RELATIVISTIC HYDRODYNAMICS

Matter fields in GR are described by the stress-energy tensor T_{ab} that enters GR equations

$$G_{ab} = 8\pi T_{ab}$$

Bianchi identities implies the conservation equation (divergence form)

$$\boxed{\nabla_a T^{ab} = 0}$$ LOCAL!

3+1 DECOMPOSITION OF MATTER EQUATIONS FOR GENERIC STRESS-ENERGY TENSOR

The meaning of matter equations can be further understood by considering the 3+1 decomposition of T_{ab} and of $\nabla_a T^{ab} = 0$. Given the time unit normal vector n^a to the hypersurfaces Σ_t, one has the unique decomposition

$$T_{ab} = S_{ab} + n_a p_b + n_b p_a + E\, n_a n_b$$

where:

- S_{ab} ~~[illegible]~~ is the stress energy tensor projection on Σ_t
- p^a is the momentum density of the matter measured by Eulerian observers
- E is the energy matter density measured by Eulerian obs

Let us now consider projections along n^a and orthogonal of the $\nabla_a T^a{}_b = 0$:

- $n^b \nabla_a T^a{}_b = 0 \quad \rightarrow \quad \mathcal{L}_m E = -\alpha\left(D_i p^i - KE - K_{ij}S^{ij}\right) - 2p^i D_i \alpha$
- $\gamma^c{}_a \nabla_b T^b{}_c = 0 \quad \rightarrow \quad \mathcal{L}_m P_i = -\alpha D_j S^j{}_i - S_{ij} D^j \alpha + \alpha K P_i - E D_i \alpha$

these equations express energy and momentum conservation as observed by Eulerian frames. The Newtonian limit

$$\gamma_{ij} \mapsto f_{ij} \;;\; D_i \mapsto \mathcal{D}_i \;;\; K_{ij} \mapsto 0 \;;\; \alpha \mapsto \sqrt{1+2\Phi} \simeq 1+\Phi$$

$|S_{ij}| \ll E$: non relativistic matter $v^2 \ll c^2$

$$\partial_t E + D_i P^i = -P^j D_j \Phi$$
$$\partial_t P_i + D_j S^j_i = -E\, D_i \Phi$$

These are conservation laws with an additional source term given by the gravitational field.
Note that P^i (momentum density) in the first equation play the role of a energy flux: ...

PERFECT FLUIDS

In order to procede with the description of matter we must introduce a model.
Let us consider a perfect fluid i.e. a continuum medium characterised by:

- u^a 4-velocity field → velocity of fluid element at each point
- isotropic pressure in the fluid frame ~~[illegible]~~

Specifically:

$$T_{ab} = (e+p)\, u_a u_b + p\, g_{ab} = e\, u_a u_b + p\,(u_a u_b + g_{ab})$$

Note that in a local frame co-moving with fluids one has the components:

$$T_{\mu\nu} = \begin{pmatrix} e & & & \\ & p & & \\ & & p & \\ & & & p \end{pmatrix}.$$

Above:

- $e = T_{ab} u^a u^b$ is the energy density of the fluid in the fluid frame

  often one uses the notation $e = \rho(1+\epsilon)$
  in which
  ρ : rest-mass density
  ϵ : internal energy

  to understand this notation one needs to think about a fluid as the continuum limit of some particle species, say ~~[illegible]~~ baryons if $m_B^{(A)}$ is the mass of each baryon and $n^{(A)}$ is the number density of baryon of specie A, one has:

$$e c^2 = \underbrace{\sum_A m_B^{(A)} n^{(A)} c^2}_{\rho} + \underbrace{e_{int}\, c^2}_{\rho\epsilon}$$

Note that in <u>non-relativistic regime</u> ($v^2 \ll c^2$) the energy density is dominated by the rest-mass term:

$$e = \rho(1+\epsilon) \approx \rho \qquad \epsilon \ll 1$$

and similarly

$$e c^2 \ggg p .$$

$$\left.\begin{aligned}[e] &= M L^{-3}\\ [p] &= M L T^{-2} L^{-2}\end{aligned}\right\} (e+p) \mapsto \left(e + \frac{p}{c^2}\right)$$

- p is the pressure in the fluid frame. Usually one define the <u>specific enthalpy</u>

$$(e+p) = \rho h \quad \rightarrow \quad h \equiv 1 + \epsilon + \frac{p}{\rho}$$

in Newtonian/non-relativistic regime $h \mapsto 1$.

