INITIAL DATA PROBLEM

Wout: determine γ_{ij} and K_{ij} on Σ_0 such that:

1) GR constraints are satisfied

$$R + K^2 - K_{ij} K^{ij} = 16\pi E$$
$$D_j K^j_{\ i} - D_i K = 8\pi P_i$$

2) the solution has a physical meaning, e.g. represents black-holes or neutron stars of interest for astrophysics.

Here we assume the metric fields are given, thus (E, P_i).

Constraint equations are 4 eqs $\Rightarrow$ whatever strategy one follows, one needs to prescribe some fields and solve for others.

FREE DATA — CONSTRAINED DATA

The choice of free/constrained data is based on

- Physical intuition
- Astrophysics motivations
- Mathematical necessity (linear eqs, de-coupled eqs, well-posedness)

Examples:

- Choose γ_{ij}, solve for 4 components of K_{ij}. How to choose the remaining 2 components?
- Use conformal decomposition ($\alpha = -10$)

$$\tilde{D}_i \tilde{D}^i \psi - \frac{1}{8}\tilde{R}\psi + \frac{1}{8}\hat{A}_{ij}\hat{A}^{ij}\psi^{-7} - \frac{1}{12}K^2\psi^5 + 2\pi\tilde{E}\psi^{-3} = 0$$

$$\tilde{D}_j \hat{A}^{ij} - \frac{2}{3}\psi^6 \tilde{D}^i K - 8\pi\tilde{P}^i = 0$$

- to gain intuition for free data;
- to take advantage of uniqueness properties.

Approach used in 3+1 numerical relativity, two methods: CTT and CTS.

CONFORMAL TRANSVERSE-TRACELESS (CTT) METHOD [York 1973, 1977]

This method rely on a decomposition of $\hat{A}^{ij}$ into <u>longitudinal</u> and <u>transverse</u> part:

$$\underbrace{\hat{A}^{ij}}_{\text{traceless}} = \underbrace{(\tilde{L}X)^{ij}}_{\substack{\text{traceless}\\ \text{and}\\ \text{"longitudinal"}}} + \underbrace{\hat{A}^{ij}_{TT}}_{\substack{\text{traceless}\\ \text{and}\\ \text{transverse } (\text{div } \hat{A}_{TT} = 0)}}$$

$$\tilde{D}_j \hat{A}^{ij}_{TT} = 0 \quad \wedge \quad \tilde{\gamma}_{ij} \hat{A}^{ij}_{TT} = 0$$

$$\boxed{(\tilde{L}X)^{ij} \equiv \tilde{D}^j X^i + \tilde{D}^i X^j - \frac{2}{3} \tilde{D}_k X^k \tilde{\gamma}^{ij}}$$

Given $\hat{A}^{ij}$, the longitudinal/transverse decomposition is determined by taking the divergence of $\hat{A}^{ij}$, $\tilde{D}_j \hat{A}^{ij}$, and solving for the vector X^i:

$$\tilde{D}_j (\tilde{L}X)^{ij} = \tilde{D}_j \hat{A}^{ij}$$

$$\underbrace{\tilde{\Delta}_L X^i \equiv \tilde{D}_j (\tilde{L}X)^{ij} = \tilde{D}_j \tilde{D}^j X^i + \frac{1}{3} \tilde{D}^i \tilde{D}_j X^j + \tilde{R}^i{}_j X^j}_{\text{CONFORMAL VECTOR LAPLACIAN OPERATOR}} = \tilde{D}_j \hat{A}^{ij}$$

$\exists!$ longitudinal + transverse decomposition of $\hat{A}^{ij}$ $\Longleftrightarrow$ $\exists!$ solution of conformal vector Laplacian

<u>Theorem by Cantor</u>:

- Asymptotically flat manifold
- $\dfrac{\partial^2 \tilde{\gamma}_{ij}}{\partial x^k \partial x^l} = \mathcal{O}(r^{-3})$

$\Big\}$ $\Rightarrow$ Existence and uniqueness is guaranteed.

