d^3x$$

$$\delta I[\beta^i]\Big|_{\text{fix } \gamma_{ij}, K_{ij}, \alpha} = 2\int_{\Sigma_t}\Sigma_{ij}(L\delta\beta)^{ij}\sqrt{\gamma}\,d^3x =$$

$$= 2\int_{\Sigma_t}\Sigma_{ij}\left(D^i\delta\beta^j + D^j\delta\beta^i - \frac{2}{3}D_k\delta\beta^k\gamma^{ij}\right)\sqrt{\gamma}\,d^3x \underset{\uparrow}{=}$$

$\Sigma_{ij}\gamma^{ij}=0$
(Σ_{ij} is traceless)

$$= 2\int_{\Sigma_t}\Sigma_{ij}\left(D^i\delta\beta^j + D^j\delta\beta^i\right)\sqrt{\gamma}\,d^3x \underset{\uparrow}{=}$$

Σ_{ij} is symmetric

$$= 4\int_{\Sigma_t}\Sigma_{ij}D^i\delta\beta^j\sqrt{\gamma}\,d^3x =$$

$$= 4\int_{\Sigma_t}\left[D^i(\Sigma_{ij}\delta\beta^j)\sqrt{\gamma}\,d^3x - \delta\beta^j D^i\Sigma_{ij}\right]\sqrt{\gamma}\,d^3x =$$

$$= \underbrace{4\oint_{\partial\Sigma_t}\Sigma_{ij}\delta\beta^j\sqrt{h}\,d^2x}_{=0 \text{ if } \delta\beta^j|_{\partial\Sigma_t}=0} - \int\delta\beta^j D^i\Sigma_{ij}\sqrt{\gamma}\,d^3x$$

$$= 0 \iff \boxed{D^j\Sigma_{ij}=0} \iff \boxed{\Delta_L\beta^i = 2D_j(\alpha A^{ij})}$$

Comments :

- Alternative derivation of the minimal distortion equation can be done by decomposing Σ_{ij} into a TT part and a longitudinal part (similarly to what done for the cTT equations). Remembering that Σ_{ij} is traceless, one has

$$\Sigma_{ij} = \Sigma^{TT}_{ij} + \Sigma^{L}_{ij}$$

$$D^i \Sigma_{ij} = D^i \Sigma^{TT}_{ij} + D^i \Sigma^L_{ij} = D^i \Sigma^L_{ij} \overset{\text{require}}{=} 0$$

- "Approximate" minimal distortion gauges have been introduced by [Nakamura 1994] and [Shibata 2007]. Following from :

$$D^j \Sigma_{ij} = D^j(\psi^4 \dot{\tilde{\gamma}}_{ij}) = \tilde{D}^j(\psi^6 \dot{\tilde{\gamma}}_{ij}) = 0$$

one could take :

$$\tilde{D}^j(\psi^6 \dot{\tilde{\gamma}}_{ij}) \approx D^j(\dot{\tilde{\gamma}}_{ij}) = \Delta\beta^j + \ldots = 0$$

- the <u>Dirac gauge</u>

$$\tilde{\Gamma}^i = D_j(\tilde{\gamma}^{ij}) = 0 \quad \Rightarrow \quad \Delta\beta^j + \ldots = 0$$

is very similar to a minimal distortion gauge.
It is used in <u>fully constrained</u> formulations of GR [Bonazzola+ 2003].
In these formulations the number of elliptic equations is maximized, leaving with only 2 equations (for the GW degrees of freedom) of hyperbolic type.

- <u>Gamma-freezing</u>

In the context of BSSN (free evolutions), one can observe that :

$$D_j \dot{\tilde{\gamma}}^{ij} = D_j(\partial_t \tilde{\gamma}^{ij}) = \partial_t(D_j \tilde{\gamma}^{ij}) = \partial_t\left[\tilde{\gamma}^{jk}\left(\bar{\Gamma}^i_{jk} - \tilde{\Gamma}^i_{jk}\right)\right] = \partial_t \tilde{\Gamma}^i$$

So one derives the condition :

$$\boxed{\partial_t \tilde{\Gamma}^i = 0} \iff \boxed{\tilde{\gamma}^{jk} D_j D_k \beta^i + \ldots = 0} \qquad (\underline{\text{Elliptic}})$$

— <u>Gauge-drivers</u> [Alcubierre, Brügmann + 2003]

Turn the Γ-freezing elliptic equation

- $\Delta \beta^i + \dots = 0$ (Elliptic)

into :

- $\partial_t \beta^i = k\, \partial_t \tilde{\Gamma}^i = k\, \Delta \beta^i + \dots$ $k>0$ (parabolic)