CONSERVATION OF PARTICLE NUMBERS

If one think of the fluid as made of many particles, e.g. a continuum approximation of a gas/plasma made of baryons, then one has to take into account the fundamental conservation laws of baryon number conservation, lepton number conservation, etc. these equations <u>must be added</u> to $\nabla_a T^a{}_b = 0$.

Let us consider a <u>simple fluid</u> made of a single species of baryon, a conservation law for the baryon number density is given by

$$\nabla_a (n u^a) = 0$$

that can also be written $\rho = m_B n \quad m_B = \text{const}$ as

$$\nabla_a (\rho u^a) = 0 .$$

Analogous equations can be added in more complicated situations.

EQUATIONS FOR RELATIVISTIC PERFECT FLUIDS

Let us consider the projection along u^a and perpendicular of the $\mathrm{div}\,\mathbf{T}=0$ equation.

<u>Projection along $\underline{u}$</u> :

$$0 = u_b \nabla_a T^{ab} = u_b \nabla_a \left(h\rho u^a u^b + P g^{ab} \right) =$$

$$= u_b \nabla_a (h\rho) u^a u^b + u_b h\rho \nabla_a (u^a u^b) + u_b \nabla_a (P g^{ab}) =$$

$$= - u^a \nabla_a (h\rho) \underbrace{u_b u^b}_{=-1} + h\rho \left(\underbrace{u_b u^b}_{=-1} \nabla_a u^a + \underbrace{u_b u^a \nabla_a u^b}_{=0 \,\leftarrow\, u^a \nabla_a (u_b u^b) = 0} \right) + u^a \nabla_a P =$$

$$= - u^a \nabla_a (h\rho) - h\rho \nabla_a u^a + u^a \nabla_a P$$

$$\boxed{- \nabla_a (h\rho u^a) + u^a \nabla_a P = 0}$$

using $h\rho = e + P$ we can write

$$0 = -\nabla_a \left[(e+P) u^a \right] + u^a \nabla_a P =$$

$$= - u^a \nabla_a (e + \cancel{P}) - (e+P) \nabla_a u^a + \cancel{u^a \nabla_a P} =$$

$$\Rightarrow \quad \boxed{u^a \nabla_a e = - (e+P) \nabla_a u^a = - \rho h \nabla_a u^a}$$

Using the conservation equation $\nabla_a (\rho u^a) = 0$ we can also write

$$u^a \nabla_a e = - \rho h \nabla_a u^a = + h u^a \nabla_a \rho$$

Projection $\perp$ to $\underline{u}$

$$\perp^a{}_b = (\delta^a{}_b + u^a u_b).$$

$\perp \text{div}\, T = 0 \rightarrow$ Relativistic Euler equation

$$\boxed{\rho h \underbrace{u^b \nabla_b u_a}_{} = -\nabla_a P - u_a u^c \nabla_c P}$$

acceleration $\Rightarrow$ "m" . "a" = "F"

Comments :

- $P = \text{const} \rightarrow \underline{a} = \nabla_{\underline{u}} \underline{u} \text{ "=" } u^b \nabla_b u_a = 0$

  geodesic equation!