The CTT equations are:

$$\boxed{\begin{array}{l}\tilde{D}_i\tilde{D}^i\psi - \frac{1}{8}\tilde{R}\psi + \frac{1}{8}\left[(\tilde{L}X)_{ij} + \hat{A}^{TT}_{ij}\right]\left[(\tilde{L}X)^{ij} + \hat{A}^{ij}_{TT}\right]\psi^{-7} - \frac{1}{12}K^2\psi^5 + 2\pi\tilde{E}\psi^{-3} = 0 \\ \tilde{\Delta}_L X^i - \frac{2}{3}\psi^6\tilde{D}^i K = 8\pi\tilde{P}^i\end{array}}$$

with:

FREE DATA : $\tilde{\gamma}_{ij}$, $\hat{A}^{ij}_{TT}$, K, $\tilde{E}$, $\tilde{P}^i$

CONSTRAINED DATA : ψ, X^i

Notes:

- ψ is given by the Lichnerowicz equation, elliptic and quasi-linear
- X^i is given by the $\tilde{\Delta}_L X^i = \dots$ equation, elliptic and linear
- Under the assumption $K = \text{const}$ (constant mean curvature -CMC- hypersurfaces) the system <u>partially decouples</u>:

$$\text{CMC} \Rightarrow \tilde{\Delta}_L X^i = 8\pi\tilde{P}^i \;;\; X^i \to \text{Lichnerowicz eq.}$$

CTT, CONFORMALLY FLAT AND TIME SYMMETRIC INITIAL DATA

Simplest choice of free data:

- $\tilde{\gamma}_{ij} = f_{ij}$: conformally flat
- $\hat{A}^{ij}_{TT} = 0$
- $K = 0$: maximal slicing
- $\tilde{E} = 0 = \tilde{P}^i$: vacuum

$\Rightarrow$

- $\tilde{D}_i = \mathcal{D}_i \to \mathcal{D}_i\mathcal{D}^i = \Delta$ Laplacian
- $\tilde{R} = 0$
- $\tilde{L} = L \to \Delta_L$ flat vector Laplacian

& the equations become:

$$\boxed{\begin{array}{l}\Delta\psi + \frac{1}{8}(LX)_{ij}(LX)^{ij}\psi^{-7} = 0 \\ \Delta_L X^i = \Delta X^i + \frac{1}{3}\mathcal{D}^i\mathcal{D}_j X^j = 0\end{array}}$$

Asymptotic flatness implies the following boundary conditions :

$$\psi = 1 \qquad\qquad r \to +\infty \qquad\qquad (i_0)$$
$$X^i = 0$$

The solution to the initial data problem now depends on the topology of Σ_0; different choices can lead to inner boundary conditions, thus determining different solutions to the boundary value problem with the elliptic equations.

Examples :

$\Sigma_0 = \mathbb{R}^3$

No inner boundary conditions.

$$\left.\begin{array}{lll} \Delta_L X^i = 0 & & \text{on } \Sigma_0 \\ X^i = 0 & & \text{on } \partial\Sigma_0 \end{array}\right\} \Rightarrow X^i \equiv 0$$

$$X^i \equiv 0 \Rightarrow (LX)^{ij} = 0 \rightarrow \quad \left.\begin{array}{lll} \Delta\psi = 0 & & \text{on } \Sigma_0 \\ \psi = 1 & & \text{on } \partial\Sigma_0 \end{array}\right\} \Rightarrow \psi \equiv 1$$

Solution is flat space :

$$\gamma_{ij} = f_{ij} \quad ; \quad K_{ij} = 0$$

$\Sigma_0 = \mathbb{R}^3 - B$

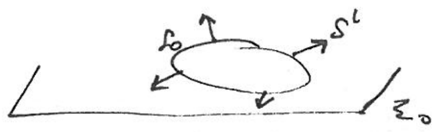
S_0 sphere delimiting the ball B.

Inner boundary condition, let us choose $\underline{X}|_{S_0} = 0$.

$$\left.\begin{array}{lll} \Delta_L X^i = 0 & & \text{on } \Sigma_0 \\ X^i = 0 & & \text{on } \partial\Sigma_0 \\ X^i = 0 & & \text{on } S_0 \end{array}\right\} \Rightarrow X^i \equiv 0$$

$$\left.\begin{array}{lll} \Delta\psi = 0 & & \text{on } \Sigma_0 \\ \psi = 1 & & \text{on } \partial\Sigma_0 \\ \psi = 1 & & \text{on } S_0 \end{array}\right\} \Rightarrow \psi = 1 \quad \text{... let us choose another boundary condition}$$

γ_{ij} has a closed minimal surface.



