HYPERBOLIC EQUATIONS AND WELL-POSEDNESS

HYPERBOLIC EQS IN GR

- weak field GR, gravitational waves $\Box h_{ij} = \dots$
- Perturbation theory of black holes, gravitational waves
  Regge-Wheeler-Zerilli equation (spherical background) $(\partial_{tt} - \partial_{r_* r_*}) \phi_{em} = V_e \phi_{em} + S_{em}$
- 3+1 GR, harmonic gauge $\Box g_{\mu\nu} = \dots$
- 3+1 GR, matter equations for perfect fluids $D_\mu T^{\mu\nu} = 0 \rightarrow \partial_t u + \partial_i F^{(i)}(u) = s$
  Balance law for hydrodynamics

Hyperbolic "wave" equations naturally arise in causal theories. Basic model:

$$\boxed{\partial_{tt}\phi - c^2 \Delta \phi = 0} \quad *$$

linear, $c \in \mathbb{R}$, 2$^{\text{ND}}$ order equation (in t and x^i). First order form in time:

$$\Pi := \partial_t \phi \quad \Rightarrow$$

$$\boxed{\begin{aligned} \partial_t \phi &= \Pi \\ \partial_t \Pi &= c^2 \Delta \phi \end{aligned}}$$

Fully first order form:

$$\Pi := \partial_t \phi \quad , \quad \Psi_i := c\,\partial_i \phi \quad \Rightarrow$$

$$\boxed{\begin{aligned} \partial_t \phi &= \Pi \\ \partial_t \Pi &= c \sum_i \partial_i \Psi_i \\ \partial_t \Psi_i &= c\, \partial_i \Pi \end{aligned}}$$

Note the first equation does not contain spatial derivatives; it is a trivial equation usually omitted from the analysis.

the above system can be written in the general form:

$$\boxed{\partial_t u + M^i \partial_i u = S}$$

where $u = (\Pi, \Psi_i)$ is a collection ("vector") of field : STATE VECTOR, and M^i are matrices of constant coefficients.

A toy model for the equation above is the <u>advection equation</u>:

$$\boxed{\partial_t u + c\, \partial_x u = 0}$$

In 1D the advection equation has general solution $u(x-ct)$, so that the IVP with initial state $u(t=0,x) = u_0(x)$ has solution $u(t,x) = u_0(x-ct)$.
Note the general solution of the wave equation $*$ is a combination (in 1D) of two elementary waves "left"/"right" that correspond to the solution of the fully first order system once diagonalized into "advection equations" (more later).

In general, we consider the model for hyperbolic equations

$$\boxed{\partial_t u + M^i(u)\, \partial_i u = S}$$

where now the matrices M^i can depend on the solution (and on space and time).
Another form of the equation is

$$\boxed{\partial_t u + \partial_i F^i(u) = S}$$

where F^i are the FLUXES and the matrices M^i are simply the Jacobians:

$$M^i_{ab} := \frac{\partial F^i_a}{\partial u_a} .$$

WELL-POSEDNESS

Given a Cauchy or IVP problem one would wish to know if the solution

- exists;
- is unique;
- depends continuously on the initial data.

Well-posed IVP $\stackrel{def}{=}$ if exist a norm $\|\cdot\|$ such that

$$\| u(t,x) \| \leq K e^{\alpha t} \| u(0,x) \|$$

with K and α independent on the initial data..

→ the solution (norm) is bounded in time for all initial data (or a certain class).

<u>Example 1a</u>

$$\begin{cases} \partial_t u = -\partial_x^2 u \\ u(t=0,x) = e^{ikx} \end{cases}$$

"inverse heat equation" (note the wrong sign)

look for solutions $u(t,x) = e^{\sigma t + ikx}$:

$$\sigma e^{\sigma t + ikx} = -(ik)^2 e^{\sigma t + ikx} \qquad \Rightarrow \quad \sigma = k^2$$

$$u = e^{k^2 t + ikx}$$

grows exponentially in a way that depends on the initial frequency of the Fourier mode ... ILL-POSED.