Properties of relativistic perfect fluid equations

- Newton limit

$$ds^2 = -\left(1 + 2\frac{\phi}{c^2}\right)dt^2 + \left(1 - 2\frac{\phi}{c^2}\right)\delta_{ij}dx^i dx^j \quad ;$$

$$-1 = u^\alpha u^\beta g_{\alpha\beta} \Rightarrow u^\alpha = c^{-1}\frac{dx^\alpha}{d\tau} \simeq (u^0, v^i) \quad \text{with} \quad \left(u^0, u^0 \frac{v^i}{c}\right)$$

$d\tau$: fluid proper time

$$v^i \equiv \frac{dx^i}{dt}.$$

$$u^0 = 1 - \frac{\Phi}{c^2} + \frac{v_i v^i}{c^2} \quad ;$$

$$u^a \nabla_a \mapsto \partial_t + v^i \partial_i \quad ;$$

$$e \simeq \rho \quad ; \quad h \to 1$$

$$\boxed{\begin{aligned} &\partial_t \rho + v^i \partial_i \rho = 0 \\ &\partial_t v^i + v^k \partial_k v^i + \frac{1}{\rho}\partial_i P + \partial_i \phi = 0 \end{aligned}}$$

- Equation of state

variables : u^b, e, p | u^b, ρ, ϵ, p

equations : Euler, $\underline{u}$-projection, ? | Euler, u-projection, baryon conservation, ?

the system must be close by an equation of type

$$\boxed{p = \tilde{p}(\rho, \epsilon)}$$

Note this is relation in the fluid's comoving frame.

this relation is guaranteed by the 1st law of thermodynamics :

$$dU = T dS - p dV + \mu dN$$

internal energy $U = \epsilon V$; entropy $S = sV$; volume ; number of particles $N = nV$

temperature $T = \left.\frac{\partial U}{\partial S}\right|_{V,N}$; pressure $p = \left.\frac{\partial U}{\partial V}\right|_{S,N}$; chemical potential $\mu = \left.\frac{\partial U}{\partial N}\right|_{S,V}$

dividing by V one can write ($dN = 0$) :

$$d\epsilon = T ds - p\, d\left(\frac{1}{n}\right)$$

or alternative form :

$$de = h d\rho + \rho T ds$$

- Perfect fluid evolution is adiabatic

let us combine the 1st law of thermodynamics with the $\underline{u}$-projection :

$$\left.\begin{aligned} \nabla_a e &= h \nabla_a \rho + T \nabla_a S \\ u^a \nabla_a e &= + h u^a \nabla_a \rho \end{aligned}\right\} \quad 0 = + T u^a \nabla_a s \qquad \forall T$$

$$\boxed{u^a \nabla_a s = 0} \quad \text{if } T \neq 0$$

The specific entropy is conserved along fluid lines!

Note now that:

$$\rho h u^b \nabla_b u_a = \rho u^b \nabla_b (h u_a) - \rho u_a u^b \nabla_b h$$

So one can write Euler equation as:

$$\rho u^b \nabla_b (h u_a) \underbrace{- \rho u_a u^b \nabla_b h + u_a u^b \nabla_b p}_{} + \nabla_a p = 0$$

$$u^a u^b (\rho \nabla_b h + \nabla_b p) =$$

$$dp = \rho dh - \rho T ds$$

$$= u^a u^b \rho T \nabla_b s = \rho T u^a u^b \nabla_b s = 0$$

so

$$\boxed{u^b \nabla_b (h u_a) = - \rho^{-1} \nabla_a p}$$

or, considering $\mathcal{L}_{\underline{u}} u_a = u^b \nabla_b u_a + u_b \nabla_a u^b = u^b \nabla_b u_a$,

$$\boxed{\mathcal{L}_{\underline{u}} u_a = -\rho^{-1} \nabla_a p - \nabla_a h}$$

SYMMETRIES OF RELATIVISTIC PERFECT FLUIDS

K^a a killing vector

T^{ab} a symmetric tensor

$\Big\} \Rightarrow J_a \equiv K^b T_{ba}$ is a conserved quantity, $\nabla_a J^a = 0$

"Komar currents"

Euler equation + K^a killing vector $\Rightarrow$

$$\boxed{\mathcal{L}_{\underline{u}} (h u_a K^a) = -\frac{1}{\rho} \mathcal{L}_{\underline{K}} p - \mathcal{L}_{\underline{K}} h}$$

if the fluids has symmetries generated by $\underline{K}$, then $\mathcal{L}_{\underline{K}}$ (fluid quantities) $= 0 \Rightarrow$

$$\mathcal{L}_{\underline{u}} (h u_a K^a) = u^a \nabla_a (h u_b K^b) = 0$$

$\Rightarrow$ $h u_b K^b$ <u>is conserved along fluid lines</u>.