S_0 is minimal $\Longleftrightarrow$ $D_i s^i \big|_{S_0} = 0$, s^i unit normal vector

Note : $D_i s^i = 0$ analogous to $\nabla_a n^a = 0$ (maximal slicing).

$$D_i s^i = \psi^{-6} \tilde{D}_i (\psi^6 s^i) \underset{\uparrow}{=} \psi^{-6} \mathcal{D}_i (\psi^6 s^i) = 0$$

$$\tilde{\gamma}_{ij} = f_{ij}$$

unit normal vec.* with respect to $\tilde{\gamma}_{ij}$ $\quad \tilde{s}^i \equiv \psi^{-2} s^i \quad \rightarrow \quad \boxed{\mathcal{D}_i (\psi^4 \tilde{s}^i)\Big|_{S_0} = \frac{1}{\sqrt{f}} \frac{\partial}{\partial x^i} \left(\sqrt{f}\, \psi^4 \tilde{s}^i \right)\Big|_{S_0} = 0}$

This is an equation for ψ giving an inner boundary at the ball surface.

Introduce spherical coords :

- $x^i = (r, \theta, \varphi)$
- $f_{ij} = \mathrm{diag}(1, r^2, r^2 \sin^2\theta)$, $\sqrt{f} = r^2 \sin\theta$
- S_0 is a sphere of radius $r = a$
- $s^i = (1, 0, 0)$

the equation becomes :

$$\frac{1}{r^2} \partial_r \left(\psi^4 r^2 \right)\Big|_{r=a} = \left(\frac{\partial \psi}{\partial r} + \frac{\psi}{2r} \right)\Big|_{r=a} = 0$$

which is a mixed Neumann-Dirichlet type of condition.

$$\left.\begin{array}{lll} \Delta \psi = 0 & & \text{on } \Sigma_0 \\ \psi = 1 & & \text{on } \partial\Sigma_0 \\ r^{-2} \partial_r (\psi^4 r^2)\Big|_{r=a} & & \text{on } S_0 \end{array}\right\} \Rightarrow \psi = 1 + \frac{a}{r}$$

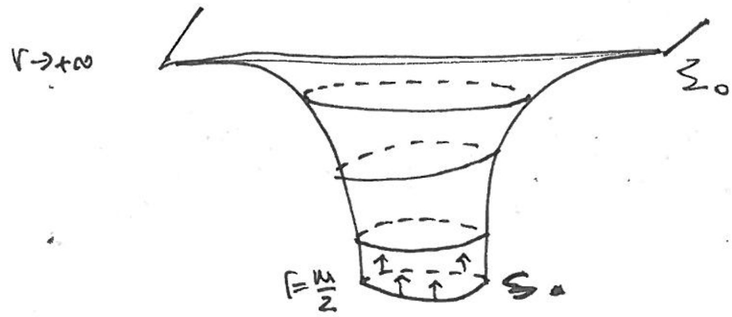
the meaning of the parameter a can be found by computing the ADM mass

* $\tilde{\gamma}(\tilde{s}, \tilde{s}) = \psi^{-4} \tilde{\gamma}(s, s) = \gamma(s, s) = 1$

$$M_{ADM} = -\frac{1}{2\pi} \lim_{r\to+\infty} \oint_{r=\text{const}} \frac{\partial\Psi}{\partial r}\, r^2 \sin\theta \, d\theta\, d\varphi = -\frac{1}{2\pi} \lim_{r\to+\infty} 4\pi r^2 \partial_r \left(1+\frac{a}{r}\right) = 2a$$

$\Rightarrow \qquad \Psi = 1 + \dfrac{m}{2r}$ This is Schwarzschild spacetime in isotropic coords ($K_{ij}=0$)

Embedding diagram :



$\mathbb{R}^3$

$\Sigma_0 = S^2 \times \mathbb{R}$

A solution with this topology can be constructed from the previous one. The basic idea is to extend the range $r\in[\frac{m}{2},+\infty)$ to $r\in(0,+\infty)$ by gluing another copy of Σ_0 at S smoothly. The transformation is

