HYPERBOLIC FREE-EVOLUTION SCHEMES

GENERALIZED HARMONIC GAUGE (GHG) AND SYMM. HYP. SCHEME

Consider the Ricci tensor expression in some coordinate system

$$R_{ab} = \underbrace{-\frac{1}{2} g^{cd} \partial_c \partial_d g_{ab} + \nabla_{(a}\Gamma_{b)}}_{(*)} + g^{cd} g^{ef} \left(\partial_e g_{ca} \partial_f g_{db} - \Gamma_{ace} \Gamma_{bdf} \right)$$

where $\Gamma_a = g^{bc} \Gamma_{abc}$ and Γ_{abc} are the christoffel symbols, and

$$\nabla_a \Gamma_b = \partial_a \Gamma_b - g^{cd} \Gamma_{cab} \Gamma_d .$$

The terms $(*)$ are those containing the 2^{ND} derivatives of the metric:

$$g^{cd} \partial_c \partial_d g_{ab} \cong \Box_g g_{ab}$$

$\nabla_{(a}\Gamma_{b)}$ can break hyperbolicity of Einstein equations in vacuum:

$$\boxed{0 = R_{ab} \cong -\frac{1}{2} \Box g_{ab} + \ldots}$$

Consider now harmonic coordinates x^α [de Donder 1921]:

$$\Box x^\alpha = 0$$

which means

$$0 = g^{ab} \nabla_a \nabla_b x^\alpha \underset{\substack{\uparrow \\ x^\alpha \text{ is not a vector}}}{=} g^{ab} \nabla_a (\partial_b x^\alpha) = g^{ab} \Big[\partial_a (\underbrace{\partial_b x^\alpha}_{\delta_b^\alpha}) - \underbrace{\partial_e x^\alpha}_{\delta_e^\alpha} \Gamma^e_{ab} \Big] =$$

$$= 0 - g^{ab} \Gamma^\alpha_{ab} =: -\Gamma^\alpha$$

$$\Gamma^a = 0 \quad \Rightarrow \quad \boxed{g^{cd} \partial_c \partial_d g_{ab} = 2 g^{cd} g^{ef} \left(\partial_e g_{ca} \partial_f g_{bd} - \Gamma_{ace} \Gamma_{bdf} \right)}$$

GR is a manifestly hyperbolic system in HG.

More in general one can use a <u>generalized harmonic gauge</u> (GHG) in the form:

$$-\Gamma^a = H^a$$

where $H^a(x^\alpha)$ can be

- a prescribed function
- a field that satisfies symm. hyp. equations

and obtain:

$$R_{ab} = -\frac{1}{2}\Box g_{ab} + \nabla_{(a}H_{b)} + \text{low order derivative terms}$$

which is also symm. hyperbolic $\rightarrow$ GHG system.

Define the quantity

$$\boxed{Z_a := H_a + \Gamma_a}$$

if we consider Z_a an <u>algebraic constraint</u> the GHG system can be written:

$$\begin{cases} 0 = R_{ab} - \nabla_{(a}Z_{b)} \\ 0 = Z_a \end{cases}$$

How does the constraint evolve?

$$0 = \mathrm{Tr}\left[R_{ab} - \nabla_{(a}Z_{b)}\right] = g^{ab}R_{ab} - g^{ab}\nabla_{(a}Z_{b)} = R - \frac{1}{2}g^{ab}(\nabla_a Z_b + \nabla_b Z_a) =$$

$$= R - \frac{1}{\cancel{2}}\cancel{2}\nabla_a C^a$$

$$0 = R_{ab} - \nabla_{(a}Z_{b)} - \mathrm{Tr}[\dots]\frac{1}{2}g_{ab}$$

$$= R_{ab} - \frac{1}{2}Rg_{ab} - \nabla_{(a}Z_{b)} + \frac{1}{2}\nabla^d Z_d\, g_{ab}$$

$$0 = \nabla^a\left\{R_{ab} - \nabla_{(a}Z_{b)} - \mathrm{Tr}[\dots]\frac{1}{2}g_{ab}\right\} =$$

$$0 = \underbrace{\nabla^a R_{ab} - \frac{1}{2} g_{ab} \nabla^a R}_{=0,\ \text{Bianchi identity}} - \frac{1}{2} \nabla^a \nabla_a Z_b - \frac{1}{2} \nabla^a \nabla_b Z_a + \frac{1}{2} g_{ab} \nabla^a \nabla^d Z_d =$$

$$= -\frac{1}{2} \nabla^a \nabla_a Z_b + \frac{1}{2} g_{ab} \nabla^a \nabla^d Z_d \underbrace{- \frac{1}{2} \nabla^a \nabla_b Z_a}_{} =$$