Note:

- similar equation to geodesics : $u_b K^b = \text{const}$
  Here the fluid has an additional "h" due to the fact the fluid has pressure.

- AXISYMMETRIC FLOW

  $\underline{K} = \partial_\varphi$ in coordinate (t, r, θ, φ) ; $\underline{K} = (0,0,0,1)$

  $h u_a K^a = h u_\varphi = \text{const}$

  Newtonian limit :

  $h \mapsto 1$

  $h u_\varphi \mapsto \Omega r^2 \qquad \Omega = \frac{d\phi}{dt}$

- STATIONARY FLOW

$$\underline{k} = \underline{\partial_t}$$

$$h u_t = \text{const}$$

Newtonian limit :

$$\left(1+\epsilon+\frac{p}{\rho}\right)\left(1+\phi+\frac{1}{2}v^2\right) \overset{(*)}{\approx} \boxed{\frac{1}{2}v^2+\phi+\epsilon+\frac{p}{\rho}}$$

retain c^{-2} terms ~~[illegible]~~

$\Rightarrow$ Bernoulli equation:

$$u^a\nabla_a\left(h u_b k^b\right) = u^a\partial_a\left(h u_b k^b\right) = \frac{u^0}{c}\underbrace{\partial_t\left(h u_b k^b\right)}_{=0} + \frac{v^i}{c}\partial_i\left(h u_b k^b\right)$$

$$\boxed{v^i\partial_i\left(\frac{1}{2}v^2+\phi+\epsilon+\frac{p}{\rho}\right)=0}$$

(*)

$$h k^b u_b \underset{\substack{\uparrow \\ k^b=(1,0,0,0)}}{=} h u_0 \cong -\left(1+\epsilon+\frac{p}{\rho}\right)\left(1+\frac{\phi}{c^2}+\frac{v^2}{2c^2}+\ldots\right)$$

$$H := \frac{e_{int}+P}{m_B n} \rightarrow h = m_B c^2\left(1+\frac{H}{c^2}\right)$$

$$\cong -m_B c^2\left(1+\frac{H}{c^2}\right)\left(1+\frac{\phi}{c^2}+\frac{v^2}{2c^2}\right)$$

$$\cong -m_B c^{2}\left(c^2+H+\frac{1}{2}\frac{v^2}{c^2}+\phi\right)$$

– IRROTATIONAL FLOW

Vorticity tensor $\overset{def}{=}$ $\Omega_{ab} \equiv \nabla_a(h u_b) - \nabla_b(h u_a)$

Irrotational fluid iff $\Omega_{ab} = 0$

$$\Rightarrow \boxed{h u_a = \nabla_a \Psi} \qquad \text{potential of the flow}$$

the flow potential satisfies the following equation derived from the baryon mass conservation:

$$0 = \nabla_a(\rho u^a) = \nabla_a\left(\frac{\rho}{h}\nabla^a \Psi\right) = \frac{\rho}{h}\nabla_a\nabla^a\Psi + \nabla^a\Psi \cdot \nabla_a\left(\frac{\rho}{h}\right)$$

• <u>An irrotational flow is isentropic or has zero temperature</u>

Consider

$$u^a \Omega_{ab} = \underbrace{u^a \nabla_a (h u_b)}_{\text{Euler eq} \rightarrow = -\frac{1}{\rho}\nabla_b p} - u^a \nabla_b (h u_a) =$$

$$= -\frac{1}{\rho}\nabla_b p - u^a \nabla_b (h u_a) =$$

$$= -\frac{1}{\rho}\nabla_b p - u^a \nabla_b h \cdot u_a - u^a h \nabla_b u_a =$$

$$= -\frac{1}{\rho}\nabla_b p + \nabla_b h - h \underbrace{u^a \nabla_b u_a}_{=0}$$

Carter-Lichnerowicz equation:

$$\boxed{u^a \Omega_{ab} = -\frac{1}{\rho}\nabla_b p + \nabla_b h = T\nabla_b S = 0}$$