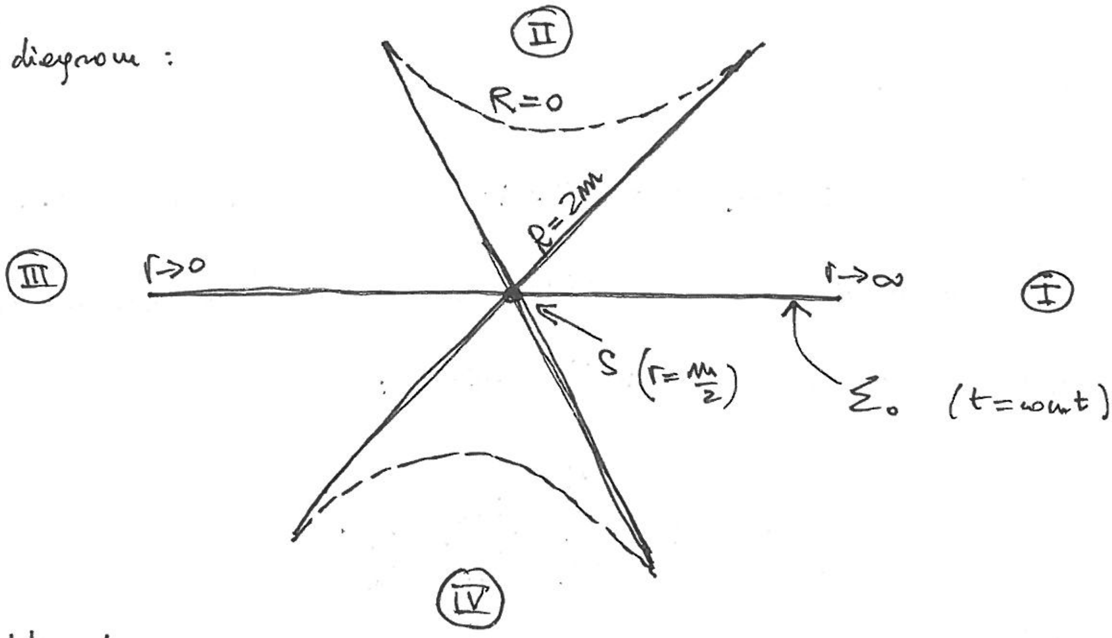
$$r \longrightarrow r' = \frac{m^2}{4r}$$

and the metric "on the other side" has exactly the same form (the transformation is an ISOMETRY).

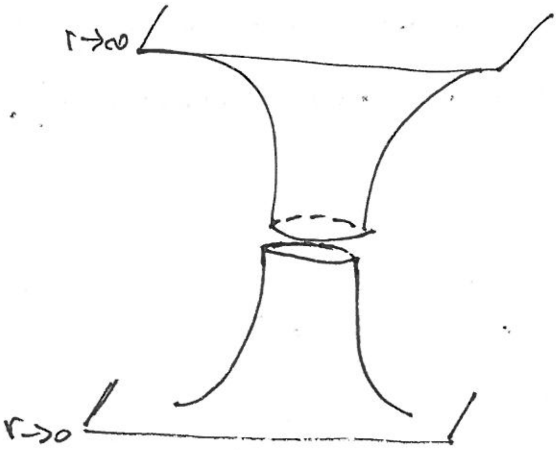
The region $r\to 0$ is not a center, but another asymptotically flat end. We obtain a slice of Schwarzschild connecting region I and III of the Kruskal diagram.

The sphere S is the Einstein-Rosen bridge.

Kruskal diagram :



Embedding diagram:



<u>Moment time symmetry</u>

Note that all these spacetimes have

$$\left.\begin{array}{l}\hat{A}^{ij}_{TT}=0\\ X^i=0\end{array}\right\} \Longrightarrow \boxed{K_{ij}=0}$$

From $K_{ij}=0 \Rightarrow \partial_t \gamma_{ij}=0$ at <u>$t=0$</u>

So the slice is <u>momentarily</u> static (Nothing guarantees that K_{ij} remains zero at $t>0$ ...)

This also means that ds^2 is invariant under $t \longleftrightarrow -t$, and that $\beta^i=0$.

These data are called also <u>time-symmetric</u>.

Multiple black holes

$\Delta\psi = 0$ is a linear equation $\Longrightarrow$ we can superpose solutions.

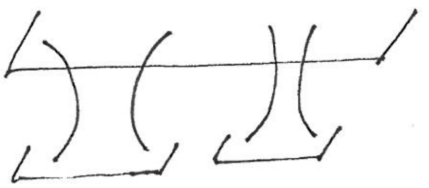
Multiple black hole solutions with a moment of time-symmetry can be constructed by considering solution ansatz :

$$\psi = 1 + \sum_{a=1}^{N} \frac{M_a}{(x^i - c_a^i)}$$

where c_e^i are the coordinate "center" of the black holes.

The construction of these data is non trivial and the precise form of the solution depends on the topology (boundary conditions).