$$= g^{ac} \nabla_c \nabla_b Z_a \underset{\substack{\uparrow \\ \text{Ricci identity}}}{=} g^{ac} (\nabla_b \nabla_c Z_a) - g^{ac} R^f{}_{acb} Z_f = \nabla_b \nabla^a Z_a - R^f{}_b Z_f$$

$$= -\frac{1}{2} \nabla^a \nabla_a Z_b + \cancel{\frac{1}{2} \nabla_b \nabla^d Z_d} - \cancel{\frac{1}{2} \nabla_b \nabla^a Z_a} + \frac{1}{2} R_{bf} Z^f =$$

$$= + \frac{1}{\cancel{2}} \nabla^a \nabla_a Z_b + \frac{1}{\cancel{2}} R_{bf} Z^f = 0$$

This equation guarantees that for $Z_a = 0 = \dot{Z}_a$ the constraint $Z_a = 0$ remains during the evolution.

We have already found this equation: the Z4 system!

Indeed GHG is formally equivalent to the Z4 system, the difference is that:

- Z4 : there is no gauge fix, the field Z_a is evolved;

- GHG : $Z_a = 0$ is an algebraic constraint.

If one takes Z4 and fix $Z_a = H_a + \Gamma_a$, then one obtains a GR solution in harmonic gauge.

The Hamiltonian and momentum constraints in GHG can be formulated using a 4-momentum constraint:

$$\boxed{C_a := \left(R_{ab} - \frac{1}{2} g_{ab} R\right) n^b = 0}$$

First-order formulation [Alvi 2002] [Lindblom et al. 2006]:

$$\Pi_{ab} := -n^c \partial_c g_{ab} \qquad \Phi_{iab} := \partial_i g_{ab}$$

Principal part:

$$\begin{cases} \partial_t g_{ab} - \beta^k \partial_k g_{ab} \cong 0 \\ \partial_t \Pi_{ab} - \beta^k \partial_k \Pi_{ab} \cong 0 \\ \partial_t \Phi_{iab} - \beta^k \partial_k \Phi_{iab} + \alpha\, \partial_i \Pi_{ab} \cong 0 \end{cases}$$

define: $\quad u = (g_{ab}, \Phi_{iab}, \Pi_{ab}) \quad \rightarrow \quad \partial_t u + A^k \partial_k u = s$

Notes:

- In numerical applications the constraint $C_{iab} := \partial_i g_{ab} - \Phi_{iab}$ is observed to grow. Numerical evolutions are different from mathematical statement (necessary but not sufficient) as additional complications can arise from discretization errors.

- The formulation can be augmented with constraints damping terms, as we have discussed with the Z4 scheme.

- Pretorius 2006 breakthrough used GHG with constraint damping in 2nd-order form.

- The above formulation can produce shocks since it is not linearly degenerate ($\nabla \lambda_i \cdot r_i = 0$). For example the (ti)-component is

$$\partial_t \beta^i - \beta^k \partial_k \beta^k \cong 0$$

i.e. Burger-type of equation.

- A formulation that is i) 1^{st}-order symm-hyperbolic ii) linearly degenerate and with C_{iab} numerically stable in different tests has been proposed by Lindblom et al 2006.

FROM GHG AND "ADM" TO BSSN

- GHG system is hyperbolic but with gauge fixed.
- 3+1 GR "ADM" is weakly hyperbolic but with gauge freedom (α, β^i).

How can we build something strongly/symmetric hyperbolic given a generic gauge?
(note we assume the gauge is in strongly/symmetric hyperbolic form, or can be put)

Let us consider the 3+1 decomposition and follow some of the ideas of GHG.

$$(\partial_t - \mathcal{L}_\beta)^2 \gamma_{ij} \cong R_{ij} + \dots$$

$$\cong \underbrace{-\frac{1}{2} \Delta \gamma_{ij}}_{\text{this is a "good" term}} + \underbrace{\gamma_{ik} \partial_j \partial_\ell \gamma^{k\ell} + \gamma_{jk} \partial_i \partial_\ell \gamma^{k\ell}}_{\text{these are problematic terms, exactly as for the derivation of GHG from 4D equations ...}} + \dots$$

[Nakamura, Oohara, Kojima 1987] $\rightarrow$ introduce auxiliary variables

$$F^k \cong \partial_\ell \gamma^{k\ell}$$

such that the system is strongly hyperbolic <u>without specifying the gauge</u>.

The problem reduces to finding an equation for F^k ...