• <u>integral of motion for irrotational fluids in presence of Killing vectors</u>

Let us define the 1-form:

$$w_a \equiv h\, u_a = \nabla_a \Psi$$

Given a Killing vector k^a one has

$$k^b \nabla_{[b} w_{a]} = k^b \nabla_b w_a - k^b \nabla_a w_b =$$

$$= k^b \nabla_b w_a - w_b \nabla_a k^b + \nabla_a (k^b w_b) =$$

$$= \mathcal{L}_{\underline{k}}\, w_a + \nabla_a (k^b w_b)$$

This relation is called <u>Cartan identity</u> and holds in general for <u>any</u> n-form $\underline{\underline{w}}$ and vector $\underline{u}$:

$$\mathcal{L}_{\underline{u}}\, \underline{\underline{w}} = \underline{u} \cdot d\underline{\underline{w}} + d(\underline{u} \cdot \underline{\underline{w}})$$

where d is the exterior derivative (antisymmetric derivative combination)
$\cdot$ is the contraction over close indexes.

In the specific case of an irrotational fluid one has

$$\Omega_{ab} = \nabla_{[b} w_{a]} = 0$$

and for a Killing vector $\mathcal{L}_{\underline{k}}\, w_a = 0$, so that

$$0 = \mathcal{L}_{\underline{k}}\, w_a + \nabla_a (k^a w_b) = 0 + \nabla_a (k^a w_b) \Rightarrow$$

$$\boxed{h\, k^a u_a = \text{const}}$$

not only a constant along fluid lines but the same constant for all fluid lines.

- Newtonian limit and meaning of irrotational flow

Define a kinematical vorticity vector :

$$\Omega^a \equiv \frac{1}{2}\varepsilon^{abcd} u_b \nabla_c u_d$$

Properties :

- $\Omega^a u_a = 0$
- $\Omega_{ab} = 0 \Rightarrow \Omega^a = 0$, irrotational flow
- $\underline{\Omega} \simeq \frac{1}{c} \nabla \times \underline{v}$ in the Newtonian limit

Indeed in Newtonian hydrodynamics irrotational fluids are characterized by

$$\underline{\Omega} = \vec{\nabla} \times \underline{v} = 0 .$$

Using Euler equation one can show that

$$\partial_t \underline{\Omega} = \vec{\nabla} \times (\underline{v} \times \underline{\Omega}) - \vec{\nabla} \times \left(\frac{\vec{\nabla} p}{\rho} \right)$$

the above equation implies that if the fluid is

$$\left.\begin{array}{lll} \text{i) isentropic}, & ds = 0 \to \frac{\vec{\nabla} p}{\rho} = \vec{\nabla} h & \to \vec{\nabla} \times \vec{\nabla} h = 0 \\ \text{ii) barotropic}, & p = \mathcal{P}(\rho) \to \nabla p \times \nabla \rho = 0 & \\ \text{iii) incompressible}, & \rho = \text{const} \; ; \; \vec{\nabla} \times \vec{\nabla} p = 0 & \end{array}\right\} (*)$$

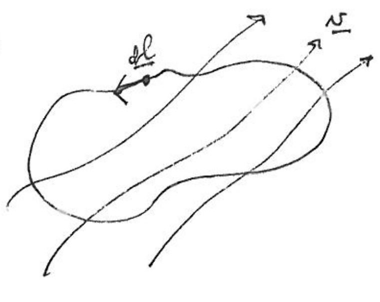
then

$$\partial_t \underline{\Omega} = \vec{\nabla} \times (\underline{v} \times \underline{\Omega})$$

and an initially irrotational fluid, $\underline{\Omega}(t=0) = 0$, remains irrotational.