Examples $N=2$:

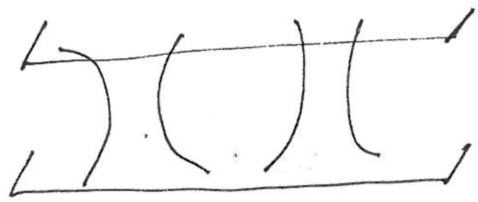


Brill-Lindquist

N+1 A.F. universes

(simple superposition)

$\mathbb{R}^3 - \{O_1, O_2\}$



Misner

2 isometric universes

(method of "images")

$\mathbb{R}^3 -$ "2 balls"

Misner data have been used for head-on collisions, with boundary conditions adapted to the horizons.

These are different data and have different content in terms of GW.

CTT, CONFORMALLY FLAT, BOWEN-YORK INITIAL DATA

Let us consider vacuum solutions with:

$$\tilde{\gamma}_{ij} = f_{ij} \quad ; \quad \hat{A}^{ij}_{TT} = 0 \quad ; \quad K = 0 \quad ; \quad \tilde{E} = \tilde{P}^i = 0$$

and choose: $\Sigma_0 = \mathbb{R}^3 \setminus \{0\}$

the topology is $S^2 \times \mathbb{R}$.

The removed point is called <u>puncture</u>.

Instead of the trivial solution $X^i = 0$, Bowen-York proposed the following solution to the $\Delta_L X^i = 0$:

$$\boxed{X^i = -\frac{1}{4r}\left(7 f^{ij} P_j + \frac{P_j x^j x^i}{r^2}\right) - \frac{1}{r^3}\epsilon^{ij}{}_k S_j x^k}$$

where <u>x^i are Cartesian coordinates</u>;

$O = (0,0,0)$;

P_i and S_j are <u>parameters</u>.

For $r^2 = x^2 + y^2 + z^2 \neq 0$ the solution is regular.

For the above solution one has:

$$\hat{A}^{ij} = (LX)^{ij} = \frac{3}{2r^3}\left[x^i P^j + x^j P^i + \left(f^{ij} - \frac{x^i x^j}{r^2}\right) P_k x^k\right] +$$

$$+ \frac{3}{r^5}\left(\epsilon^{ik}{}_l S_k x^l x^j + \epsilon^{jk}{}_l S_k x^l x^i\right)$$

where at $r \to +\infty$

$$\frac{3}{2} r^{-3}\left[x^i P^j + x^j P^i + \left(f^{ij} - \frac{x^i x^j}{r^2}\right) P_k x^k\right] \sim O(r^{-2})$$

$$\frac{3}{r^5}\left[\epsilon S x x + \epsilon S x x\right] \sim O(r^{-3})$$

Physical meaning of the parameters

ADM momentum ($\psi \sim 1$, $K = 0$) ~~[illegible]~~ :

$$P_i^{ADM} = \frac{1}{8\pi} \lim_{r \to +\infty} \oint_{r=\text{const}} \hat{A}_{ik} x^k r \sin\theta \, d\theta \, d\varphi =$$

$$= \frac{1}{8\pi} \lim_{r \to +\infty} \oint_{r=\text{const}} \frac{3}{2} r^{-2} \Big[x_i P_k x^k + \underbrace{x_k x^k}_{=r^2} P_i - \underbrace{\Big(\underbrace{f_{ik} x^k}_{=x_i} - x_i \frac{\overbrace{x_k x^k}^{=r^2}}{r^2} \Big)}_{=0} P_j x^j \Big] \cdot \sin\theta \, d\theta \, d\varphi =$$

$$+ \text{ term } O(r^{-3}) =$$

$$= \frac{1}{8\pi} \lim_{r \to +\infty} \oint_{r=\text{const}} \frac{3}{2} P_k \frac{x_i x^k}{r^2} + \oint_{r=\text{const}} \frac{3}{2} P_i =$$

$$= \frac{3}{16\pi} P_k \underbrace{\oint_{r=\text{const}} \frac{x^i x^k}{r^2} \sin\theta \, d\theta \, d\varphi}_{} + \frac{3}{16\pi} P_i \cdot 4\pi =$$

$$\int \frac{x^i x^k}{r^2} d\Omega = \delta^{ik} \int \frac{(x^k)^2}{r^2} d\Omega = \delta^{ik} \frac{1}{3} \int \frac{r^2}{r^2} d\Omega = \frac{4\pi}{3} \delta^{ik}$$

$$= \frac{3}{16\pi} \left(\frac{4\pi}{3} + 4\pi \right) P_i = P_i$$

A similar calculation shows that $J_i^{(ADM)} = S_i$.

Note that the Cartesian coordinates employed by Bowen-York belong to the quasi-isotropic gauge since the metric is conformally flat.