... a solution is to combine the evolution equation for γ_{ij}

$$\partial_t F^k \sim \partial_t \partial_i \gamma^{ki} \sim \partial_i \partial_t \gamma^{ki} \sim \partial_i K^{ki}$$

and e.g. use the constraint (momentum)

... analogously one can remember the Z4 system and $F^k \sim Z^k$ so that

$$\partial_t F^k \sim \partial_t Z^k \propto (\text{mom. constraint}) -$$

Let us now play the same game with the 3+1 conformal decomposition of ADM eqs.:

$$\Delta^k_{ij} := \tilde{\Gamma}^k_{ij} - \underbrace{\bar{\Gamma}^k_{ij}}_{\Gamma^i_{jk} \text{ of the flat metric}} = \frac{1}{2}\tilde{\gamma}^{kl}\left(D_i\tilde{\gamma}_{kl} + D_j\tilde{\gamma}_{il} - D_l\tilde{\gamma}_{ij}\right)$$

REM: Δ^i_{jk} depend on the background metric.

$$\tilde{R}_{ij} = D_k\Delta^k_{ij} - \Delta^k_{il}\Delta^l_{kj} \cong -\frac{1}{2}\tilde{\gamma}^{kl}\left(D_k D_l\tilde{\gamma}_{ij} + \tilde{\gamma}_{ik} D_i D_l\tilde{\gamma}^{kl} + \tilde{\gamma}_{jk} D_i D_l \tilde{\gamma}^{kl}\right) + \ldots$$

and again

$$(\partial_t - \partial_\beta)^2\,\tilde{\gamma}_{ij} \cong \tilde{R}_{ij} + \ldots$$

[Shibata & Nakamura 1995]
[Baumgarte & Shapiro 1996] $\longrightarrow$ $\boxed{-\tilde{\Gamma}^l := D_l\tilde{\gamma}^{kl}}$

Auxiliary conformal variables

To obtain an equation one can either use Z4 (Z4c) or follow the ideas sketched above:

$$\partial_t\tilde{\gamma}^{ij} = 2\alpha\tilde{A}^{ij} + \beta^k D_k\tilde{\gamma}^{ij} - \tilde{\gamma}^{kj} D_k\beta^i - \tilde{\gamma}^{ik} D_k\beta^j + \frac{2}{3} D_k\beta^k\,\tilde{\gamma}^{ij}$$

take the divergence:

$$D_i(\partial_t\tilde{\gamma}^{ij}) = \partial_t D_i\tilde{\gamma}^{ij} = \partial_t(-\tilde{\Gamma}^j) = 2\alpha\underbrace{D_i\tilde{A}^{ij}}_{(\cdot)} + \ldots$$

use the mom. constraint:

$$(\cdot)\quad D_j\hat{A}^{ij} = -6\hat{A}^{ij} D_j\ln\psi + \frac{2}{3}\tilde{D}^i K + 8\pi\psi^4 p^i = D_j\tilde{A}^{ij} + \Delta^i_{jk}\tilde{A}^{kj}$$

The full BSSN (or BSSNOK) system is

$$
\begin{cases}
\mathcal{L}_m \psi = \frac{\psi}{6} \tilde{D}_i \beta^i - \alpha K \\
\mathcal{L}_m \tilde{\gamma}_{ij} = -2\alpha \tilde{A}_{ij} - \frac{2}{3} \tilde{D}_k \beta^k \tilde{\gamma}_{ij} \\
\mathcal{L}_m K = \dots \\
\mathcal{L}_m \tilde{A}_{ij} = \dots \\
\mathcal{L}_m \tilde{\Gamma}^i = \frac{2}{3} D_k \beta^k \tilde{\Gamma}^i + \tilde{\gamma}^{jk} D_j D_k \beta^i + \frac{1}{3} \tilde{\gamma}^{ij} D_j D_k \beta^k - 2 \tilde{A}^{ij} D_j \alpha + \\
\qquad - 2\alpha \left[8\pi \psi^4 p^i - \tilde{A}^{jk} \Delta^i_{jk} - 6 \tilde{A}^{ij} D_j \ln\psi + \frac{2}{3} \tilde{\gamma}^{ij} D_j K \right]
\end{cases}
$$

+ gauge (to be specified)

+ Hamiltonian and momentum constraints $\quad C_0 = 0 = C_i$

+ Algebraic constraints

$$
\begin{cases}
\det \tilde{\gamma}_{ij} = f \\
\tilde{\gamma}^{ij} \tilde{A}_{ij} = 0 \\
\tilde{\Gamma}^i + D_j \tilde{\gamma}^{ij} = 0
\end{cases}
$$

<u>Note</u> : This is analogous to Z4 but with an equation less $[\dot{\Theta} \simeq C_0]$ and with $\tilde{\Gamma}^i$ playing the role of Z_i.