- $\partial_t^2 \beta^i = k\, \partial_t \tilde{\Gamma}^i - (\eta - \partial_t \ln k)\, \partial_t \beta^i$ $k, \eta > 0$ (hyperbolic)

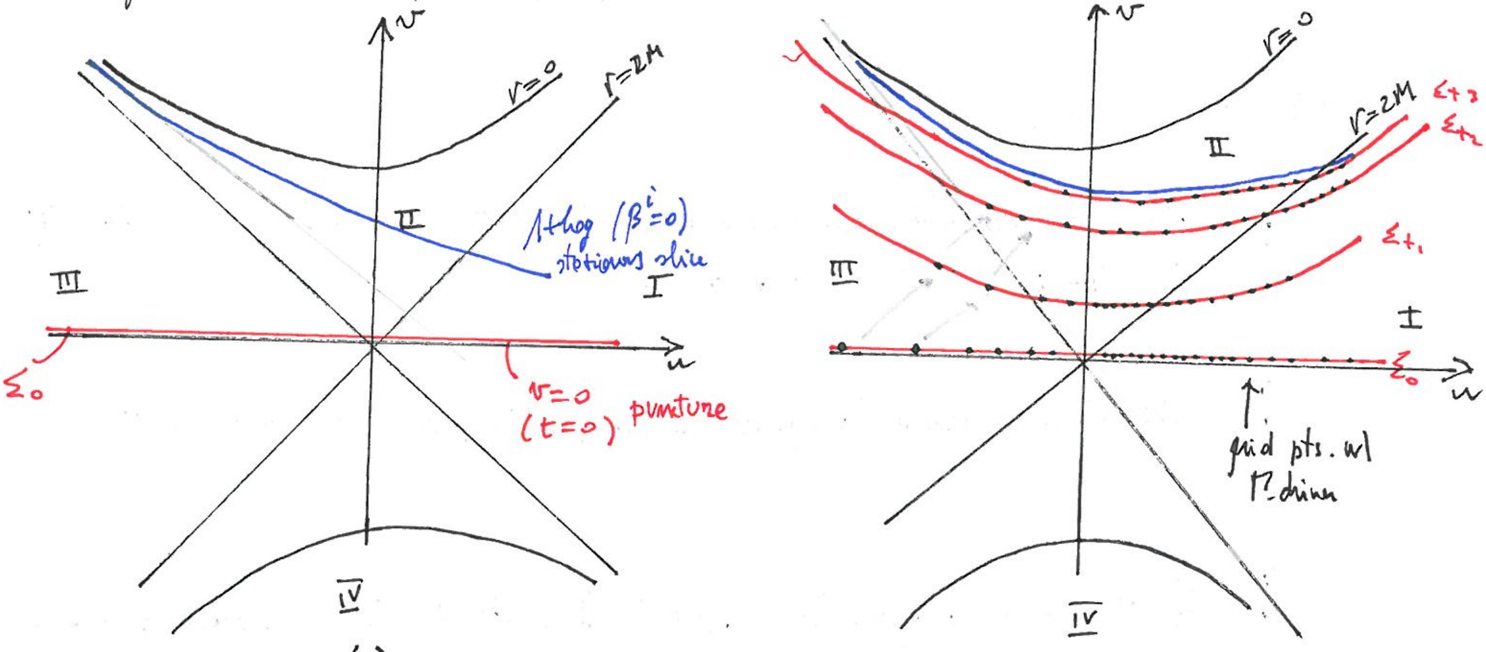
"Γ-driver"

Various versions : $\partial_t \mapsto \partial_t - \beta^i \partial_i$
fully first order
etc.

→ Hyperbolic Γ-drivers are employed for BBH puncture simulations and BNS.

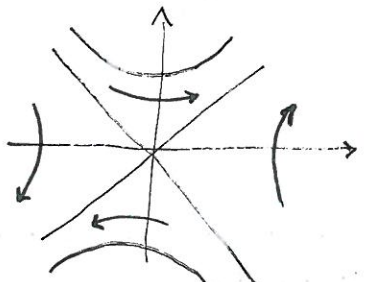
ROLE OF 1+log AND Γ-DRIVER IN BLACK-HOLE AND GRAVITATIONAL COLLAPSE SIMUL.

Consider <u>puncture black-hole initial data</u> for a single, nonspinning hole.
In the Kruskal diagram the initial slice Σ_0 is the $v=0$ slice connecting region I with region III:



(a) (b)

Σ_0 is, in general, not stationary. Cf. discussion on initial data with CTS.
The evolution of Σ_0 depends on the choice of lapse.
Any spacelike slice can be evolved into a time-independent slice if lapse and shift are chosen such that $t^a = \alpha n^a + \beta^a$ coincides with the timelike Killing vector field.
The directions of a Killing vector that is future pointing in region I are



and they imply that one cannot have a stationary foliation extending from I to III if <u>$\alpha > 0$ everywhere</u>.