Example: Schwarzschild limit

$P_i = S_i = 0 \Rightarrow \underline{X}^i = 0 \Rightarrow$ Schwarzschild in isotropic coords.

Example: Kerr limit?

$P_i = 0 \wedge S_i \neq 0$ DOES NOT correspond to Kerr!

[Garat & Price]^{2000} showed that there exist <u>no</u> Kerr foliation:

i) is axisymmetric;

ii) reduces to slices of $t = $ const Schwarzschild;

iii) is conformally flat.

$\rightarrow$ Bowen-York data represent black hole solution that is <u>not stationary</u>
i.e. it is a black-hole "plus" some GWs ...

<u>Solution of Ham. constraint</u>

Since $\hat{A}^{ij} \neq 0$ the Lichnerowicz equation needs to be solved ~~analytically~~ numerically.

Possible approaches:

- Generalized Misner-type data [Cook 1990]. Solve using the Neumann-Dirichlet condition at the throats, but using non-vanishing curvature.

- Generalized Brill-Lindquist data [Brandt & Brügmann 1996] or <u>punctures</u>.
Solve on $\mathbb{R}^3$ by analytically separating the singular behaviour.
Ansatz:

$$\psi = \psi_{BL} + u \quad , \text{ with } \psi_{BL} := \sum_{a=1}^{N} \frac{m}{|x^i - c_a^i|}$$

the equation for ψ_{BL} is:

$$\Delta \psi_{BL} = 0 \quad \text{on } \mathbb{R}^3 \setminus \{0_i\}$$

Lichnerowicz equation for u gives:

$$\Delta u + \frac{\hat{A}_{ij}\hat{A}^{ij}}{8\psi_{BL}^{7}}\left(1+\frac{u}{\psi_{BL}}\right)^{-7} = 0 \qquad \text{on } \mathbb{R}^3$$

with outer boundary conditions given by A.F. :

$$u \approx 1 + \frac{k}{r} \quad \text{or} \quad \partial_r u = \frac{1-u}{r} \qquad r \to +\infty$$

and no special boundary conditions at the punctures. Note in fact that

$$\psi_{BL} \sim |x^i - c_a^i|^{-1}, \quad x^i \approx c_a^i$$

$$\hat{A}_{ij}\hat{A}^{ij} \sim |x^i - c_e^i|^{p}, \quad x^i \approx c_e^i \quad \text{with } p = \begin{cases} -6 & S_i \neq 0 \\ -4 & S_i = 0 \end{cases}$$

$$\frac{\hat{A}_{ij}\hat{A}^{ij}}{8\psi_{BL}^{7}} \sim |x^i - c_e^i|^{q}, \quad x^i \approx c_e^i \quad \text{with } q = \begin{cases} +1 & S_i \neq 0 \\ +3 & S_i = 0 \end{cases}$$

which imply:

$$\Delta u = 0 \quad \text{at} \quad x^i \approx c_e^i$$

and $\exists!$ solution on $\mathbb{R}^3$.

As Brill-Lindquist data, each black-hole has a separate A.F. region.

CONFORMAL THIN SANDWICH (CTS) METHOD [York 1999]

Consider the "kinematical equation" for γ_{ij} with the $a=-4$ conformal scaling for the extrinsic curvature:

$$\mathcal{L}_m \tilde{\gamma}^{ij} = 2\alpha \bar{A}^{ij} + \frac{2}{3} \tilde{D}_k \beta^k \tilde{\gamma}^{ij}$$

$$= \partial_t \tilde{\gamma}^{ij} - \mathcal{L}_\beta \tilde{\gamma}^{ij} = \partial_t \tilde{\gamma}^{ij} + (\tilde{L}\beta)^{ij} + \frac{2}{3} \tilde{D}_k \beta^k \tilde{\gamma}^{ij}$$

It can be re-written as an equation for A^{ij}:

$$\bar{A}^{ij} = (2\alpha)^{-1} \left[\partial_t \tilde{\gamma}^{ij} + (\tilde{L}\beta)^{ij} \right]$$

and used as an "alternative decomposition" to the TT case.