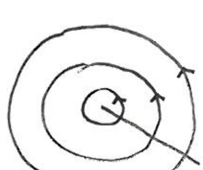
Moreover, Stokes theorem implies that the circulation along any close path γ is zero :

$$C_\gamma = \oint_\gamma \underline{v} \cdot d\underline{l} = \int_{A_\gamma} \nabla \times \underline{v} \cdot d\underline{a} = 0 \quad \forall \gamma$$



Examples

- Rigid body rotation $\underline{v} \propto r$



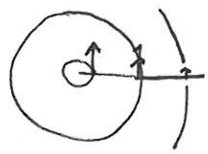
$\Rightarrow \underline{\Omega} \neq 0$

- Parallel flow with shear

$\Rightarrow \underline{\Omega} \neq 0$

- Rotation with $v \propto \frac{1}{r}$

  "irrotational vortex"



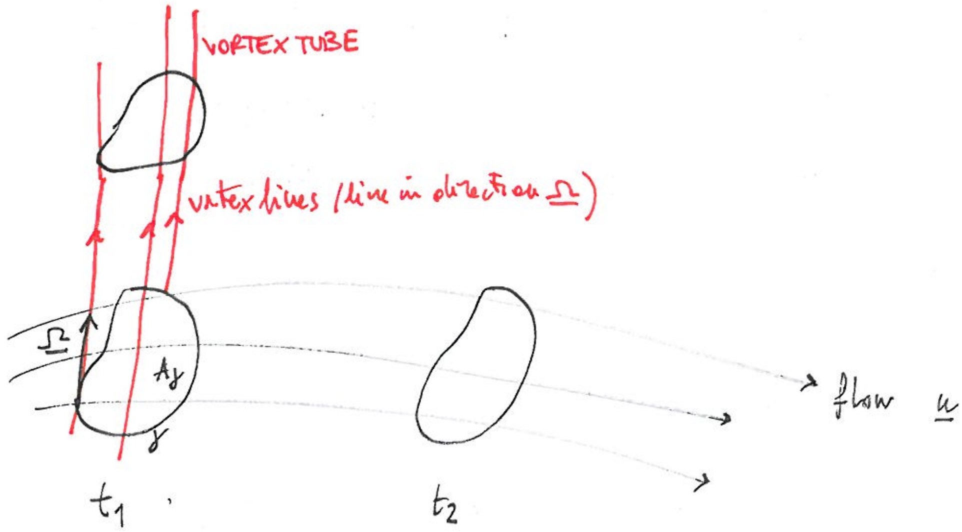
$\Rightarrow \underline{\Omega} = 0$

Kelvin circulation theorem:

$$\frac{dC}{dt} = \int_{A_\gamma} \left[\partial_t \underline{\Omega} + \nabla \times (\underline{v} \times \underline{\Omega}) \right] \cdot d\underline{a} = \int_{A_\gamma} \frac{\nabla \rho \times \nabla p}{\rho^2} \cdot d\underline{a} = \oint_\gamma \frac{\nabla p}{\rho} \cdot d\underline{l}$$

If satisfies one of ⊛, then $\boxed{\frac{dC}{dt} = 0}$.

This is also expressed as "the number of vortex lines through any area element remain constant" :



CONSERVATIVE FORM OF RELATIVISTIC HYDRODYNAMICS EQUATIONS

Euler equations are <u>nonlinear</u>, their solution is challenging since well-posedness can only proven locally. In astrophysical situations one encounters cases/applications in which solutions are non-unique and non-smooth, e.g. "shocks", "explosions", etc. These solutions are <u>physical</u> and they emerge from the "zero" viscosity limit of viscous fluids. To handle these solutions one needs to consider <u>weak form</u> of the equations (integral form); robust numerical approaches are based on 1^{st} order hyperbolic formulations in <u>conservative form</u>:

$$\partial_t u + \partial_i F^i = s .$$

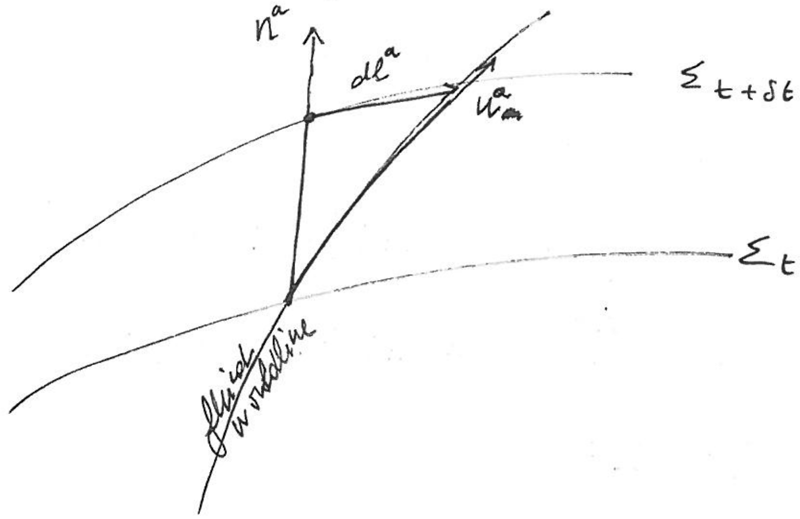
The key step to derive relativistic hydrodynamics in conservative form is to consider the 3+1 decomposition of T_{ab} based on <u>Eulerian observers</u> [Marti 1991, Banyuls + 1997].

Let us start writing the relation between the energy measured by the Eulerian observers and the fluid variables in the comoving frame:

$$E = n^a n^b T_{ab} = \rho h \, (u_a n^a)^2 - P$$

E: energy density in Eulerian frame

$(u_a n^a)$: what is this quantity?



Lorentz factor between the Eulerian and fluid frames:

$$d\tau_E = W \, d\tau_F$$

Velocity composition

$$u^a d\tau_F = n^a d\tau_E + (dl)^a$$

Contract with n^a:

$$n^a u_a \, d\tau_F = \underbrace{n^a n_a}_{=-1} d\tau_E + \underbrace{n_a (dl)^a}_{=0}$$

$$\Rightarrow n^a u_a = -\frac{d\tau_E}{d\tau_F} = -W$$

Fluid velocity relative to Eulerian observers: $V^a \equiv \frac{dl^a}{d\tau_E}$

$u^a = W(n^a + V^a)$ "3+1" decomposition of the fluid velocity

$$-1 = u^a u_a = W^2 \Big(\underbrace{n_a n^a}_{=-1} + 2 \underbrace{n^a V^a}_{=0} + V^a V_a \Big) \Rightarrow \boxed{W^{-2} = 1 - V^2}$$

components: $$V^i = \frac{\gamma^i{}_a u^a}{-n_a u^a} = \frac{\gamma^i{}_a u^a}{\alpha u^0} = \frac{1}{\alpha}\left(\frac{u^i}{u^0} + \beta^i \right)$$

Eulerian and fluid comoving quantities

$$E = \rho h W^2 - p$$

$$S_a = -\gamma^b_a n^c T_{bc} = (e+p) W^2 V_a = \rho h W^2 V_a$$

$$S_{ab} = \gamma^c_a \gamma^d_b T_{cd} = \rho h W^2 u_a u_b + p \gamma_{ab}$$

<u>Continuity equation</u>:

$$0 = \nabla_a (\rho u^a) = \frac{1}{\sqrt{-g}} \partial_a (\sqrt{-g} \rho u^a) = \frac{1}{\sqrt{-g}} \left[\partial_t (\alpha \sqrt{\gamma} \rho u^t) + \partial_i (\alpha \sqrt{\gamma} \rho u^i) \right] =$$

$$= \frac{1}{\sqrt{-g}} \Big\{ \partial_t (\sqrt{\gamma} \underbrace{\rho W}_{\equiv D}) + \partial_i \Big[\sqrt{\gamma} \rho \underbrace{\alpha u^0}_{W} \underbrace{(\alpha v^i - \beta^i)}_{\tilde{v}^i} \Big] \Big\} =$$

$$= \frac{1}{\sqrt{-g}} \left[\partial_t (\sqrt{\gamma} D) + \partial_i (\sqrt{\gamma} D \tilde{v}^i) \right]$$

$$\boxed{\partial_t(\sqrt{\gamma}D) + \partial_i(\sqrt{\gamma}D\hat{v}^i) = 0}$$

Energy equation:

$$0 = n_a \nabla_b T^{ab} = \nabla_a(T^{ab} n_a) - T^{ab}\nabla_a n_b = \dots =$$

$$= \partial_t(\sqrt{\gamma}E) + \partial_i\left[\sqrt{\gamma}(\alpha S^i - \beta^i E)\right] - \alpha\sqrt{\gamma}\, T^{ab}\nabla_b n_a =$$

$$= \partial_t(\sqrt{\gamma}E) + \partial_i\left[\sqrt{\gamma}(\alpha S^i - \beta^i E)\right] + \alpha\sqrt{\gamma}\left(K_{ij}S^{ij} - S^k\partial_k \ln\alpha\right)$$

$$\boxed{\partial_t(\sqrt{\gamma}E) + \partial_i\left[\sqrt{\gamma}(\alpha S^i - \beta^i E)\right] = -\alpha\sqrt{\gamma}\left(K_{ij}S^{ij} - S^k\partial_k \ln\alpha\right)}$$

Note that no source term depend on ∂_t, unlike other formulations.

Momentum equation:

$$0 = \nabla_a T^a{}_b = \dots = \frac{1}{\sqrt{-g}}\left[\partial_a\left(\sqrt{-g}\,T^a{}_b\right) - \frac{1}{2}T^{cd}\partial_b g_{cd}\right]$$

$b \to j$:

$$\boxed{\partial_t(\sqrt{\gamma}S_j) + \partial_i\left[\sqrt{\gamma}\left(S_j\hat{v}^i + p\,\delta^i_j\right)\right] = \sqrt{\gamma}\left(\frac{1}{2}\alpha S^{ik}\partial_j\gamma_{ik} + S_i\partial_j\beta^i - E\,\partial_j\alpha\right)}$$

The system is thus in conservation form with

$$u = \sqrt{\gamma}\,(D, S_j, E) \qquad F^i = \sqrt{\gamma}\,(D\hat{v}^i,\ S_j\hat{v}^i + p\,\delta^i_j,\ \alpha S^i - \beta^i E)$$

and source terms.

Note:

- D-equation $\Rightarrow$ $M_b \equiv \int d^3x \sqrt{\gamma}\,D$ is a conserved quantity, rest mass
- E-equation is often substituted with an equation for $\tau \equiv E - D$

- The system is <u>strongly hyperbolic</u> once closed by an EOS that is <u>causal</u> i.e. $c_s^2 = \frac{\partial p}{\partial \rho} < 1$

  characteristic fields and speeds are known from $M^i = \frac{\partial F^i}{\partial u}$.

  Note that diagonalization of M^i matrices is not trivial due to the fact that there exist primitive and conservative variables, and that the EOS closes the system.

- One needs to consider <u>primitive variables</u> $(\rho, \epsilon, p, v^i) = w$ and <u>conserved variables</u> $(D, S_j, \tau) = u$.

  Since one solves for u but T_{ab} is expressed in terms of w, a procedure that uses the EOS must be used to compute $u \longmapsto w$.

  This is in general a numerical procedure.

- <u>Special relativity limit</u>:

  $\alpha \to 1$; $\beta^i \mapsto 0$; $\gamma_{ij} \mapsto f_{ij}$;

  one obtains a conservative system with:

  $$u = (D, S_j, \tau) \quad \text{or} \quad (D, S_j, E)$$

  $$F^i = (Dv^i, S_j v^i + p\delta^i_j, \tau v^i + pv^i)$$

  and <u>sources $= 0$</u>.

  the special rel. limit $\Rightarrow$ energy and momentum conservation.

- <u>Newtonian limit</u>

  $v \ll c$; $W \mapsto 1$; $h \mapsto 1$; $e \to \rho$

  $$u = (\rho, \rho v_j, e)$$

  $$F^i = (\rho v^i, \rho v^i v_j + p\delta^i_j, (e+p)v^i)$$

and, in presence of porosity (or conservative terms):

$$S = \left(0, -\rho\,\partial_j \Phi\,,\; -\rho v^i \partial_i \Phi\right)$$