<u>Example 1b</u>

$$\partial_t u = + \partial_x^2 u$$

Correct sign! $\Rightarrow \quad \sigma = -k^2$

The Fourier mode is damped (except $k=0$)

<u>Example 2a</u>

$$\partial_t^2 \phi = - \partial_x^2 \phi$$

Laplace equation in which one dimension is taken as "time" first order form: $\qquad u_1 := \partial_t \phi \,, \quad u_2 := \partial_x \phi \quad \Rightarrow$

$$\partial_t u_1 = - \partial_x u_2$$
$$\partial_t u_2 = + \partial_x u_1$$

Fourier mode evolution: $\qquad \phi = \phi_0 \, e^{\kappa t + ikx} \quad \Rightarrow$ the solution grows at a rate dependent on initial data $\Rightarrow$ ILL-POSED (Laplace equation has a ill-posed IVP but a well posed BVP)

Example ~~2~~2b

$$\partial_t^2 \phi = \partial_x^2 \phi \qquad \text{wave equation} \rightarrow \text{well posed.}$$

Example 3

$$\begin{aligned} \partial_t u_1 &= \partial_x u_1 + \lambda \partial_x u_2 \\ \partial_t u_2 &= \partial_x u_2 \end{aligned} \qquad \rightarrow \qquad \partial_t u = M \partial_x u \qquad M = \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}$$

$\lambda = 0$: 2 independent advection equations, M is diagonal ✓

$\lambda \neq 0$: coupled system, M cannot be diagonalized ✗

Solution for Fourier mode initial data:

$$u_2 = A_2 \, e^{ik(t+x)}$$

$$u_1 = (\underbrace{ik\lambda A_2 t}_{} + A_1) e^{ik(t+x)}$$

grow in time and depends on initial frequency

HYPERBOLICITY

Consider the system $\partial_t u + M^i \partial_i u = 0$

with M^i $N \times N$ matrices with constant coefficients (real or complex, we take real here)

n_i : unit vector in arbitrary direction

PRINCIPAL SYMBOL $\stackrel{def}{=}$ $P(n_i) \equiv M^i n_i$

the system is

STRONGLY HYPERBOLIC iff $\begin{cases} P \text{ has \underline{real} eigenvalues } \forall n_i \\ P \text{ has a complete set of eigenvectors } \forall n_i \end{cases}$

WEAKLY HYPERBOLIC iff $\begin{cases} P \text{ has real eigenvalues } \forall n_i \\ P \text{ does \underline{not} have a complete set of eigenvectors} \end{cases}$

For a strongly hyperbolic system one has:

$$P r_a = \lambda_a r_a \qquad \text{or} \qquad PR = \Lambda R$$

where R is the matrix of right eigenvectors (columns) and $\Lambda = \mathrm{diag}(\lambda_a)$.

SYMMETRIZER $\stackrel{def}{=}$ $H \equiv (R^{-1})^{+} R^{-1}$

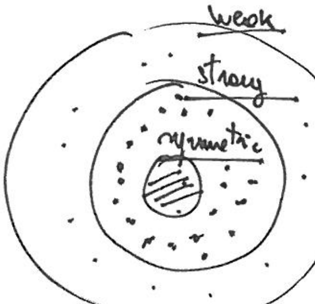
Properties:

- symmetric (Hermitian) and positive definite
- $HP = P^{+}H^{+} = P^{+}H$
  i.e. HP is also symmetric. Proof:

$$HP = (R^{-1})^{+} R^{-1} P \underset{\substack{\uparrow \\ R^{-1}PR = \Lambda \\ R^{-1}P = \Lambda R^{-1}}}{=} (R^{-1})^{+} \Lambda R^{-1} = \left((R^{-1})^{+} \Lambda R^{-1}\right)^{+}$$

- Not unique: R and λ depend on normalization the system is

SYMMETRIC HYPERBOLIC $\stackrel{def}{=}$ H is independent on n:
(all M^i are symmetric)