Similarly, $\bar{A}^{ij} = \psi^{-6} \hat{A}^{ij}$, and one has

$$\hat{A}^{ij} = (2\tilde{\alpha})^{-1} \left[\partial_t \tilde{\gamma}^{ij} + (\tilde{L}\beta)^{ij} \right] \quad \text{with} \quad \tilde{\alpha} \equiv \psi^{-6} \alpha .$$

The CTS scheme employs the Lichnerowicz equation for ψ coupled to a new equation from the momentum constraint:

$$\boxed{\tilde{D}_j \left(\tilde{\alpha}^{-1} (\tilde{L}\beta)^{ij} \right) + \tilde{D}_j \left(\tilde{\alpha}^{-1} \dot{\tilde{\gamma}}^{ij} \right) - \frac{4}{3} \psi^6 \tilde{D}^i K - 16\pi \tilde{P}^i = 0}$$

with $\dot{\tilde{\gamma}}^{ij} \equiv \partial_t \tilde{\gamma}^{ij}$, and

FREE DATA : $\tilde{\gamma}_{ij}$; $\dot{\tilde{\gamma}}_{ij}$; K ; $\tilde{\alpha}$; $(\tilde{E}, \tilde{P}^i)$

CONSTRAINED DATA : ψ, β^i

The eq. for β^i is elliptic and linear and it decouple from the Lichnerowicz if $K=0$.

The presence of $\dot{\tilde{\gamma}}_{ij}$ might bring intuition on how to choose free data.

However, $\tilde{\alpha}$ has a difficult interpretation...

An extension of the CTS method is as follows [Pfeiffer & York 2003]

Consider the dynamical equation for the trace of K_{ij} :

$$\mathcal{L}_m K = -\psi^{-4}\left(\tilde{D}_i \tilde{D}^i \alpha + 2\tilde{D}_i \ln\psi \, \tilde{D}^i \alpha\right) + \alpha\left[4\pi(E+S) + \tilde{A}_{ij}\tilde{A}^{ij} + \frac{K^2}{3}\right]$$

and combine it with the identity

$$\tilde{D}_i \tilde{D}^i \alpha + 2\tilde{D}_i \ln\psi \, \tilde{D}^i \alpha = \psi^{-1}\left[\tilde{D}_i \tilde{D}^i (\alpha\psi) + \alpha \tilde{D}_i \tilde{D}^i \psi\right]$$

and with the Hamiltonian constraint $\tilde{D}_i \tilde{D}^i \psi = \dots$, one gets :

$$\boxed{\tilde{D}_i \tilde{D}^i(\tilde{\alpha}\psi^7) - (\tilde{\alpha}\psi^7)\left[\frac{1}{8}\tilde{R} + \frac{5}{12}K^2\psi^4 + \frac{7}{8}\hat{A}_{ij}\hat{A}^{ij}\psi^{-8} + 2\pi(\tilde{E} + 2\underset{\substack{\parallel \\ S\psi^8}}{\tilde{S}})\psi^{-4}\right] + \underbrace{(\partial_t K - \beta^i \tilde{D}_i K)}_{\mathcal{L}_m K}\psi^5 = 0}$$

This equation can be combined to Lichnerowicz and to the momentum constraint of CTS to obtain the EXTENDED CTS (XCTS)

FREE DATA : $\tilde{\gamma}_{ij}$; $\dot{\tilde{\gamma}}_{ij}$; K ; $\dot{K}$; $(\tilde{E}, \tilde{S}, \tilde{p}^i)$

CONSTRAINED DATA : $\psi, \tilde{\alpha}, \beta^i$

Notes :

- 3 equations system, does not decouple for $K=0$;
- Non-unique solutions for certain choices of initial data ;
- As CTS, XCTS assume $\det \tilde{\gamma}_{ij} = f$;

+ Give control on time development of initial data ;

+ Better description than CTS and CTT of stationary spacetimes since $\dot{\tilde{\gamma}}_{ij} = 0 = \dot{K}$ are necessary conditions for ∂_t being a Killing vector ;

Example: Static black hole

Consider the punctured hypersurface $\Sigma_0 = \mathbb{R}^3 - \{0\}$ with free data:

- $\hat{f}_{ij} = f_{ij}$: conformally flat
- $\dot{\tilde{\gamma}}^{ij} = 0 = \dot{K}$: "zero time derivatives"
- $K = 0$: maximal hypersurface

the XCTS becomes:

$$\left.\begin{array}{ll} 1) & \Delta\psi + \frac{1}{8}\hat{A}_{ij}\hat{A}^{ij}\psi^{-7} = 0 \\ 2) & \mathcal{D}_j\left[\tilde{\alpha}^{-1}(L\beta)^{ij}\right] = 0 \\ 3) & \Delta(\tilde{\alpha}\psi^7) - \frac{7}{8}\hat{A}_{ij}\hat{A}^{ij}\psi^{-1}\tilde{\alpha} = 0 \end{array}\right\} \text{ on } \Sigma_0$$

with A.F. outer boundary conditions on $\partial\Sigma_0$.