Note also that the choice of the shift does <u>not</u> affect the slicing.
The shift determines how spatial coordinates are carried from one slice to the next.
In a simulation that employs a discretization, the shift tells how the distribution of grid points is carried from one slice to the next.

Main questions:

- How does Σ_0 (punctured black hole) evolve under 1+log slicing?
- Does Σ_0 ~~[illegible]~~ evolve to the trumpet slice?
- What is the role of the Γ-driver shift?

Consider first $\beta^i \equiv 0$.

- Σ_0 is symmetric under reflections about the v-axis, i.e. the metric γ_{ij} is invariant under $u \longleftrightarrow -u$
- the spatial geometry in region III is <u>identical</u> to the one in region I
- if one choose a symmetric $\alpha(t=0)$, then the <u>geometry must remain symmetric during the time evolution!</u>
- Σ_0 <u>cannot</u> evolve to the trumpet geometry.

Problem: numerical experiments indicate that the evolution of Σ_0
- reaches a quasi-stationary state
- the quasi-stationary state is very close to the trumpet

Consider now β^i evolved with Γ-driver.

- Numerically the stationary state is quickly reached and simulations are long-term stable. Consider a finite-difference simulation for clarity.

Initially grid points are <u>not</u> uniformly distributed in u since they are unif. distributed in r.

During the evolution β^r get large values close to the puncture $r \sim 0$ $\Rightarrow$ the Γ-driver drives points (coordinate points) from III to II.

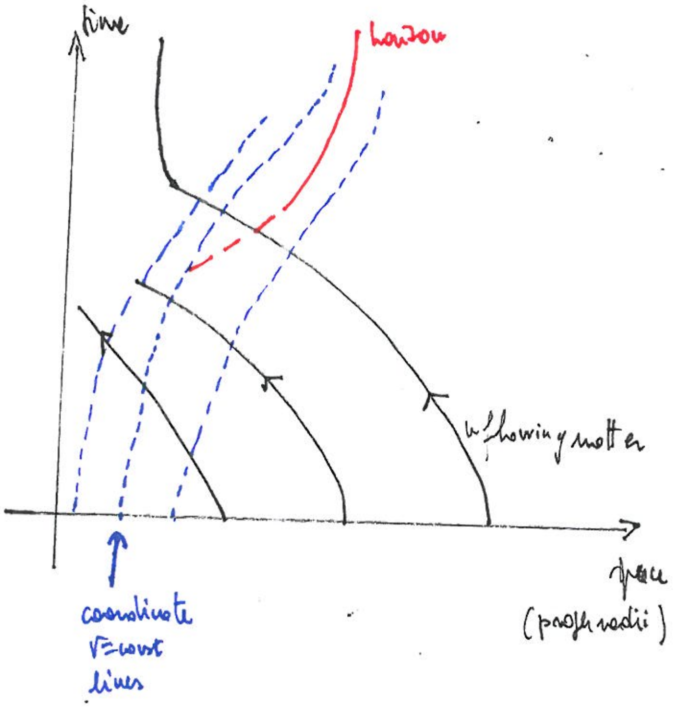
This is very fast (in fact <u>superlinear</u>) because

$$\partial_t \beta^i \sim \mu_S \tilde{\Gamma}^i \qquad \text{with} \quad \mu_S \sim \alpha^{-2}$$

- $\Sigma_\circ$ evolves to an approximately quasistationary state close (but not the same) to trumpet.
  An incredibly high resolution would be needed to continue to resolve region III.
  In fact this resolution can not be reached even with 4+1 codes.
  The points in region II and III can now evolve "close" to timelike Killing vector.

- In puncture evolution the spacetime inside the horizon is not resolved.
  The slice terminate somewhere in region II due to the lack of resolution
  this effect is realized by the shift evolution equation.
  Comparing to excision techniques that employ sophisticated algorithms, the puncture evolution is simpler and effective. Sometimes it is referred as natural excision.

Q: What happens in case of gravitational collapse?



Now the situation is different:
the initial slice does not extend from region I to III,
it is regular at $t=0$ due to the presence of matter.

However, the 1+log slicing works in a similar way:
when $\alpha \to 0$ ($\alpha \lesssim 0.3$) points well-inside the horizon (region II) are pushed to larger proper radii, effectively de-resolving region II.

Notes:

- the simulation reaches a quasi-stationary state very close to the trumpet geometry
- Rest-mass "falls-off" the grid

[Thierfelder+ 2011]