WELL-POSEDNESS OF SYMMETRIC HYPERBOLIC SYSTEMS

Use the symmetriser to construct a <u>norm</u> (energy):

$$\langle u, v \rangle := u^+ H v$$

$$\| u \|^2 := \langle u, u \rangle = u^+ H u$$

Consider a Fourier mode $u = \tilde{u}(t)\, e^{ik\, x \cdot n}$ and the evolution of the norm:

$$\partial_t \|u\|^2 = \partial_t (u^+ H u) = \partial_t (u^+) H u + u^+ H \partial_t (u) =$$

$$\partial_t u = M^i \partial_i u \rightarrow e^{ik\, x\cdot n} \partial_t \tilde{u} = M^i \left(\tilde{u} \partial_i e^{ik\, x \cdot n} \right) = M^i \tilde{u}\, ik n_i e^{ik\, x\cdot n}$$

$$= e^{ik\, x \cdot n}\, ik P \tilde{u}$$

$$= ik \tilde{u}^+ P^+ H \tilde{u} - ik \tilde{u}^+ H P \tilde{u} =$$

$$= ik \tilde{u}^+ \left(P^+ H - H P \right) \tilde{u} = 0$$

The energy (norm) is conserved and the solution is bounded.

Moreover for a strongly hyperbolic system one can consider the following projections:

CHARACTERISTIC FIELDS $\stackrel{def}{=}$ $\quad w := R^{-1} u$

- In 1D one has:

$$R^{-1} \cdot \dot{u} = R^{-1} \cdot M \partial_x u$$

$$\partial_t (R^{-1} u) = R^{-1} P \partial_x u$$

$$\partial_t w = \Lambda R^{-1} \partial_x u$$

$$\partial_t w = \Lambda \partial_x w$$

the original system is diagonalized into "elementary waves".

- In multi-D the system does not decouple in general since the eigenfunctions depend on n_i.

- To prove that a system is strongly hyperbolic it is also possible to recast it in the form above using linear combination of the original variables

Comments for real life

- $M^i = M^i(t, x, u)$ — one needs linearization around a background solution; the hyperbolicity analysis will at most prove well-posedness locally; the energy norm will in general grow in time but in a way that can be bounded ~~by the~~ independently on the initial data.

- $S(u) \neq 0$ and P.P. — complex system of equations are often analyzed in principal part (highest derivatives); source terms and lower derivatives terms do not affect hyperbolicity in simple cases in which proofs can be given.

WAVE EQ. ANALYSIS

Consider the fully first order form and Cartesian coordinates and

$n_i = (n_x, n_y, n_z)$

The principal symbol is:

$$P(n_i) = c \begin{pmatrix} 0 & -n_x & -n_y & -n_z \\ -n_x & 0 & 0 & 0 \\ -n_y & 0 & 0 & 0 \\ -n_z & 0 & 0 & 0 \end{pmatrix}$$

P is symmetric $\Rightarrow$ $H = \mathbb{1}$ => wave eq. is symmetric hyperbolic

Eigenvalues: $\lambda = (c, -c, 0, 0)$

From eigenvectors one can compute the characteristic variables associated to the speeds λ:

$$\lambda = \pm c \qquad w_\pm = \Pi \mp \sum_i n_i \Psi_i$$

$$\lambda = 0 \qquad w_{3,4} = \sum_i s_i^{(3,4)} \Psi_i \qquad \text{with} \quad s_i^{(3,4)} n^i = 0$$

→ characteristics ("modes") with speed $\pm c$ propagate along n_i

→ transverse modes do not propagate.

Energy ~~density~~ density (norm):

$$\|u\|^2 = \Pi^2 + \sum_i (\Psi_i)^2 = (\partial_t \phi)^2 + c^2 \sum_i (\partial_i \phi)^2$$

Integrated energy and conservation:

$$\frac{d}{dt} E = \frac{d}{dt} \int d^3x \, \|u\|^2 = \frac{d}{dt} \int d^3x \left[(\partial_t \phi)^2 + c^2 \sum_i (\partial_i \phi)^2 \right] = \frac{d}{dt} \int d^3x \left[\Pi^2 + \sum_i (\Psi_i)^2 \right] =$$

$$= 2 \int d^3x \Big(\Pi \underbrace{\partial_t \Pi}_{= c \sum_i \partial_i \Psi_i} + \sum_i \Psi_i \underbrace{\partial_t \Psi_i}_{c\, \partial_i \Pi} \Big) = 2c \sum_i \underbrace{\int \partial_i (\Pi \Psi_i) \, d^3x}_{\int \text{divergence}} = 0$$

for data with compact support.