- $\beta^i \equiv 0$ is a solution of 2)

$$\left.\begin{array}{l} \beta^i = 0 \\ \dot{\tilde{\gamma}}^{ij} = 0 \end{array}\right\} \Rightarrow \hat{A}^{ij} = 0 \Rightarrow \begin{array}{l} 1) \ \Delta\psi = 0 \\ 3) \ \Delta(\tilde{\alpha}\psi^7) = 0 \end{array}$$

- $\psi = 1 + \frac{m}{2r}$ is a solution of 1) and it is regular on Σ_0 (punctured)

- A solution of 3) is

$$\tilde{\alpha}\psi^7 = 1 + \frac{a}{r}$$

with a undetermined constant. The lapse is

$$\alpha = \psi^6\tilde{\alpha} = (\tilde{\alpha}\psi^7)\psi^{-1} = \left(1 + \frac{a}{r}\right)\psi^{-1} = \left(1 + \frac{a}{r}\right)\left(1 + \frac{m}{2r}\right)^{-1} = \frac{r+a}{r + \frac{m}{2}}$$

The value of "a" can be determined at the puncture:

$$\frac{r+a}{r+\frac{M}{2}} \approx \frac{2a}{M} \qquad r\to 0$$

$$\alpha_0 = \lim_{r\to 0} \alpha = ?$$

- Natural choice : $\alpha_0 = 1 \Rightarrow a = \frac{M}{2} \Rightarrow$ <u>Geodesic slicing</u> $\alpha \equiv 1$
- Other choice : $\alpha_0 = -1 \Rightarrow a = -\frac{M}{2} \Rightarrow \alpha = \dfrac{1-\frac{M}{2r}}{1+\frac{M}{2r}}$

Notes:

- both choices lead to momentarily static initial slice for Schwarzschild;
- their time-development is however different;
- $\alpha \equiv 1$ (geodesic slicing) will give a nontrivial evolution;
- $\alpha = \dfrac{1-\frac{M}{2r}}{1+\frac{M}{2r}}$ has a <u>trivial</u> evolution in time.

  in this case t is the standard Schwarzschild coordinate $\to \partial_t$ is a Killing vector.
- the choice $\alpha_0 = -1$ contradicts the assumption $\alpha > 0$ for the lapse. Near the A.F. flat end $r\to 0$ the coordinate time is running backwards.

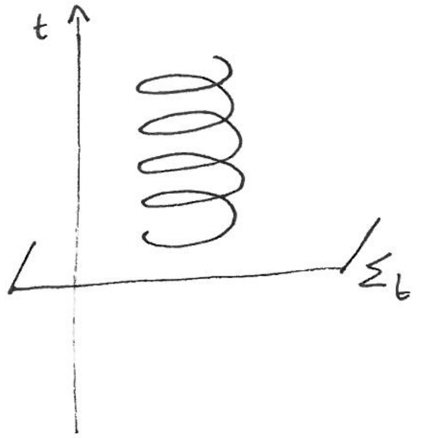
INITIAL DATA FOR BINARY SYSTEMS

Gravitational waves emission leads to orbit circularization.
Coalescence simulations might need to start with "circular" initial data.
What is the symmetry of circular binary?
How to impose a binary is in circular orbit?

Stationary spacetimes have ∂_t as killing vector.
Rotational symmetry is impose by a killing vector ∂_φ.
Circular orbits have <u>none</u> of the two symmetries, but the spacetime must be invariant if one moves of an angle $d\phi$ in a time dt because

$$d\phi = \Omega\, dt \qquad \Omega = \text{const, orbital frequency.}$$

Circular orbits can be imposed by imposing a HELICAL Killing vector



that must be the combination of ∂_t and ∂_φ.
An inertial observer would write:

$$\xi^a = \left(\frac{\partial}{\partial t}\right)^a + \Omega \left(\frac{\partial}{\partial \phi}\right)^a$$

and one would demand $\mathcal{L}_\xi g = 0$.

If now one chooses a <u>corotating</u> frame with $(\partial_t)^a = \xi^a$, then in this frame the helical Killing vector can be easily imposed by asking

$$\dot{\gamma}_{ij} = 0 = \dot{K}_{ij} .$$

the XCTS formalism would naturally account for helical killing vec.